Persistence and stability of a stage-structured predator-prey model with time delays

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Abstract

A delayed predator-prey model with stage structure for the predator is proposed and investigated. Sufficient conditions are derived for persistence, the local and global asymptotic stability of a positive equilibrium of the model. Numerical simulations are presented to illustrate the feasibility of our main results.

Keywords: Stage structure; Predator-prey model; Time delay; Persistence; Stability

1. Introduction

Predator-prey systems are very important in the models of multi-species population dynamics and have been studied by many authors [1–3]. It is assumed in the classical predator-prey model that each individual predator admits the same ability to attack prey. This assumption is obviously unrealistic for many animals. In the natural world, there are many species whose individuals have a life history that take them through two stages, immature and mature, where immature predators are raised by their parents, and the rate they attack at prey and the reproductive rate can be ignored. Stage-structured models have received much attention in recent years. Mathematical analyses...
for single species models with stage structure were performed by Aiello and Freedman [4], Freedman and Wu [5,6], Kuang and So [7], and Brauer and Ma [8].

In [4], Aiello and Freedman built and studied a time delay model of single species growth with stage structure as follows:

\[
\begin{align*}
\dot{x}_i(t) &= ax_m(t) - \gamma x_i(t) - \alpha e^{-\gamma s} x_m(t - \tau), \\
\dot{x}_m(t) &= \alpha e^{-\gamma s} x_m(t - \tau) - \beta x_m^2(t), \quad t > \tau,
\end{align*}
\]

(1.1)

where \(x_i(t)\) is the immature population density, \(x_m(t)\) is the mature population density. \(a > 0\) represents the birth rate, \(\gamma > 0\) is the immature death rate, \(\beta > 0\) is the mature death and overcrowding rate, \(\tau\) is the time to maturity. The term \(\alpha e^{-\gamma s} x_m(t - \tau)\) represents those immatures who were born at time \(t - \tau\) and survive at time \(t\) (with the immature death rate \(\gamma\)), and exit from the immature population and enter the mature population.

Recently, Wang and Chen [9] and Magnusson [10] proposed and investigated predator-prey models with stage structure for the predator to analyze the influence of a stage structure for the predator on the dynamics of predator-prey models. But these models ignore the duration time of immature predators. In [11], a stage-structured ratio-dependent predator-prey model was proposed and investigated by Wang et al. to show the effect of the duration time of immature predator on the global dynamics of predator-prey system. Sufficient conditions are obtained in [11] for the permanence and global stability of a positive equilibrium of the proposed model.

An important problem in predator-prey theory and related topics in mathematical ecology, concerns the global stability of ecological system with time delays. However, most of the global stability or convergence results appearing so far for delayed ecological systems require that the instantaneous negative feedbacks dominate both delayed feedback and interspecific interactions. Such a requirement is rarely met in real systems since feedbacks are generally delayed. This leads to the standing question: Under what conditions will the global stability of a nonnegative steady state of a delay differential system persist when time delays involved in some part of the negative feedbacks are small enough? Kuang [12] presented a partial answer to this open question for Lotka–Volterra-type systems.

Motivated by the works of Aiello and Freedman [4], Wang et al. [11], and Kuang [12], in the present paper, we study the combined effects of stage structure for predator and time delay due to negative feedback of prey on the global dynamics of predator-prey model. To do so, we study the following delayed system:
\[ \begin{align*}
\dot{x}(t) &= x(t)(r_1 - a_{11}x(t - \tau_1) - a_{12}y_2(t)), \\
y_1(t) &= x(t)y_2(t) - \gamma y_1(t) - ax(t - \tau_2)y_2(t - \tau_2), \\
y_2(t) &= ex(t - \tau_2)y_2(t - \tau_2) - a_{22}y_2^2(t),
\end{align*} \tag{1.2} \]

where \( x(t) \) denotes the density of the prey at time \( t \), \( y_1(t) \) is the density of immature individual predators at time \( t \), and \( y_2(t) \) denotes the density of mature individual predators at time \( t \). \( r_1, a_{11}, a_{12}, a_{22}, \alpha \), and \( \gamma \) are positive constants. \( r_1 \) is the intrinsic growth rate of the prey, \( a_{11} \) is the density-dependent coefficient of the prey, \( a_{12} \) is the capturing rate of mature predators, \( \alpha / a_{12} \) is the rate of conversion of nutrients into the reproduction rate of the mature predator, \( \gamma \) is the death rate of the immature predators, and \( a_{22} \) is the death and overcrowding rate of the mature predator. \( \tau_1 \geq 0 \) and \( \tau_2 > 0 \) are constant delays. \( \tau_1 \) denotes the delay due to the negative feedback of the prey species. The term \( ex(t - \tau_2)y_2(t - \tau_2) \) stands for the number of immature predators that were born at time \( t - \tau_2 \) which still survive at time \( t \) and are transferred from the immature stage to the mature stage at time \( t \). It is assumed in (1.2) that immature individual predators do not feed on prey and do not have the ability to reproduce.

The initial conditions for system (1.2) take the form of

\[ \begin{align*}
x(\theta) &= \phi(\theta), & y_1(\theta) &= \psi_1(\theta), \\
\phi(\theta) &\geq 0, & \psi_1(\theta) &\geq 0, & \theta &\in [-\tau, 0], \\
\phi(0) &> 0, & \psi_1(0) &> 0, & i &= 1, 2, \tag{1.3}
\end{align*} \]

where \( \tau = \max\{\tau_1, \tau_2\} \), \( \Phi = (\phi(\theta), \psi_1(\theta), \psi_2(\theta)) \in C([-\tau, 0], R^3_{+0}) \), the Banach space of continuous functions mapping the interval \([-\tau, 0]\) into \( R^3_{+0} \), where we define

\[ R^3_{+0} = \{(x_1, x_2, x_3) : x_i \geq 0, \ i = 1, 2, 3\} \]

and the interior of \( R^3_{+0} \):

\[ R^3_+ = \{(x_1, x_2, x_3) : x_i > 0, \ i = 1, 2, 3\}. \]

For continuity of the initial conditions, we require

\[ y_1(0) = \int_{-\tau_2}^{0} ax e^{\lambda s} \phi(s) \psi_2(s) \, ds. \tag{1.4} \]

The organization of this paper is as follows. In Section 2, sufficient conditions are established for the positivity of solutions and the persistence of system (1.2) with initial conditions (1.3) and (1.4). In Section 3, we are concerned with the local stability of a positive equilibrium of system (1.2). An estimation on the length of delays for which the local stability is maintained is also provided. In Section 4, a set of easily verifiable sufficient conditions are derived for the
global asymptotic stability of a positive equilibrium of system (1.2) by means of a suitable Lyapunov functional. Finally, numerical simulations are presented to illustrate our main results.

2. Uniform persistence

We first show the positivity and boundedness of solutions of system (1.2) with initial conditions (1.3) and (1.4).

Lemma 2.1. Solutions of system (1.2) with initial conditions (1.3) and (1.4) are positive for all $t \geq 0$.

Proof. Let $(x(t), y_1(t), y_2(t))$ be a solution of system (1.2) with initial conditions (1.3) and (1.4). Let us consider $y_2(t)$ for $t \in [0, \tau_2]$. It follows from the third equation of system (1.2) that

$$y_2(t) = xe^{-\tau_2 \phi(t - \tau_2)\psi_2(t - \tau_2)} - a_{22}y_2^2(t)$$

since $\phi(\theta) \geq 0$, $\psi_2(\theta) \geq 0$ for $\theta \in [-\tau_2, 0]$. Therefore, a standard comparison argument shows

$$y_2(t) \geq \frac{y_2(0)}{1 + a_{22}y_2(0)t},$$

i.e. $y_2(t) > 0$ for $t \in [0, \tau_2]$.

By the first equation of system (1.2), we obtain

$$x(t) = x(0) \exp \left\{ \int_0^t [r_1 - a_{11}x(s - \tau_1) - a_{12}y_2(s)] ds \right\} > 0 \quad \text{for} \quad t \geq 0.$$

From (1.4) and the second equation of system (1.2), $y_1(t)$ can be written as

$$y_1(t) = \int_{t-\tau_2}^t xe^{-\tau(s-s')}x(s)y_2(s)ds.$$

Hence the positivity of $x(t), y_2(t)$ on $[-\tau_2, \tau_2]$ implies that of $y_1(t)$ for $t \in [0, \tau_2]$.

In a similar way we treat the intervals $[\tau_2, 2\tau_2], \ldots, [n\tau_2, (n + 1)\tau_2], \ n \in \mathbb{N}$. Thus, $x(t) > 0$, $y_1(t) > 0$ and $y_2(t) > 0$ for all $t \geq 0$. This completes the proof.

Lemma 2.2 (Song and Chen, 2001 [13]). Consider the following equation:

$$\dot{x}(t) = ax(t - \tau) - bx(t) - cx^2(t),$$

where $a, b, c, \tau > 0$, $x(t) > 0$ for $t \in [-\tau, 0]$. We have

(i) If $a > b$, then $\lim_{t \to +\infty} x(t) = (a - b)/c$. 


(ii) If $a < b$, then $\lim_{t \to +\infty} x(t) = 0$.

**Lemma 2.3.** Let $z(t) = (x(t), y_1(t), y_2(t))$ denote any positive solution of system (1.2) with initial conditions (1.3) and (1.4). Then there exits a $T > 0$ such that

$$0 < x(t) \leq M_1, \quad 0 < y_i(t) \leq M_{i+1} \quad (i = 1, 2) \quad \text{for } t \geq T,$$

where

$$M_1 = \frac{r_1}{a_{11}} e^{r_1 \tau_1}, \quad M_2 = \frac{x_1 M_3}{\gamma} (1 - e^{-\gamma \tau_2}), \quad M_3 > M_3^* = \frac{x_1}{a_{11} a_{22}} e^{r_1 \tau_1 - \gamma \tau_2}. \quad (2.4)$$

**Proof.** Suppose $z(t) = (x(t), y_1(t), y_2(t))$ is a solution of system (1.2) which satisfies (1.3) and (1.4). According to the first equation of system (1.2), it follows from the positivity of the solution that

$$\frac{dx_1(t)}{dt} \leq x_1(t)(r_1 - a_{11} x_1(t - \tau_1)). \quad (2.5)$$

A standard comparison argument (see, for example, the proof of Lemma 2.1 in [14]) shows that there is a $T_1 > 0$ such that

$$x(t) \leq \frac{r_1}{a_{11}} e^{r_1 \tau_1} := M_1 \quad \text{for } t \geq T_1. \quad (2.6)$$

It follows from (2.6) and the third equation of system (1.2) that for $t \geq T_1 + \tau_2$,

$$\frac{dy_2(t)}{dt} \leq ax^{-\gamma \tau_2} M_1 y_2(t - \tau_2) - a_{22} y_2^2(t).$$

We consider the following equation:

$$\frac{du(t)}{dt} = ax^{-\gamma \tau_2} M_1 u(t - \tau_2) - a_{22} u^2(t).$$

By Lemma 2.2 we derive

$$\lim_{t \to +\infty} u(t) = \frac{ax^{-\gamma \tau_2} M_1}{a_{22}} := M_3^*.$$

By comparison, there exists an $M_3 > M_3^*$ and a $T_1^* > T_1 + \tau_2$ such that $y_2(t) \leq M_3$ for $t \geq T_1^*$.

Set $T = T_1^* + \tau_2$. It follows from (2.2) that for $t \geq T$,

$$y_1(t) = \int_{t - \tau_2}^t ax^{-\gamma(t-s)} x(s) y_2(s) ds \leq \int_{t - \tau_2}^t ax^{-\gamma(t-s)} M_1 M_3 ds = \frac{x_1 M_3}{\gamma} (1 - e^{-\gamma \tau_2}).$$

The proof is complete. □
Theorem 2.1. Suppose that system (1.2) satisfies the following:

(H1) \( a_{11}a_{22} - a_{12}ae^{r_1 \tau_1 - r_2} > 0 \).

Then system (1.2) is uniformly persistent.

Proof. Suppose \( z(t) = (x(t), y_1(t), y_2(t)) \) is a solution of system (1.2) with initial conditions (1.3) and (1.4). According to the first equation of system (1.2), if (H1) holds, then we have that for \( t \geq T \),

\[
\frac{dx(t)}{dt} \geq x(t)\left[r_1 - a_{12}M_3 - a_{11}x(t - \tau_1)\right].
\]  

(2.7)

Notice that if (H1) holds, then \( r_1 - a_{12}M_3 > 0 \). Thus, \( M_3 \) can be chosen to sufficiently close to \( M_3^* \) such that \( r_1 > a_{12}M_3 \). Similar to the proof of Theorem 2.1 of [15], one can easily verify that there exists a \( T_2 > T \) such that for \( t > T_2 \),

\[
x(t) \geq \frac{r_1 - a_{12}M_3}{a_{11}} \exp\{(r_1 - a_{11}M_1 - a_{12}M_3)\tau_1\} := m_1.
\]  

(2.8)

From (2.8) and the third equation of (1.2), we obtain that for \( t > T_2 + \tau_2 \),

\[
\frac{dy_2(t)}{dt} \geq ae^{-r_2}m_1y_2(t - \tau_2) - a_{22}y_2^2(t).
\]

We consider the following equation:

\[
\frac{du(t)}{dt} = ae^{-r_2}m_1u(t - \tau_2) - a_{22}u^2(t).
\]

By Lemma 2.2 we derive

\[
\lim_{t \to +\infty} u(t) = \frac{ae^{-r_2}m_1}{a_{22}} := m_3^*.
\]

By comparison, there exists an \( m_3, 0 < m_3 < m_3^* \) and a \( T_3 > T_2 \) such that \( y_2(t) > m_3 \) for \( t > T_3 \).

It follows from (2.2) that for \( t \geq T_3 + \tau_2 \),

\[
y_1(t) = \int_{t-\tau_2}^{t} ae^{-\gamma(t-s)}x(s)y_2(s) \, ds \geq \int_{t-\tau_2}^{t} zm_1m_3e^{-\gamma(t-s)} \, ds
\]

\[
= \frac{zm_1m_3}{2\gamma} (1 - e^{-\gamma\tau_2}) := m_2.
\]

(2.9)

We now let

\[
D = \{(x,y_1,y_2) \mid m_1 \leq x \leq M_1, \ m_2 \leq y_1 \leq M_2, \ m_3 \leq y_2 \leq M_3\}.
\]

Then \( D \) is a bounded compact region in \( R^3_+ \) which has positive distance from coordinate planes. From what has been discussed above, we obtain that there exists a \( T^* > T_3 \), if \( t \geq T^* \), then every positive solution of system (1.2) with
initial conditions (1.3) and (1.4) eventually enters and remains in the region $D$. The proof is complete. \quad \square

3. Local stability

In this section, we shall establish sufficient conditions for the local stability of a positive equilibrium of (1.2) with two different delays. In general, the local stability of the positive equilibrium of delay autonomous system can be studied by analyzing the roots of the characteristic equation of the corresponding linear variation system and the analysis involves complicated calculation (see, for example, [16]). In this paper, by constructing Lyapunov functionals, we are able to study the local asymptotic stability of the positive equilibrium of system (1.2).

System (1.2) has a unique positive equilibrium, say $E^* (x^*, y_1^*, y_2^*)$, where

$$x^* = \frac{r_1 a_{22}}{a_{11} a_{22} + a_{12} e^{-\gamma_2}}, \quad y_1^* = \frac{x(1 - e^{-\gamma_2}) x y_2^*}{\gamma}, \quad y_2^* = \frac{r_1 x e^{-\gamma_2}}{a_{11} a_{22} + a_{12} e^{-\gamma_2}}.$$

It is easy to see that if we know the properties of $x(t)$ and $y_2(t)$, then the properties of $y_1(t)$ can be obtained from (2.2) and the initial conditions $y_1(0) = \int_{-\tau_1}^{0} x e^{-\gamma_2} \phi(s) \psi_2(s) \, ds$. Therefore, in the following we need only to consider the following subsystem:

$$\begin{align*}
\dot{x}(t) &= x(t)(r_1 - a_{11} x(t - \tau_1) - a_{12} y_2(t)), \\
\dot{y}_2(t) &= x e^{-\gamma_2} x(t - \tau_2) y_2(t - \tau_2) - a_{22} y_2(t)
\end{align*}$$

with initial conditions

$$\begin{align*}
x(\theta) &= \phi(\theta), \quad y_2(\theta) = \psi_2(\theta), \\
\phi(\theta) &\geq 0, \quad \psi_2(\theta) \geq 0, \quad \theta \in [-\tau, 0], \\
\phi(0) &> 0, \quad \psi_2(0) > 0,
\end{align*}$$

where $(\phi(\theta), \psi_2(\theta)) \in C([-\tau, 0], R^2_{\geq 0})$.

System (3.1) has a unique positive equilibrium $E^*_1 (x^*, y_2^*)$. Linearizing system (3.1) at $E^*_1 (x^*, y_2^*)$, we obtain

$$\begin{align*}
\dot{N}_1(t) &= AN_1(t - \tau_1) + BN_2(t), \\
\dot{N}_2(t) &= CN_1(t - \tau_2) + DN_2(t - \tau_2) + D_1 N_2(t),
\end{align*}$$

where

$$\begin{align*}
A &= -a_{11} x^*, \quad B = -a_{12} x^*, \quad C = x e^{-\gamma_2} y_2^*, \\
D &= x e^{-\gamma_2} x^*, \quad D_1 = -2 a_{22} y_2^*.
\end{align*}$$
It is noticed that the local asymptotic stability of the positive equilibrium \( E^*_C(x^*_C, y^*_C) \) of system (3.1) follows from that of the zero solution of system (3.3) (see [17, Theorem 4.2, p. 26]).

**Theorem 3.1.** The positive equilibrium \( E^*_C \) of system (1.2) is locally asymptotically stable provided that

\[
\begin{align*}
(H2) \quad & a_1 > 0, \quad a_2 > 0, \\
n \quad & a_1 = \frac{x e^{-\tau_2} y^*_C}{a_{12}} \{2a_{11} - x^*[a_{11}(2a_{11} + a_{12})\tau_1 + a_{12}(a_{11} + xe^{-\tau_2})\tau_2]\}, \\
n \quad & a_2 = y_C^2\{2a_{22} - xe^{-\tau_2}x^*[a_{11}\tau_1 + (a_{11} + 2a_{12} + xe^{-\tau_2} + 6a_{22})\tau_2]\}. \quad (3.5)
\end{align*}
\]

**Proof.** The first equation of (3.3) can be rewritten as

\[
\frac{\dot{N}_1(t)}{A} = AN_1(t) + BN_2(t) - A \int_{t - \tau_1}^t \dot{N}_1(u) \, du
\]

\[
= AN_1(t) + BN_2(t) - A \int_{t - \tau_1}^t (AN_1(u - \tau_1) + BN_2(u)) \, du. \quad (3.6)
\]

Define

\[
W_{11}(t) = N_1^2(t). \quad (3.7)
\]

Calculating the derivative of \( W_{11}(t) \) along solutions of (3.3), we obtain

\[
\frac{d}{dt} W_{11}(t) = 2N_1(t) \left\{ AN_1(t) + BN_2(t) - A \int_{t - \tau_1}^t (AN_1(u - \tau_1) + BN_2(u)) \, du \right\}
\]

\[
= 2AN_1^2(t) + 2BN_1(t)N_2(t) - 2AN_1(t)
\]

\[
\times \int_{t - \tau_1}^t (AN_1(u - \tau_1) + BN_2(u)) \, du.
\]

Using the inequality \( a^2 + b^2 \geq 2ab \), we get

\[
\frac{d}{dt} W_{11}(t) \leq 2AN_1^2(t) + 2BN_1(t)N_2(t) + A(A + B)\tau_1 N_1^2(t)
\]

\[
+ A \int_{t - \tau_1}^t [AN_1^2(u - \tau_1) + BN_2^2(u)] \, du. \quad (3.8)
\]

Define

\[
W_{12}(t) = A \int_{t - \tau_1}^t \int_v^t [AN_1^2(u - \tau_1) + BN_2^2(u)] \, du \, dv. \quad (3.9)
\]
It follows from (3.8) and (3.9) that
\[
\frac{d}{dt}(W_{11}(t) + W_{12}(t)) \leq 2AN_1^2(t) + 2BN_1(t)N_2(t) + A(A + B)\tau_1N_1^2(t) + A\tau_1[AN_1^2(t - \tau_1) + BN_2^2(t)].
\] (3.10)

We define
\[
W_1(t) = W_{11}(t) + W_{12}(t) + W_{13}(t),
\] (3.11)

where
\[
W_{13}(t) = A^2\tau_1 \int_{t_1}^t N_1^2(l) \, dl.
\] (3.12)

Then from (3.10)–(3.12) we obtain
\[
\frac{dW_1(t)}{dt} \leq [2A + A(2A + B)\tau_1]N_1^2(t) + 2BN_1(t)N_2(t) + AB\tau_1N_2^2(t).
\] (3.13)

Similarly, the second equation of (3.3) can be rewritten as
\[
\dot{N}_2(t) = (D + D_1)N_2(t) + CN_1(t) - C \int_{t - \tau_2}^t \dot{N}_1(u) \, du - D \int_{t - \tau_2}^t \dot{N}_2(u) \, du
\]
\[
\quad = (D + D_1)N_2(t) + C\int_{t - \tau_2}^t [AN_1(u - \tau_1) + BN_2(u)] \, du
\]
\[
\quad - D \int_{t - \tau_2}^t [CN_1(u - \tau_2) + DN_2(u - \tau_2) + D_1N_2(u)] \, du.
\] (3.14)

Define
\[
W_{21}(t) = N_2^2(t).
\] (3.15)

Then along the solution of (3.3), we derive
\[
\frac{d}{dt} W_{21}(t)
\]
\[
= 2N_2(t) \left\{ (D + D_1)N_2(t) + CN_1(t) - C \int_{t - \tau_2}^t [AN_1(u - \tau_1) + BN_2(u)] \, du 
\]
\[
\quad - D \int_{t - \tau_2}^t [CN_1(u - \tau_2) + DN_2(u - \tau_2) + D_1N_2(u)] \, du \right\}
\]
\[
= 2(D + D_1)N_2^2(t) + 2CN_1(t)N_2(t) - 2CN_2(t)
\]
\[
\times \int_{t - \tau_2}^t [AN_1(u - \tau_1) + BN_2(u)] \, du - 2DN_2(t)
\]
\[
\times \int_{t - \tau_2}^t [CN_1(u - \tau_2) + DN_2(u - \tau_2) + D_1N_2(u)] \, du.
\] (3.16)
Using the inequality \(a^2 + b^2 \geq 2ab\), we get
\[
\frac{d}{dt} W_{21}(t) \leq 2(D + D_1)N_2^2(t) + 2CN_1(t)N_2(t) - C(A + B)\tau_2 N_2^2(t)
\]
\[+ D(C + D - D_1)\tau_2 N_2^2(t) - C \int_{t-\tau_2}^{t} [AN_1^2(u - \tau_1) + BN_2^2(u)] \, du
\]
\[+ D \int_{t-\tau_2}^{t} [CN_1^2(u - \tau_2) + DN_2^2(u - \tau_2) - D_1 N_2^2(u)] \, du. \tag{3.17}
\]
Define
\[
W_{22}(t) = -C \int_{t-\tau_2}^{t} \int_{v}^{t} [AN_1^2(u - \tau_1) + BN_2^2(u)] \, du \, dv
\]
\[+ D \int_{t-\tau_2}^{t} \int_{v}^{t} [CN_1^2(u - \tau_2) + DN_2^2(u - \tau_2) - D_1 N_2^2(u)] \, du \, dv.
\]
It follows from (3.17) and (3.18) that
\[
\frac{d}{dt} (W_{21}(t) + W_{22}(t)) \leq 2(D + D_1)N_2^2(t) + 2CN_1(t)N_2(t) - C(A + B)\tau_2 N_2^2(t)
\]
\[+ D(C + D - D_1)\tau_2 N_2^2(t) - C\tau_2 [AN_1^2(t - \tau_1) + BN_2^2(t)]
\]
\[+ D\tau_2 [CN_1^2(t - \tau_2) + DN_2^2(t - \tau_2) - D_1 N_2^2(t)]. \tag{3.19}
\]
Define
\[
W_2(t) = W_{21}(t) + W_{22}(t) + W_{23}(t), \tag{3.20}
\]
in which
\[
W_{23}(t) = -AC\tau_2 \int_{t-\tau_1}^{t} N_1^2(l) \, dl + D\tau_2 \int_{t-\tau_2}^{t} [CN_1^2(l) + DN_2^2(l)] \, dl. \tag{3.21}
\]
Then it follows from (3.19)–(3.21) that
\[
\frac{d}{dt} W_2(t) \leq 2(D + D_1)N_2^2(t) + 2CN_1(t)N_2(t) - C(A + 2B)\tau_2 N_2^2(t)
\]
\[+ D(C + 2D - 2D_1)\tau_2 N_2^2(t) + C(D - A)\tau_2 N_1^2(t). \tag{3.22}
\]
Let
\[
W(t) = -\frac{C}{B} W_1(t) + W_2(t). \tag{3.23}
\]
Then it follows from (3.13), (3.22) and (3.23) that
\[
\frac{d}{dt} W(t) \leq - \frac{C}{B} \left( |2A + (2A + B)\tau_1|N_1^2(t) + 2BN_1(t)N_2(t) + AB\tau_1N_2^2(t) \right) \\
+ 2(D + D_1)N_2^2(t) + 2CN_1(t)N_2(t) - C(A + 2B)\tau_2N_2^2(t) \\
+ D(C + 2D - 2D_1)\tau_2N_2^2(t) + C(D - A)\tau_2N_1^2(t) \\
:= -\alpha_1N_1^2(t) - \alpha_2N_2^2(t),
\]
where \( \alpha_1 \) and \( \alpha_2 \) are defined in (3.5).

By assumption (H2), we have \( \alpha_1 > 0, \alpha_2 > 0 \). According to the Lyapunov theorem (see [17, Theorem 5.1, p. 27]), we see that the zero solution of (3.3) is uniformly asymptotically stable. Accordingly, the positive equilibrium \( E^* \) of system (1.2) is uniformly asymptotically stable. This completes the proof. \( \square \)

**Corollary 3.1.** The positive equilibrium \( E^*(x^*, y_1^*, y_2^*) \) of system (1.2) is locally asymptotically stable provided that \( 0 \leq \max\{\tau_1, \tau_2\} \leq \tau_L \), where
\[
\tau_L = \min \left\{ \frac{2a_{11}}{r_1[2a_{11}(a_{11} + a_{12}) + x a_{12}]} , \frac{2(a_{11}a_{22} + x a_{12})}{x r_1(2a_{11} + 2a_{12} + x + 6a_{22})} \right\}.
\]

**Remark 1.** Noting that \( D + D_1 = -a_{22}y_2^* \), from the proof of Theorem 3.1 it is easy to see that the positive equilibrium of the “instantaneous” (when \( \tau_1 = 0, \tau_2 = 0 \)) system (1.2) is locally uniformly asymptotically stable. Then the local uniform asymptotic stability of \( E^* \) of the delayed model (1.2) is preserved for small \( \tau_1 \) and \( \tau_2 \) satisfying (H2).

### 4. Global stability

The objective of this section is to formulate the global asymptotic stability of the positive equilibrium \( E^*(x^*, y_1^*, y_2^*) \) of system (1.2). The method of proof is to construct a suitable Lyapunov functional for system (1.2).

**Theorem 4.1.** In addition to (H1), assume further that

\( (H3) \) \( A_1 > 0, A_2 > 0, \)
where
\[ A_1 = a_{11} - \frac{1}{2}a_{11}\tau_1 M_1(2a_{11} + a_{12}) - \frac{1}{2}a_{11}a_{12}M_1\tau_2 \]
\[ - \frac{1}{2m_3}xa_{12}\tau_2 y_2^2 M_1 e^{-\tau_2}, \]
\[ A_2 = a_{12} \left\{ \frac{a_{22}}{x} e^{\tau_2} - \frac{1}{2}a_{11}M_1\tau_1 - \frac{1}{2}M_1\tau_2(a_{11} + 2a_{12}) \right\} \]
\[ - \frac{1}{2m_3}\tau_2 M_1 [2xe^{-\tau_2}M_1 + xe^{-\tau_2}y_2^2 + 2a_{22}(M_3 + y_2^2)] \}
(4.1)

Then the positive equilibrium \( E^* \) of system (1.2) is globally asymptotically stable.

**Proof.** Let \( z(t) = (x(t), y_1(t), y_2(t)) \) be any positive solution of system (1.2) with initial conditions (1.3) and (1.4).

We first consider the following subsystem:
\[ \dot{x}(t) = x(t)(r_1 - a_{11}x(t - \tau_1) - a_{12}y_2(t)), \]
\[ \dot{y}_2(t) = xe^{-\tau_2}x(t - \tau_2)y_2(t - \tau_2) - a_{22}y_2^2(t) \]
with initial conditions (3.2).

System (4.2) can be rewritten as
\[ \dot{x}(t) = x(t)(r_1 - a_{11}x(t) - a_{12}y_2(t)) \]
\[ + a_{11}x(t) \int_{t-\tau_1}^t x(u)(r_1 - a_{11}x(u - \tau_1) - a_{12}y_2(u)) \, du \]
\[ \dot{y}_2(t) = y_2(t)(xe^{-\tau_2}x(t) - a_{22}y_2(t)) \]
\[ - xe^{-\tau_2}y_2(t) \int_{t-\tau_2}^t x(u)(r_1 - a_{11}x(u - \tau_1) - a_{12}y_2(u)) \, du \]
\[ - xe^{-\tau_2}x(t - \tau_2) \int_{t-\tau_2}^t [xe^{-\tau_2}x(u - \tau_2)y_2(u - \tau_2) - a_{22}y_2^2(u)] \, du. \]
(4.3)

Noting that \( E^*_1(x^*, y_2^*) \) is a positive equilibrium of system (4.3), we can rewrite (4.3) into
\[ \dot{x}(t) = x(t)[-a_{11}(x(t) - x^*) - a_{12}(y_2(t) - y_2^*)] \]
\[ + a_{11}x(t) \int_{t-\tau_1}^t x(u)[-a_{11}(x(u - \tau_1) - x^*) - a_{12}(y_2(u) - y_2^*)] \, du \]
\[ \dot{y}_2(t) = y_2(t)[-xe^{-\tau_2}(x(t) - x^*) - a_{22}(y_2(t) - y_2^*)] \]
\[ - xe^{-\tau_2}y_2(t) \int_{t-\tau_2}^t x(u)[-a_{11}(x(u - \tau_1) - x^*) - a_{12}(y_2(u) - y_2^*)] \, du \]
\[ - xe^{-\tau_2}x(t - \tau_2) \int_{t-\tau_2}^t [xe^{-\tau_2}x(u - \tau_2)(y_2(u - \tau_2) - y_2^*)] \]
\[ + xe^{-\tau_2}y_2^2(x(u - \tau_2) - x^*) - a_{22}(y_2(u) + y_2^*)(y_2(u) - y_2^*)] \, du. \]
(4.4)
Let
\[ V_1(t) = x(t) - x^* - x^* \ln \frac{x(t)}{x^*} + c \left( y_2(t) - y_2^* - y_2^* \ln \frac{y_2(t)}{y_2^*} \right), \]
(4.5)
where \( c = (a_2/x) e^{\gamma t} \).

Calculating the derivative of \( V_1(t) \) along solutions of (4.4), it follows that
\[
\frac{dV_1(t)}{dt} = (x(t) - x^*) \frac{\dot{x}(t)}{x(t)} + c (y_2(t) - y_2^*) \frac{\dot{y}_2(t)}{y_2(t)}
\]
\[ = -a_{11}(x(t) - x^*)^2 - ca_{22}(y_2(t) - y_2^*)^2 + a_{11}(x(t) - x^*) \]
\[ \times \int_{t-t_0}^t x(u)[-a_{11}(x(u - \tau_1) - x^*) - a_{12}(y_2(u) - y_2^*)] \, du \]
\[ - cxe^{-\gamma t} (y_2(t) - y_2^*) \int_{t-t_2}^t x(u)[-a_{11}(x(u - \tau_1) - x^*) \]
\[ - a_{12}(y_2(u) - y_2^*)] \, du - cxe^{-\gamma t} x(t - \tau_2) \frac{(y_2(t) - y_2^*)}{y_2(t)} \]
\[ \times \int_{t-t_2}^t \left[ xe^{-\gamma t} x(u - \tau_2) (y_2(u - \tau_2) - y_2^*) + xe^{-\gamma t} y_2^*(x(u - \tau_2) - x^*) \right] \]
\[ - a_{22}(y_2(u) + y_2^*) (y_2(u) - y_2^*)] \, du. \]
(4.6)

By using the inequality \( a^2 + b^2 \geq 2ab \), we derive
\[
\frac{dV_1(t)}{dt} \leq -a_{11}(x(t) - x^*)^2 - ca_{22}(y_2(t) - y_2^*)^2 + \frac{1}{2} a_{11}(x(t) - x^*)^2
\]
\[ \times \int_{t-t_0}^t (a_{11} + a_{12}) x(u) \, du + \frac{1}{2} a_{11} \int_{t-t_1}^t x(u)[a_{11}(x(u - \tau_1) - x^*)^2 \]
\[ + a_{12}(y_2(u) - y_2^*)^2] \, du + \frac{1}{2} cxe^{-\gamma t} (y_2(t) - y_2^*)^2 \int_{t-t_2}^t (a_{11} + a_{12}) x(u) \, du \]
\[ + \frac{1}{2} cxe^{-\gamma t} \int_{t-t_1}^t x(u)[a_{11}(x(u - \tau_1) - x^*)^2 + a_{12}(y_2(u) - y_2^*)^2] \, du \]
\[ + \frac{1}{2} cxe^{-\gamma t} x(t - \tau_2) \frac{(y_2(t) - y_2^*)^2}{y_2(t)} \int_{t-t_2}^t \left[ xe^{-\gamma t} x(u - \tau_2) + xe^{-\gamma t} y_2^* \right] \]
\[ + a_{22}(y_2(u) + y_2^*)] \, du + \frac{1}{2} cxe^{-\gamma t} x(t - \tau_2) \frac{1}{y_2(t)} \]
\[ \times \int_{t-t_2}^t \left[ xe^{-\gamma t} x(u - \tau_2) (y_2(u - \tau_2) - y_2^*) + xe^{-\gamma t} y_2^*(x(u - \tau_2) - x^*)^2 \right] \]
\[ + a_{22}(y_2(u) + y_2^*) (y_2(u) - y_2^*)] \, du. \]
(4.7)

By Theorem 2.1, we have \( 0 < x(t) \leq M_1, \, m_3 < y_2(t) \leq M_3 \) for \( t \geq T^* \). This, together with (4.7) for \( t \geq T^* + \tau \), leads to
Define

\[
\frac{dV_1(t)}{dt} \leq -a_{11}(x(t) - x^*)^2 - ca_{22}(y_2(t) - y_2^*)^2 + \frac{1}{2}a_{11}\tau_1 M_1 (a_{11}
\]

\[+ a_{12}(x(t) - x^*)^2 + \frac{1}{2}a_{11}\tau_1 \int_{t-\tau_1}^t [a_{11}(x(u - \tau_1) - x^*)^2 + a_{12}(y_2(u)
\]

\[- y_2^*)^2] \, du + \frac{1}{2}c\tau_2 M_1 \tau_2 e^{-\tau_2 c(a_{11} + a_{12})(y_2(t) - y_2^*)^2 + \frac{1}{2}c\tau_2 M_1 e^{-\tau_2 c}
\]

\[\times \int_{t-\tau_2}^t [a_{11}(x(u - \tau_1) - x^*)^2 + a_{12}(y_2(u) - y_2^*)^2] \, du
\]

\[+ c\tau_2 M_1 \frac{1}{2m_3} \int_{t-\tau_2}^t \left[ e^{-\tau_2 c} M_1 (y_2(t - \tau_2) - y_2^*)^2
\]

\[+ c\tau_2 M_1 \frac{1}{2m_3} \int_{t-\tau_2}^t \left[ e^{-\tau_2 c} M_1 (y_2(t - \tau_2) - y_2^*)^2
\]

\[+ e^{-\tau_2 c} y_2^*(x(u - \tau_2) - x^*)^2 + a_{22}(M_3 + y_2^*)(y_2(t) - y_2^*)^2] \, du. \quad (4.8)
\]

It then follows from (4.8) and (4.9) that for \( t \geq T^* + \tau, \)

\[
\frac{dV_2(t)}{dt} + \frac{V_2(t)}{dt} \leq -a_{11}(x(t) - x^*)^2 - ca_{22}(y_2(t) - y_2^*)^2
\]

\[+ \frac{1}{2}a_{11}\tau_1 M_1 (a_{11} + a_{12})(x(t) - x^*)^2 + \frac{1}{2}a_{11}\tau_1 [a_{11}(x(t - \tau_1) - x^*)^2
\]

\[+ a_{12}(y_2(t) - y_2^*)^2] + \frac{1}{2}c\tau_2 M_1 \tau_2 e^{-\tau_2 c(a_{11} + a_{12})(y_2(t) - y_2^*)^2}
\]

\[+ \frac{1}{2}c\tau_2 M_1 \tau_2 e^{-\tau_2 c(a_{11}(x(t - \tau_1) - x^*)^2 + a_{12}(y_2(t) - y_2^*)^2]}
\]

\[+ c\tau_2 M_1 \frac{1}{2m_3} \left[ e^{-\tau_2 c M_1 (y_2(t - \tau_2) - y_2^*)^2 + e^{-\tau_2 c} y_2^*(x(t - \tau_2) - x^*)^2
\]

\[+ a_{22}(M_3 + y_2^*)(y_2(t) - y_2^*)^2] \right]. \quad (4.10)
\]
We now define

\[ V(t) = V_1(t) + V_2(t) + V_3(t), \]

where

\[ V_3(t) = \frac{1}{2} a_{11} M_1 \tau_1 \int_{t-\tau_1}^{t'} (x(l) - x^*)^2 \, dl + \frac{1}{2} c x a_{11} M_1 \tau_2 e^{-\gamma \tau_2} \int_{t-\tau_2}^{t'} (x(l) - x^*)^2 \, dl \]

\[ + c x \tau_2 e^{-\gamma \tau_2} M_1 \frac{1}{2m_3} \int_{t-\tau_2}^{t} [x \epsilon^{-\gamma \tau_2} M_1 (y_2(l) - y_2^*)^2 \]

\[ + x \epsilon^{-\gamma \tau_2} y_2^*(x(l) - x^*)^2] \, dl. \]  

Then it follows from (4.10)–(4.12) that for \( t \geq T^* + \tau, \)

\[ \frac{dV(t)}{dt} \leq -a_{11}(x(t) - x^*)^2 - c a_{22}(y_2(t) - y_2^*)^2 + \frac{1}{2} a_{11} \tau_1 M_1 (a_{11} \]

\[ + a_{12})(x(t) - x^*)^2 + \frac{1}{2} a_{11} \tau_1 M_1 \tau_2 (a_{11}(x(t) - x^*)^2 + a_{12}(y_2(t) - y_2^*)^2] \]

\[ + \frac{1}{2} \epsilon x \tau_2 e^{-\gamma \tau_2} (a_{11} + a_{12})(y_2(t) - y_2^*)^2 + \frac{1}{2} \epsilon x x \tau_2 e^{-\gamma \tau_2} [a_{11}(x(t) - x^*)^2 \]

\[ + a_{12}(y_2(t) - y_2^*)^2] + \frac{c x \tau_2 M_1}{2m_3} \epsilon^{-\gamma \tau_2} [x \epsilon^{-\gamma \tau_2} M_1 + x \epsilon^{-\gamma \tau_2} y_2^* \]

\[ + a_{22}(M_3 + y_2^*) (y_2(t) - y_2^*)^2 + c x \tau_2 e^{-\gamma \tau_2} M_1 \frac{1}{2m_3} [x \epsilon^{-\gamma \tau_2} M_1 (y_2(t) - y_2^*)^2 \]

\[ + x \epsilon^{-\gamma \tau_2} y_2^*(x(t) - x^*)^2 + a_{22}(M_3 + y_2^*)(y_2(t) - y_2^*)^2] \]

\[ = -A_1(x(t) - x^*)^2 - A_2(y_2(t) - y_2^*)^2, \] 

where \( A_1 \) and \( A_2 \) are defined in (4.1). We can conclude from the Lyapunov Theorem (see [17, Corollary 5.2]) that the positive equilibrium \( E_1^*(x^*, y_2^*) \) of system (4.2) is globally asymptotically stable and thus we have

\[ \lim_{t \to \infty} x(t) = x^*, \quad \lim_{t \to \infty} y_2(t) = y_2^*. \]  

By L’Hospital’s Rule, it follows directly from (2.2) and (4.14) that

\[ \lim_{t \to \infty} y_1(t) = \lim_{t \to \infty} \int_{t-\tau_2}^{t} x e^{-\gamma(t-s)} \, ds = \lim_{t \to \infty} \frac{\int_{t-\tau_2}^{t} e^{\gamma t} x(s) y_2(s) \, ds}{e^{\gamma t}} \]

\[ = \gamma \lim_{t \to \infty} \frac{e^{\gamma t} x(t) y_2(t) - e^{\gamma t} x(t-\tau_2) y_2(t-\tau_2)}{x(t-\tau_2) y_2(t-\tau_2)} \]

\[ = \gamma \lim_{t \to \infty} \frac{x(t) y_2(t) - e^{-\gamma \tau_2} x(t-\tau_2) y_2(t-\tau_2)}{y_2^*} = y_2^*. \]

This completes the proof. \( \square \)

Finally, we give an example to illustrate the feasibility of our main results.
Example. Consider the following delayed predator-prey model:

\[
\begin{align*}
\dot{x}(t) &= x(t)(10 - 10x(t - \tau_1) - y_2(t)), \\
\dot{y}_1(t) &= 3x(t)y_2(t) - 0.1y_1(t) - 3e^{-0.1\tau_2}x(t-\tau_2)y_2(t-\tau_2), \\
\dot{y}_2(t) &= 3e^{-0.1\tau_2}x(t-\tau_2)y_2(t-\tau_2) - 10y_2^2(t).
\end{align*}
\] (4.15)

System (4.15) has a unique positive equilibrium \(E^*\) \((100/(100 + 3e^{-0.1\tau_2}), 90000e^{-0.1\tau_2}(1 - e^{-0.1\tau_2})/(100 + 3e^{-0.1\tau_2})^2, 30/(100e^{0.1\tau_2} + 3))\). By Theorem 2.1, it is easy to see that system (4.15) is uniformly persistent if \(10\tau_1 - 0.1\tau_2 < \ln(100/3)\). By Corollary 3.1, we know that the positive equilibrium \(E^*\) of system (4.15) is locally asymptotically stable provided that \(\max\{\tau_1, \tau_2\} < 103/1275\). Set \(M_1 = e^{10\tau_1}, M_3 = 0.30001e^{10\tau_1 - 0.1\tau_2}, y_2^* = 30/(100e^{0.1\tau_2} + 3), m_3 = 0.029999(10 - M_3)e^{(10-M_1-M_3)\tau_1 - 0.1\tau_2}\). By Theorem 4.1, we see that the positive equilibrium \(E^*\) of system (4.15) is globally stable provided that

\[
105M_1\tau_1 + 5M_1\tau_2 + \frac{3}{2m_3}y_2^*M_1\tau_2e^{-0.1\tau_2} < 10,
\]
\[
5M_1\tau_1 + 6M_1\tau_2 + \frac{M_1\tau_1}{2m_3}(6M_1e^{-0.1\tau_2} + 3y_2^*e^{-0.1\tau_2} + 20(M_3 + y_2^*)) < \frac{10}{3}e^{0.1\tau_2}.
\]

Using L.F. Shampine and S. Thompson’s program dde23 in solving DDEs [18], numerical simulation shows that if \(\max\{\tau_1, \tau_2\} < 103/1275\), the positive equilibrium \(E^*\) of (4.15) is locally asymptotically stable (see Fig. 1). In fact, after testing a large range of initial data, it seems that \(E^*\) is also globally stable for \(\max\{\tau_1, \tau_2\} < 103/1275\). and further numerical simulations suggest that it remains so at least for \(\tau_1 < 0.15, \tau_2 < 30\) but the density of the adult predator becomes very low with the increasing of the time delay \(\tau_2\) (see Fig. 2). For \(\tau_1 \geq 0.168\), numerical simulation shows that a stability switch occurs, the positive equilibrium \(E^*\) becomes unstable (see Fig. 3). Further increasing of the delay \(\tau_2\) may result in the adult predator going to extinction. These results suggest that the bounds derived in previous sections are somewhat conservative.

Remark 2. In this paper, based on the work of Aiello and Freedman [4], we proposed and discussed a predator-prey model with stage structure for the predator and time delay due to the negative feedback of the prey. By constructing suitable Lyapunov functionals, sufficient conditions are established for both the local and global stability of the positive equilibrium of the model. By Theorems 3.1 and 4.1, we see that the positive equilibrium of system (1.2) with initial conditions (1.3) and (1.4) will be locally and globally asymptotically stable respectively provided that time delays \(\tau_1\) and \(\tau_2\) are sufficiently small.
Fig. 1. The temporal solution found by numerical simulation of system (4.15) with $\tau_1 = 0.01$, $\tau_2 = 0.05$.

Fig. 2. The temporal solution found by numerical simulation of system (4.15) with $\tau_1 = 0.15$, $\tau_2 = 30$. 
We would like to mention here that numerical simulations in the example show that our results in Theorems 3.1 and 4.1 have room for improvement. We leave these for future work.

References