



# Eigenfunction expansions in $L^2$ spaces for boundary value problems on time-scales

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## Abstract

Let  $\mathbb{T} \subset \mathbb{R}$  be a bounded time-scale, with  $a = \inf \mathbb{T}$ ,  $b = \sup \mathbb{T}$ . We consider the weighted, linear, eigenvalue problem

$$\begin{aligned}
 -(pu^\Delta)^\Delta(t) + q(t)u^\sigma(t) &= \lambda w(t)u^\sigma(t), \quad t \in \mathbb{T}^{\kappa^2}, \\
 c_{00}u(a) + c_{01}u^\Delta(a) &= 0, \quad c_{10}u(\rho(b)) + c_{11}u^\Delta(\rho(b)) = 0,
 \end{aligned}$$

for suitable functions  $p$ ,  $q$  and  $w$  and  $\lambda \in \mathbb{R}$ . Problems of this type on time-scales have normally been considered in a setting involving Banach spaces of continuous functions on  $\mathbb{T}$ . In this paper we formulate the problem in Sobolev-type spaces of functions with generalized  $L^2$ -type derivatives. This approach allows us to use the functional analytic theory of Hilbert spaces rather than Banach spaces. Moreover, it allows us to use more general coefficient functions  $p$ ,  $q$ , and weight function  $w$ , than usual, viz.,  $p \in H^1(\mathbb{T}^\kappa)$  and  $q, w \in L^2(\mathbb{T}^\kappa)$  compared with the usual hypotheses that  $p \in C_{\text{rd}}^1(\mathbb{T}^\kappa)$ ,  $q, w \in C_{\text{rd}}^0(\mathbb{T}^{\kappa^2})$ . Further to these conditions, we assume that  $p \geq c > 0$  on  $\mathbb{T}^\kappa$ ,  $C \geq w \geq c > 0$  on  $\mathbb{T}^{\kappa^2}$ , for some constants  $C > c > 0$ . These conditions are similar to the usual assumptions imposed on Sturm–Liouville, ordinary differential equation problems. We obtain a min–max characterization of the eigenvalues of the above problem, and various eigenfunction expansions for functions in suitable function spaces. These results extend certain aspects of the standard theory of self-adjoint operators with compact resolvent to the above problem, even though the linear operator associated with the left-hand side of the problem is not in fact self-adjoint on general time-scales.

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## 1. Introduction

Let  $\mathbb{T}$  be a bounded time-scale (a non-empty, closed subset of  $\mathbb{R}$ ), with  $a = \inf \mathbb{T}$ ,  $b = \sup \mathbb{T}$ . We consider the weighted, linear, eigenvalue problem

$$-(pu^\Delta)^\Delta(t) + q(t)u^\sigma(t) = \lambda w(t)u^\sigma(t), \quad t \in \mathbb{T}^{\kappa^2}, \quad (1.1)$$

$$c_{00}u(a) + c_{01}u^\Delta(a) = 0, \quad c_{10}u(\rho(b)) + c_{11}u^\Delta(\rho(b)) = 0, \quad (1.2)$$

for suitable functions  $p$ ,  $q$  and  $w$  and  $\lambda \in \mathbb{R}$ , and constants  $c_{ij}$ , with  $|c_{j0}| + |c_{j1}| > 0$ ,  $j = 0, 1$ . Problems of this type on time-scales have normally been considered in a setting involving Banach spaces of continuous functions on  $\mathbb{T}$ . In this paper we formulate the problem in the Sobolev-type spaces introduced in [10]. This approach allows us to use the functional analytic theory of Hilbert spaces rather than Banach spaces. Moreover, it allows us to use more general coefficient functions  $p$ ,  $q$ , and weight function  $w$ , than usual, viz.,  $p \in H^1(\mathbb{T}^\kappa)$  and  $q, w \in L^2(\mathbb{T}^\kappa)$  compared with the usual hypotheses that  $p \in C_{\text{rd}}^1(\mathbb{T}^\kappa)$  and  $q, w \in C_{\text{rd}}^0(\mathbb{T}^{\kappa^2})$ . Further to these conditions, we assume that  $p \geq c > 0$ , on  $\mathbb{T}^\kappa$ ,  $C \geq w \geq c > 0$  on  $\mathbb{T}^{\kappa^2}$ , for some constants  $C > c > 0$ . These conditions are similar to the usual assumptions imposed on Sturm–Liouville, ordinary differential equation problems.

We give a brief resume of time-scale notation and terminology in Section 2, in particular the function spaces used here—for now, we give a brief introduction to our results.

Naturally, an *eigenvalue* of (1.1)–(1.2) is a number  $\lambda$  such that there exists a non-trivial *eigenfunction*  $u$  satisfying these equations. It will be shown that there exists a collection of geometrically simple eigenvalues of the problem and we give a min–max characterization of these eigenvalues. We also obtain an eigenfunction expansion for a general class of functions defined on  $\mathbb{T}$ . Essentially, our results extend the standard functional–analytic results for self-adjoint operators with compact resolvent to the eigenvalue problem (1.1)–(1.2). However, the differential operator induced by the left-hand side of (1.1), together with the boundary conditions (1.2), is not ‘formally self-adjoint,’ in general (we will discuss this more fully below), and the right-hand side contains the function  $u^\sigma$  rather than  $u$ . Thus, this is not quite a standard eigenvalue problem, and it is not immediately obvious that the usual self-adjoint results should even hold. In fact, a combination of the properties of both the left and right-hand sides of (1.1) yields some results similar to those one expects from a self-adjoint problem, but some non-self-adjoint features also arise.

A similar problem, but in the standard Banach space setting of continuous functions on  $\mathbb{T}$ , is considered in [1] (the weight function  $w \equiv 1$  in [1], and an additional technical assumption is imposed at the end points of  $\mathbb{T}$ ). The existence of a sequence of geometrically simple eigenvalues is proved in [1], together with a min–max characterization of the eigenvalues and a characterization of the number of nodes of the corresponding eigenfunctions. However, no eigenfunction expansion is obtained in [1]. Thus, in this sense, the current paper and [1] consider complementary spectral properties of the problem (1.1)–(1.2). Moreover, the stricter continuity conditions on the coefficients in [1] mean that some of the results obtained here generalize the corresponding results in [1].

**2. Preliminary results**

*2.1. Function spaces*

In this section we describe some preliminary results on various function spaces (in particular, on  $L^2$  and Sobolev-type spaces) that will be required below. We first briefly recall some basic definitions and results concerning time-scales. Further general details can be found in, for example, [2,3,5,7,8,10].

The time-scale  $\mathbb{T}$  is a non-empty, closed and bounded subset of  $\mathbb{R}$ . Let

$$a := \inf\{s \in \mathbb{T}\}, \quad b := \sup\{s \in \mathbb{T}\}.$$

Define the jump operators  $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$  by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad t \in \mathbb{T},$$

where, in this definition, we write  $\inf \emptyset = a, \sup \emptyset = b$ , so that  $\rho(a) = a$  and  $\sigma(b) = b$ . A point  $t \in \mathbb{T}$  is said to be *left-dense*, *left-scattered*, *right-dense*, *right-scattered* if  $\rho(t) = t, \rho(t) < t, \sigma(t) = t, \sigma(t) > t$ , respectively. We endow  $\mathbb{T}$  with the subspace topology inherited from  $\mathbb{R}$ .

Now suppose that  $u : \mathbb{T} \rightarrow \mathbb{R}$ . Continuity of  $u$  is defined in the usual manner, while  $u$  is said to be *rd-continuous* on  $\mathbb{T}$  if it is continuous at all right-dense points in  $\mathbb{T}$  and has finite left-sided limits at all left-dense points. We let  $C_{rd}^0(\mathbb{T})$  (respectively  $C^0(\mathbb{T})$ ) denote the set of rd-continuous (respectively continuous) functions  $u : \mathbb{T} \rightarrow \mathbb{R}$ , and let

$$\|u\|_{0,\mathbb{T}} := \sup\{|u(t)| : t \in \mathbb{T}\}, \quad u \in C_{rd}^0(\mathbb{T}).$$

With this norm these spaces are Banach spaces.

We assume throughout that  $\rho(b) > a$ , so that  $\mathbb{T}$  must contain at least 3 points. Now define the sets

$$\mathbb{T}^\kappa := \mathbb{T} \setminus (\rho(b), b], \quad \mathbb{T}^{\kappa^2} := \mathbb{T} \setminus (\rho^2(b), b].$$

These sets are closed (their construction successively removes isolated maximal points from  $\mathbb{T}$ , if they exist), so they are time-scales and we can also define the above spaces and norms using  $\mathbb{T}^\kappa$  and  $\mathbb{T}^{\kappa^2}$  instead of  $\mathbb{T}$ .

A function  $u : \mathbb{T} \rightarrow \mathbb{R}$  is *differentiable at  $t \in \mathbb{T}^\kappa$*  if there exists a number  $u^\Delta(t)$  with the following property: for any  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$s \in \mathbb{T} \quad \text{and} \quad |t - s| < \delta \implies |u(\sigma(t)) - u(s) - u^\Delta(t)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s|.$$

If  $u$  is differentiable at every  $t \in \mathbb{T}^\kappa$  then  $u$  is said to be *differentiable*. It can be shown that if  $u$  is differentiable at  $t$  then  $u$  is continuous at  $t$ , and so, if  $u$  is differentiable then  $u \in C^0(\mathbb{T})$ . Let  $C_{rd}^1(\mathbb{T})$  (respectively  $C^1(\mathbb{T})$ ) denote the set of functions  $u \in C^0(\mathbb{T})$  which are differentiable and for which  $u^\Delta \in C_{rd}^0(\mathbb{T}^\kappa)$  (respectively  $u^\Delta \in C^0(\mathbb{T}^\kappa)$ ). With the norm

$$\|u\|_{1,\mathbb{T}} := \|u\|_{0,\mathbb{T}} + \|u^\Delta\|_{0,\mathbb{T}^\kappa}, \quad u \in C_{rd}^1(\mathbb{T}),$$

these spaces are again Banach spaces.

We will also require various  $L^2$  and Sobolev-type spaces. Most of the required definitions and results are described more fully in [10], here we will simply summarize the main points. Fundamental to the construction of these spaces is a Lebesgue-type integral defined in [10]; we use the notation  $\int_s^t u \Delta$ , to denote the Lebesgue integral of a function  $u$  between  $s, t \in \mathbb{T}$  (when it is defined). That is, we use the same notation for the Lebesgue-type integral defined in [10]

as is commonly used in the time-scale literature for a Riemann-type integral defined in terms of anti-derivatives. A detailed discussion of the Lebesgue-type integral and its relationship with the usual time-scale integral is given in [10]. With the Lebesgue integral defined, we let  $L^1(\mathbb{T})$  consist of functions  $u : \mathbb{T}^\kappa \rightarrow \mathbb{R}$  for which  $|u|$  is Lebesgue-integrable on  $\mathbb{T}$ , and let

$$L^2(\mathbb{T}) := \{u \in L^1(\mathbb{T}) : |u|^2 \in L^1(\mathbb{T})\},$$

$$\|u\|_{\mathbb{T}} := \left( \int_a^b |u|^2 \Delta \right)^{1/2}, \quad u \in L^2(\mathbb{T}).$$

It is shown in [10] that  $L^2(\mathbb{T})$  is complete with respect to the norm  $\|\cdot\|_{\mathbb{T}}$ .

Next, define the norm  $\|\cdot\|_{1,\mathbb{T}}$  on  $C_{\text{rd}}^1(\mathbb{T})$  by

$$\|u\|_{1,\mathbb{T}}^2 := \|u\|_{\mathbb{T}}^2 + \|u^\Delta\|_{\mathbb{T}}^2, \quad u \in C_{\text{rd}}^1(\mathbb{T}),$$

and define the space  $H^1(\mathbb{T}) \subset L^2(\mathbb{T})$  to be the completion of  $C^1(\mathbb{T})$  with respect to the norm  $\|\cdot\|_{1,\mathbb{T}}$  (see Definition 4.1 and Remark 4.2 in [10]). The space  $H^1(\mathbb{T})$  is a time-scale analogue of the usual Sobolev space  $H^1(I)$  on a real interval  $I$ . The following lemma is proved in [10].

**Lemma 2.1.**  *$u \in H^1(\mathbb{T})$  if and only if there exists a function  $u^{\Delta g} \in L^2(\mathbb{T})$  such that the following condition holds: there exists a sequence  $(u_n)$  in  $C^1(\mathbb{T})$  such that  $u_n \rightarrow u$  and  $u_n^\Delta \rightarrow u^{\Delta g}$  in  $L^2(\mathbb{T})$ . If  $u \in H^1(\mathbb{T})$ , then the function  $u^{\Delta g}$  is unique (in the  $L^2(\mathbb{T})$  sense). Also,*

- (a) if  $u \in C^1(\mathbb{T})$  then  $u^{\Delta g} = u^\Delta$ ;  
 (b)  $H^1(\mathbb{T}) \subset C^0(\mathbb{T})$ .

For any  $u \in H^1(\mathbb{T})$  the function  $u^{\Delta g}$  in Lemma 2.1 is called the *generalized derivative* of  $u$ .

To represent the boundary value problem (1.1)–(1.2) in a functional–analytic setting we introduce some further spaces. Let

$$H^2(\mathbb{T}) := \{u \in C^1(\mathbb{T}) : u^\Delta \in H^1(\mathbb{T}^\kappa)\},$$

with the norm

$$\|u\|_{2,\mathbb{T}}^2 := \|u\|_{\mathbb{T}}^2 + \|u^\Delta\|_{1,\mathbb{T}^\kappa}^2, \quad u \in H^2(\mathbb{T}),$$

and let

$$X := \{u \in H^2(\mathbb{T}) : u \text{ satisfies (1.2)}\}, \quad Z := L^2(\mathbb{T}^\kappa),$$

with the norms  $\|\cdot\|_X := \|\cdot\|_{2,\mathbb{T}}$ ,  $\|\cdot\|_Z := \|\cdot\|_{\mathbb{T}^\kappa}$ , respectively. We also define inner products on  $Z$  by

$$\langle u, v \rangle := \int_a^{\rho(b)} uv \Delta, \quad \langle u, v \rangle_w := \int_a^{\rho(b)} uvw^{-1} \Delta, \quad u, v \in Z$$

(by the boundedness assumption on  $w$  in the introduction, the function  $w^{-1}$  is bounded). With the inner product  $\langle \cdot, \cdot \rangle$ , the space  $Z$  is a Hilbert space (completeness is proved in [10]), and the norm  $\|\cdot\|_Z$  is the norm induced by this inner product. We will also use the norm  $\|u\|_w^2 := \langle u, u \rangle_w$ ,  $u \in Z$ . The boundedness assumptions on  $w$  ensure that the norms  $\|\cdot\|_Z$  and  $\|\cdot\|_w$  are equivalent.

We observe that, for consistency, these inner products and the norm  $\| \cdot \|_w$  should have subscripts  $Z$ , but we omit this to simplify the formulae below; these are the only inner products that will be used in this paper, so this will cause no ambiguity.

We will say that  $y, z \in Z$  are  $w$ -orthogonal if  $\langle y, z \rangle_w = 0$ , and a set  $\{z_n: n \geq 1\}$  in  $Z$  is  $w$ -orthonormal if  $\langle z_m, z_n \rangle_w = \delta_{mn}$ , for all  $m, n \geq 1$ , where  $\delta_{mn}$  is the Kronecker delta.

The following result is Corollary 4.13 in [10].

**Lemma 2.2.** *The embeddings  $H^{i+1}(\mathbb{T}) \hookrightarrow C^i(\mathbb{T})$ ,  $i = 0, 1$ , are compact.*

The dimension of the above spaces will be important when  $\mathbb{T}$  is a finite set. Let  $N \geq 1$  be the number of points in  $\mathbb{T}^{\kappa^2}$  ( $N$  is either a finite number or ‘ $\infty$ ’). Now, from the definitions of the above spaces it can be seen that

$$\dim C^0(\mathbb{T}) = N + 2, \quad \dim X = \dim Z = N$$

(again, we allow  $N = \infty$  here).

**Remark 2.3.** Of course, if  $N < \infty$  then any integration is simply a summation, and  $H^2(\mathbb{T}) = C^0(\mathbb{T})$  (as sets—their norms are still different). This case is illustrated in the examples in Section 3. We note that  $\dim X = N$  because the two boundary conditions (1.2) reduce the dimension of  $C^0(\mathbb{T})$  by two. There is one trivial case where this is not true, viz., when  $N = 1$ , so that  $\sigma(a) = \rho(b)$ , and the boundary conditions reduce to  $u(\sigma(a)) = u(\rho(b)) = 0$ , that is, the boundary conditions coincide. We suppose from now on that this is not the case. In fact, Assumption 2.6 which we impose below precludes this case. This situation would also not arise if we simply assumed that  $N \geq 2$ , but the case  $N = 1$  will be useful for the examples in Section 3.

Finally, in this section, for any  $t \in \mathbb{T}$ , we define the function  $\mathbf{1}_t : \mathbb{T} \rightarrow \mathbb{R}$  by

$$\mathbf{1}_t(s) := \begin{cases} 1, & s = t, \\ 0, & s \in \mathbb{T} \setminus \{t\}. \end{cases}$$

This function will be useful below for constructing various examples. We note that, if  $t$  is isolated in  $\mathbb{T}$ , then  $\mathbf{1}_t \in C^0(\mathbb{T})$ .

### 2.2. An operator formulation of the boundary value problem

In this section we use the function spaces described above to set up an operator theoretic representation of the boundary value problem (1.1)–(1.2), and describe some preliminary results for this formulation. The next section will then utilize these results to obtain various spectral properties of the problem.

To represent the right-hand side of (1.1) in terms of a linear operator we first define  $I^\sigma : C^0(\mathbb{T}) \rightarrow C^0_{\text{rd}}(\mathbb{T})$  by  $I^\sigma u := u \circ \sigma$ , for  $u \in C^0(\mathbb{T})$ . The operator  $I^\sigma$  is linear and for  $u \in C^0(\mathbb{T})$ ,

$$\|I^\sigma u\|_{0, \mathbb{T}} = \sup_{t \in \mathbb{T}} \{ |u(\sigma(t))| \} \leq \sup_{t \in \mathbb{T}} \{ |u(t)| \} = \|u\|_{0, \mathbb{T}}, \quad u \in C^0(\mathbb{T}), \tag{2.1}$$

so  $I^\sigma : C^0(\mathbb{T}) \rightarrow C^0_{\text{rd}}(\mathbb{T})$  is bounded. To deal with the specific problem (1.1)–(1.2) it is also convenient to define an operator  $I^\sigma_{XZ} : X \rightarrow Z$  by

$$I^\sigma_{XZ} u := (I^\sigma u)|_{\mathbb{T}^{\kappa^2}}, \quad u \in X,$$

where  $(I^\sigma u)|_{\mathbb{T}^{\kappa^2}}$  denotes the restriction of the function  $I^\sigma u \in C_{\text{rd}}^0(\mathbb{T})$  to the set  $\mathbb{T}^{\kappa^2}$ , and we then regard  $(I^\sigma u)|_{\mathbb{T}^{\kappa^2}}$  as an element of  $Z$ . To obtain nice spectral properties of (1.1)–(1.2) it will be helpful if the operator  $I_{XZ}^\sigma$  is injective, so we now consider this issue.

**Lemma 2.4.** *For any time-scale  $\mathbb{T}$  of the form discussed above,*

$$N(I^\sigma) = C^0(\mathbb{T}) \cap \text{span}\{\mathbf{1}_a\}, \quad N(I_{XZ}^\sigma) = X \cap \text{span}\{\mathbf{1}_a, \mathbf{1}_b\}.$$

**Proof.** Suppose that there exists  $0 \neq u \in C^0(\mathbb{T})$  such that  $I^\sigma u \equiv 0$  on  $\mathbb{T}$ . Then there exists  $t_0 \in \mathbb{T}$  such that  $u(t_0) \neq 0$ . If  $\rho(t_0) < t_0$  then  $I^\sigma u(\rho(t_0)) = u(t_0) \neq 0$ , which contradicts our choice of  $u$ . If  $\rho(t_0) = t_0 > a$  then  $t_0$  is left-dense and, by continuity, there exists  $\delta > 0$  such that if  $t \in \mathbb{T}$  and  $t_0 - \delta < t < t_0$  then  $u(t) \neq 0$ , and hence  $I^\sigma u(t) \neq 0$ , which again contradicts our choice of  $u$ , and shows that  $u \equiv 0$  on  $\mathbb{T} \setminus \{a\}$ . If  $t_0 = a$  and  $a$  is right-dense, then a similar continuity argument shows that  $u(a) = 0$ , and so proves the result in this case. Finally, if  $a$  is right-scattered then  $\mathbf{1}_a \in C^0(\mathbb{T})$ , and  $N(I^\sigma) = \text{span}\{\mathbf{1}_a\}$ , which proves the first result. The second result follows from similar arguments and the definition of  $I_{XZ}^\sigma$  by restriction to  $\mathbb{T}^{\kappa^2}$ .  $\square$

If  $a$  is right-dense then  $\mathbf{1}_a \notin C^0(\mathbb{T})$ , so we have the following corollary.

**Corollary 2.5.** *If  $a$  is right-dense then  $I^\sigma : C^0(\mathbb{T}) \rightarrow C_{\text{rd}}^0(\mathbb{T})$  is injective.*

To obtain injectivity of  $I_{XZ}^\sigma$  we will impose a further condition on the problem. If  $a$  is right-scattered, or  $b$  is left-scattered, then the boundary conditions at  $a$  or  $b$  can be written in the form

$$a_0 u(a) + a_1 u(\sigma(a)) = 0 \quad \text{or} \quad b_0 u(\rho(b)) + b_1 u(b) = 0. \quad (2.2)$$

From now on we make the following assumption.

**Assumption 2.6.** *If  $a$  is right-scattered (respectively  $b$  is left-scattered) then*

$$a_0 \neq 0 \quad (\text{respectively } b_1 \neq 0). \quad (2.3)$$

**Lemma 2.7.** *The operator  $I_{XZ}^\sigma$  is injective and compact.*

**Proof.** If  $a$  is right-dense then  $\mathbf{1}_a \notin C^0(\mathbb{T})$ , and so  $\mathbf{1}_a \notin X$ , while if  $a$  is right-scattered then Assumption 2.6 ensures that  $\mathbf{1}_a \notin X$ . Similarly,  $\mathbf{1}_b \notin X$ , so that  $N(I_{XZ}^\sigma) = \{0\}$ , by Lemma 2.4. This proves that  $I_{XZ}^\sigma$  is injective. The compactness of  $I_{XZ}^\sigma$  follows readily from Lemma 2.2.  $\square$

To simplify the notation below, from now for any  $u \in X$ , we denote  $I_{XZ}^\sigma u$  by  $u^\sigma$ . This notation is slightly different to the standard time-scale notation, where  $u^\sigma$  is usually taken to mean  $I^\sigma u$ . However, our use of  $u^\sigma$  mean  $I_{XZ}^\sigma u$  is consistent with Eq. (1.1), which holds on the set  $\mathbb{T}^{\kappa^2}$ . Thus, in the boundary value context, the present notation seems preferable to the standard notation.

To represent the left-hand side of (1.1) in terms of a linear operator we now define  $L : X \rightarrow Z$  by

$$Lu := -(pu^\Delta)^{\Delta_g} + qu^\sigma, \quad u \in X.$$

Clearly,  $L$  is a bounded, linear operator. Also, it follows readily from integration by parts (using the properties of the generalized derivative proved in Corollary 4.6 of [10]), together with the definition of the inner product  $\langle \cdot, \cdot \rangle$  and the boundary conditions (1.2) that

$$\langle Lu, v^\sigma \rangle = \langle u^\sigma, Lv \rangle, \quad u, v \in X. \quad (2.4)$$

The next result follows from Assumption 2.6 and a trivial extension of Lemma 5.3 in [10].

**Lemma 2.8.** *There exists a constant  $C$  such that*

$$\langle (L + CwI_{XZ}^\sigma)u, u^\sigma \rangle > 0, \quad 0 \neq u \in X, \quad (2.5)$$

and hence the operator  $L + CwI_{XZ}^\sigma : X \rightarrow Z$  is injective.

From now on we suppose that Lemma 2.8 holds with  $C = 0$ . This entails no loss of generality since we may add  $Cw(t)u^\sigma(t)$  to both sides of Eq. (1.1) and redefine  $q, \lambda$  to be  $q + Cw, \lambda + C$ , respectively—the resulting problem satisfies the above assumption. With injectivity of  $L$  in place, the Green's function  $g$  of  $L$  can be constructed. The following lemma is proved in Theorem 5.11 in [10].

**Lemma 2.9.** *There exists a function  $g : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$  such that  $g(t, \cdot) \in C_{\text{rd}}^0(\mathbb{T})$ ,  $g(\cdot, s) \in C_{\text{rd}}^0(\mathbb{T})$ , for all  $t, s \in \mathbb{T}$ , and for any  $z \in Z$  the boundary value problem*

$$Lu = z, \quad (2.6)$$

has a unique solution  $u \in X$  given by

$$u(t) = \int_a^{\rho(b)} g(t, \cdot)z \Delta, \quad t \in \mathbb{T}.$$

We define the Green's operator  $G : Z \rightarrow X$  by

$$Gz(t) := \int_a^{\rho(b)} g(t, \cdot)z \Delta, \quad z \in Z.$$

Lemma 2.9 yields the following result.

**Corollary 2.10.** *The operators  $L : X \rightarrow Z$ ,  $G : Z \rightarrow X$ , are invertible, bounded linear operators and  $L^{-1} = G$ ,  $G^{-1} = L$ .*

### 3. Spectral properties

A solution of (1.1)–(1.2) will be defined to be a function  $u \in X$  for which Eq. (1.1) holds on  $\mathbb{T}^{\kappa^2}$  (in the  $L^2$  sense—for such a  $u$  both sides of (1.1) belong to  $Z$ ). With the above notation, the eigenvalue problem (1.1)–(1.2) can be rewritten in the form

$$Lu = \lambda wI_{XZ}^\sigma u, \quad u \in X. \quad (3.1)$$

Now, substituting  $u = Gz$ ,  $z \in Z$ ,  $\lambda = \mu^{-1}$ , into (3.1), and defining the operator  $T := wI_{XZ}^\sigma G : Z \rightarrow Z$ , yields the eigenvalue problem

$$Tz = \mu z, \quad z \in Z. \quad (3.2)$$

**Lemma 3.1.** *The operator  $T$  is compact, self-adjoint and positive definite, with respect to the inner product  $\langle \cdot, \cdot \rangle_w$ , in the following senses,*

$$\langle Tz_1, z_2 \rangle_w = \langle z_1, Tz_2 \rangle_w, \quad z_1, z_2 \in Z, \quad (3.3)$$

$$\langle Tz, z \rangle_w > 0, \quad 0 \neq z \in Z. \quad (3.4)$$

*The eigenvalue problems (3.1) and (3.2) are equivalent, in the sense that  $\lambda$  is an eigenvalue of (3.1) if and only if  $\mu = 1/\lambda$  is an eigenvalue of (3.2), and the corresponding eigenfunctions  $u$  and  $z$  are related to each other by  $u = Gz$ .*

**Proof.** The compactness of  $T$  follows from Lemma 2.7. Next, let  $z_1, z_2 \in Z$  be arbitrary, and define  $x_1 = Gz_1, x_2 = Gz_2$ . Then

$$\langle Tz_1, z_2 \rangle_w = \langle I_{XZ}^\sigma Gz_1, z_2 \rangle = \langle x_1^\sigma, Lx_2 \rangle = \langle Lx_1, x_2^\sigma \rangle = \langle z_1, I_{XZ}^\sigma Gz_2 \rangle = \langle z_1, Tz_2 \rangle_w$$

(using (2.4)), which proves (3.3). The inequality (3.4) also follows from this calculation, using (2.5). Finally, the equivalence of the eigenvalue problems follows from the invertibility of  $I_{XZ}^\sigma$  and  $G$ .  $\square$

Now, the standard functional-analytic theory of compact, self-adjoint linear operators in Hilbert spaces (see, for example, Section 72 of [6], or Section 6.3 of [9]) yields the following result.

**Lemma 3.2.** *For each  $k = 1, \dots, N$ , the operator  $T$  has an eigenvalue  $\mu_k$  and a corresponding eigenfunction  $z_k$  such that the set of eigenfunctions  $\{z_k\}$  is  $w$ -orthonormal and the following properties hold:*

- (i)  $\mu_k > \mu_{k+1} > 0$ ; if  $N = \infty$  then  $\lim_{k \rightarrow \infty} \mu_k = 0$ ;
- (ii)  $\dim N(T - \mu_k I_Z) = 1$ ;
- (iii) for any  $z \in Z$ ,

$$z = \sum_{k=1}^N \langle z, z_k \rangle_w z_k. \quad (3.5)$$

**Remark 3.3.** If  $N < \infty$  then (3.5) holds as a simple equality at each of the points in  $\mathbb{T}^{\kappa^2}$ ; if  $N = \infty$  then the series in (3.5) convergences in the sense of the norm  $\| \cdot \|_w$ . This remark also holds for the convergence of similar series below.

**Proof of Lemma 3.2.** It follows from the uniqueness of solutions to the initial value problem for Eq. (1.1), and the equivalence of the eigenvalue problems (3.1) and (3.2), that if  $\lambda_k$  is an eigenvalue then  $\dim N(T - \mu_k I_Z) = 1$ , which proves part (ii) of the lemma.

Now suppose that  $N < \infty$  (and recall that  $\dim Z = N$ ). Then by standard linear algebra the set of eigenvectors of  $T$  spans  $Z$ . Thus, by part (ii), there must be exactly  $N$  distinct eigenvalues  $\mu_k, k = 1, \dots, N$ , and a corresponding  $w$ -orthonormal set of eigenvectors  $\{z_k: k = 1, \dots, N\}$ . The eigenvalues can clearly be ordered so that condition (i) in the lemma holds, and (3.5) follows from the  $w$ -orthonormality of the set of eigenvectors.

If  $N = \infty$  then we appeal to Proposition 72.1 in [6] or Section 6.3 of [9] to assert the existence of a sequence of eigenvalues and a  $w$ -orthonormal set of eigenvectors, which spans  $Z$  and for which (3.5) holds.  $\square$

We now re-express the results of Lemma 3.2 more directly in terms of the eigenfunctions of (1.1). Let

$$\lambda_k = 1/\mu_k, \quad u_k := Gz_k, \quad v_k := \frac{u_k}{\|wu_k^\sigma\|_w}, \quad k = 1, \dots, N.$$

**Theorem 3.4.** *For each  $k = 1, \dots, N$ ,  $\lambda_k$  is an eigenvalue of (1.1), with corresponding eigenfunction  $v_k \in X$ , and there are no other eigenvalues. The set  $\{wv_k^\sigma\}$  is  $w$ -orthonormal. The following properties hold for each  $k$ :*

- (i)  $\lambda_{k+1} > \lambda_k > 0$ ; if  $N = \infty$  then  $\lim_{k \rightarrow \infty} \lambda_k = \infty$ ;
- (ii)  $\dim N(L - \lambda_k I_{XZ}^\sigma) = 1$ ;
- (iii) for any  $z \in Z$ ,

$$z = \sum_{k=1}^N \langle z, wv_k^\sigma \rangle_w wv_k^\sigma. \tag{3.6}$$

**Proof.** By the above definitions, and the equivalence of the eigenvalue problems (3.1) and (3.2), we have

$$z_k = Lu_k = \lambda_k wu_k^\sigma,$$

and hence,  $1 = \|z_k\|_w = \lambda_k \|wu_k^\sigma\|_w$ , and from this,

$$z_k = w\lambda_k \|wu_k^\sigma\|_w \frac{u_k^\sigma}{\|wu_k^\sigma\|_w} = wv_k^\sigma. \tag{3.7}$$

Thus, the set of functions  $\{wv_k^\sigma\}$  is  $w$ -orthonormal, and the above substitution clearly turns the expansion (3.5) into (3.6).  $\square$

The next result follows immediately from (3.6) and the  $w$ -orthonormality of the set  $\{wv_k^\sigma\}$ .

**Corollary 3.5** (Bessel’s equality). *For any  $z \in Z$ ,*

$$\|z\|_w^2 = \sum_{k=1}^N \langle z, wv_k^\sigma \rangle_w^2 = \sum_{k=1}^N \langle z, v_k^\sigma \rangle^2. \tag{3.8}$$

**Remark 3.6.** If we do not impose the condition (2.3) the operator  $I_{XZ}^\sigma$  may have a non-trivial null-space, with  $\dim N(I_{XZ}^\sigma) = 1$  or  $2$ , depending on whether the points  $a$  and  $b$  are right- or left-dense, and the boundary conditions at these points. Hence, by the definition of  $T$  and the invertibility of  $G$ , we have  $\dim N(T) = \dim N(I_{XZ}^\sigma)$ . Non-zero elements of  $N(T)$  correspond to an eigenvalue  $\mu = 0$  of  $T$ . However, only non-zero eigenvalues of  $T$  yield eigenvalues of the problem (2.6) (in a sense, an eigenvalue  $\mu = 0$  of  $T$  corresponds to an eigenvalue  $\lambda = \infty$  of (2.6)). Since  $T$  must have  $N$  eigenvalues (counting multiplicity, in the usual manner), it follows that  $T$  has  $N - \dim N(I_{XZ}^\sigma)$  non-zero eigenvalues, and hence the problem (2.6) has  $N - \dim N(I_{XZ}^\sigma)$ , rather than  $N$ , eigenvalues (this phenomenon was observed in Theorem 8 in [1], although it is described in a rather different manner in [1]). In particular, if  $\dim N(I_{XZ}^\sigma) > 0$  then (2.6) does not have enough eigenfunctions to yield the eigenfunction expansion (3.6).

**Remark 3.7.** The eigenfunction expansion (3.6) has some slightly strange features.

(i) Although the eigenfunctions  $v_k$  are defined on  $\mathbb{T}$ , the expansion (3.6) only holds for functions  $z \in Z$ , rather than functions in  $L^2(\mathbb{T})$ . In fact, when  $N < \infty$ , the eigenfunctions cannot span  $L^2(\mathbb{T})$ , since  $\dim L^2(\mathbb{T}) = N + 1$  and there are only  $N$  eigenfunctions.

(ii) The eigenfunction expansion (3.6) is in terms of the functions  $v_k^\sigma$ , rather than  $v_k$ . These functions are, in general, rd-continuous but not continuous. (In fact, by Lemma 2.2,  $v_k \in C^1(\mathbb{T})$ , but it cannot be guaranteed that  $v_k^\sigma$  is any smoother than  $C_{\text{rd}}^0(\mathbb{T})$ .) Thus we cannot, in general, hope for convergence of (3.6) with respect to the norm  $|\cdot|_{0,\mathbb{T}}$ , even when  $z, w \in C^0(\mathbb{T}^{\kappa^2})$  (see Example 3.17 below), so convergence with respect to the weighted  $L^2$  norm  $\|\cdot\|_w$  is a natural form of convergence. Of course, when  $N < \infty$  the convergence is pointwise, on the finite set of points  $\mathbb{T}^{\kappa^2}$ , so will hold with respect to any norm on  $Z$ .

**Theorem 3.8.** *If  $\lambda$  is not an eigenvalue of (1.1) then  $L - \lambda w I_{XZ}^\sigma : X \rightarrow Z$  has a bounded inverse  $(L - \lambda w I_{XZ}^\sigma)^{-1} : Z \rightarrow X$ . For each  $k = 1, \dots, N$ ,*

$$R(L - \lambda_k w I_{XZ}^\sigma) = \{z \in Z : \langle z, v_k^\sigma \rangle = 0\}. \quad (3.9)$$

**Proof.** By Proposition 72.2 in [6], for any  $\mu \in \mathbb{R}$ ,

$$R(T - \mu I) = N(T - \mu I)^\perp \quad (3.10)$$

(where  $\perp$  denotes orthogonal complement with respect to  $\langle \cdot, \cdot \rangle_w$  in  $Z$ ). Now suppose that  $\lambda = \lambda_k = 1/\mu_k$  for some  $k \geq 1$ , and  $z \in Z$  satisfies  $\langle z, z_k \rangle_w = 0$ , that is,  $\langle z, w v_k^\sigma \rangle_w = 0$  (by (3.7)). Then there exists  $y \in Z$  such that  $Ty - \mu_k y = z$ , and putting  $x = Gy$  yields  $\lambda_k w I_{XZ}^\sigma x - Lx = \frac{1}{\mu_k} z$  (since  $Lx = y$ ). It can be shown similarly that if  $\lambda$  is not an eigenvalue of (1.1) then  $L - \lambda w I_{XZ}^\sigma : X \rightarrow Z$  is surjective and hence bijective (by definition, it is injective).  $\square$

**Remark 3.9.** Equation (3.10) is standard for self-adjoint operators. The result (3.9) seems similar, but is in fact fundamentally different, in the following sense. It follows from (3.10) that  $R(T - \mu I) \cap N(T - \mu I) = \{0\}$  (this result is fundamental to many of the standard spectral properties of self-adjoint operators), whereas Section 6 of [4] constructs an example of an operator  $L$ , with an eigenvalue  $\lambda = 0$ , for which  $R(L) \cap N(L) \neq \{0\}$  (clearly, such an  $L$  is not positive definite, but this could easily be arranged). In particular, for this  $L$  the algebraic multiplicity of the eigenvalue 0 is strictly greater than 1, although the geometric multiplicity is equal to 1.

We now obtain a minimization characterization of the eigenvalues.

**Theorem 3.10.** *For each  $k = 1, \dots, N$ ,*

$$\lambda_k = \min \left\{ \frac{\langle Lv, w v^\sigma \rangle_w}{\|w v^\sigma\|_w^2} : 0 \neq v \in X, \langle w v^\sigma, w v_n^\sigma \rangle_w = 0, n = 1, \dots, k-1 \right\}$$

(the orthogonality condition in the minimization is absent when  $k = 1$ ).

**Proof.** Proposition 72.1 in [6] gives a maximization characterization of the eigenvalue  $\mu_k$  in Lemma 3.2, and combining this with  $\lambda_k = 1/\mu_k$ , yields the following characterization of the eigenvalue  $\lambda_k$ ,

$$\lambda_k = \min \left\{ \frac{\langle z, z \rangle_w}{\langle Tz, z \rangle_w} : 0 \neq z \in Z, \langle z, z_n \rangle_w = 0, n = 1, \dots, k - 1 \right\} \tag{3.11}$$

(this relies on the positive-definiteness of  $T$  proved in Lemma 3.1). We will now convert this characterization into that given in the theorem.

Since  $T$  is self-adjoint and positive definite, the operator  $T^{1/2} : Z \rightarrow Z$  exists and is self-adjoint and injective (it can be constructed in the standard manner using the eigenfunction expansion (3.5)). Also, since  $G : Z \rightarrow X$  is surjective, it follows from the definition of  $T$  and from  $T = T^{1/2}T^{1/2}$  that

$$R(wI_{XZ}^\sigma) = R(T) \subset R(T^{1/2}).$$

Thus  $T^{1/2}$  has an (unbounded) inverse operator  $T^{-1/2} : D(T^{-1/2}) \subset Z \rightarrow Z$ , with domain  $D(T^{-1/2}) \supset R(wI_{XZ}^\sigma)$ . We replace  $z \in Z$  in the minimization (3.11) with  $z$  of the form

$$z = T^{-1/2}(wv^\sigma), \quad v \in X. \tag{3.12}$$

For such  $z$  the orthogonality condition in (3.11) becomes

$$0 = \mu_n^{1/2} \langle z, z_n \rangle_w = \langle z, T^{1/2}z_n \rangle_w = \langle wv^\sigma, z_n \rangle_w = \langle wv^\sigma, wv_n^\sigma \rangle_w,$$

by (3.7), that is,  $v \in X$  in (3.12) must satisfy

$$\langle wv^\sigma, wv_n^\sigma \rangle_w = 0. \tag{3.13}$$

Now, substituting (3.12) into the fraction in (3.11) yields

$$\frac{\langle z, z \rangle_w}{\langle Tz, z \rangle_w} = \frac{\langle T^{-1}(wv^\sigma), wv^\sigma \rangle_w}{\langle wv^\sigma, wv^\sigma \rangle_w} = \frac{\langle L(I_{XZ}^\sigma)^{-1}(w^{-1}wv^\sigma), wv^\sigma \rangle_w}{\|wv^\sigma\|_w^2} = \frac{\langle Lv, wv^\sigma \rangle_w}{\|wv^\sigma\|_w^2}$$

(here, we have used the injectivity of  $I_{XZ}^\sigma$  (Lemma 2.7), so that  $I_{XZ}^\sigma$  has an inverse  $(I_{XZ}^\sigma)^{-1}$ , defined on the set  $R(I_{XZ}^\sigma)$ ). Hence, we conclude that

$$\lambda_k \leq \min \left\{ \frac{\langle Lv, wv^\sigma \rangle_w}{\|wv^\sigma\|_w^2} : 0 \neq v \in X, \langle wv^\sigma, wv_n^\sigma \rangle_w = 0, n = 1, \dots, k - 1 \right\},$$

and putting  $v = v_k$  into the right-hand side shows that this inequality is in fact an equality.  $\square$

**Remark 3.11.** Theorem 3.10 is proved in Theorem 2 in [1] for continuous coefficient functions, and with  $w \equiv 1$  (and an additional condition is imposed on  $\mathbb{T}$  at  $a$  and  $b$ ). The proof used in [1] is completely different to the proof in the present setting, using the theory of self-adjoint operators.

We now return to the eigenfunction expansion result. Despite Remark 3.7, when  $z \in X$  we can amend (3.6) slightly to obtain an expansion which converges uniformly.

**Theorem 3.12.** For any  $x \in X$ ,

$$x = \sum_{k=1}^N \langle wx^\sigma, wv_k^\sigma \rangle_w v_k, \tag{3.14}$$

where convergence is in the sense of the norm  $|\cdot|_0, \mathbb{T}$ .

**Remark 3.13.** If  $N < \infty$  then (3.6) holds, as a simple equality, at each of the points in  $\mathbb{T}^{\kappa^2}$ . On dimension grounds (see Remark 3.7) it seems slightly strange, at first sight, that the expansion (3.14) can hold, for general  $u \in X$ , at the two extra points  $\rho(b)$  and  $b$  in  $\mathbb{T} \setminus \mathbb{T}^{\kappa^2}$ , compared with the expansion (3.6). However, since functions  $u \in X$  satisfy the two boundary conditions (1.2) we have  $\dim X = \dim C^0(\mathbb{T}) - 2 = N = \dim Z$ , so there is no contradiction.

**Proof of Theorem 3.12.** We first suppose that  $N = \infty$  and prove that the series in (3.14) converges (to something) with respect to  $|\cdot|_{0,\mathbb{T}}$ . From the equation  $Lv_k = \lambda_k w v_k^\sigma$  and (2.4) we obtain

$$\langle wx^\sigma, wv_k^\sigma \rangle_w = \frac{1}{\lambda_k} \langle x^\sigma, Lv_k \rangle = \frac{1}{\lambda_k} \langle Lx, v_k^\sigma \rangle = \frac{1}{\lambda_k} \langle Lx, wv_k^\sigma \rangle_w, \tag{3.15}$$

while from  $\lambda_k^{-1} v_k = G(wv_k^\sigma)$  we obtain, for arbitrary  $t \in \mathbb{T}$ ,

$$\frac{1}{\lambda_k} v_k(t) = \int_a^{\rho(b)} g(t, \cdot) w v_k^\sigma \Delta = \langle g(t, \cdot) w, v_k^\sigma \rangle,$$

and hence, by (3.8),

$$\sum_{k=1}^N \frac{1}{\lambda_k^2} v_k(t)^2 = \sum_{k=1}^N \langle g(t, \cdot) w, v_k^\sigma \rangle^2 = \|g(t, \cdot) w\|_w^2 \leq C, \tag{3.16}$$

for some  $C > 0$ . Now, by (3.15) and (3.16), for any integers  $n > m \geq 1$ ,

$$\left| \sum_{k=m}^n \langle wx^\sigma, wv_k^\sigma \rangle_w v_k(t) \right|^2 \leq \sum_{k=m}^n \langle Lx, wv_k^\sigma \rangle_w^2 \sum_{k=m}^n \frac{1}{\lambda_k^2} v_k(t)^2 \leq C \sum_{k=m}^n \langle Lx, wv_k^\sigma \rangle_w^2.$$

By (3.8), the series  $\sum_{k=1}^\infty \langle Lx, wv_k^\sigma \rangle_w^2$  converges, and hence, since  $t \in \mathbb{T}$  is arbitrary, the sequence of partial sums of the series in (3.14) forms a Cauchy sequence with respect to the norm  $|\cdot|_{0,\mathbb{T}}$ , and hence the series converges with respect to this norm. Moreover, since  $v_n \in X \subset C^0(\mathbb{T})$  for all  $n$ , the sum of this series belongs to  $C^0(\mathbb{T})$ .

We now show that the series in (3.14) actually converges to  $x$  (for  $N < \infty$  or  $N = \infty$ ). Defining  $z = wI_{XZ}^\sigma x \in Z$ , and substituting this into (3.6) and rearranging yields

$$I_{XZ}^\sigma \left( x - \sum_{k=1}^N \langle wx^\sigma, wv_k^\sigma \rangle_w v_k \right) = 0 \tag{3.17}$$

(using the strict positivity of  $w$ , together with the continuity of  $I_{XZ}^\sigma$  on  $C^0(\mathbb{T})$  and the convergence of the series in (3.14) in  $C^0(\mathbb{T})$ ). Now, by Lemma 2.7, this implies that (3.14) holds.  $\square$

**Remark 3.14.** The expansion (3.14) seems more reasonable than (3.6) when the function  $x$  is continuous, in that the expansion (3.14) is in terms of the continuous functions  $v_k$ , and the convergence is uniform. However, this expansion has the rather strange feature that the inner product terms contain the function  $wx^\sigma$ , rather than simply  $x$ , which is what one would normally expect. It seems that eigenfunction expansions for the time-scale eigenvalue problem considered here unavoidably contain features not occurring in standard eigenfunction expansions for self-adjoint problems.

We now give some simple examples. The first one illustrates the slightly strange features of the expansions (3.6) and (3.14) mentioned above.

**Example 3.15.** Let  $\mathbb{T} := \{1, 2, 3\}$ , so that  $\mathbb{T}^{\kappa^2} = \{1\}$  and  $N = 1$ . We write a function  $x : \mathbb{T} \rightarrow \mathbb{R}$  in the form  $[x_1, x_2, x_3]$ , where  $x_i = x(i)$ ,  $i = 1, 2, 3$ , that is, we identify  $H^2(\mathbb{T}) = C^0(\mathbb{T})$  with  $\mathbb{R}^3$ . Similarly, we write  $z : \mathbb{T}^{\kappa^2} \rightarrow \mathbb{R}$  in the form  $[z_1]$  (that is, we identify  $Z$  with  $\mathbb{R}$ ), and so  $Z = \text{span}\{[1]\}$ . Suppose that the boundary conditions are  $u(1) = u(3) = 0$ , and so  $X = \text{span}\{[0, 1, 0]\}$ . Also suppose that  $p \equiv 1$ ,  $q \equiv 0$ ,  $w \equiv 1$ , so that Eq. (1.1) becomes

$$-u^{\Delta\Delta}(1) = -(u(3) - 2u(2) + u(1)) = 2u(2) = \lambda u(2),$$

which yields the single eigenvalue  $\lambda = 2$ , with eigenfunction  $u = [0, 1, 0]$ . Now,  $u^\sigma = [1]$ , so that  $Z$  is spanned by  $u^\sigma$ , which illustrates the expansion (3.6), while  $X$  is clearly spanned by  $u$ , which illustrates the expansion (3.14). More explicitly, for arbitrary  $z = [z_1] \in Z$ ,  $x = [0, x_2, 0] \in X$ , the expansions (3.6), (3.14), become, respectively,

$$\begin{aligned} \langle z, u^\sigma \rangle u^\sigma &= z_1 [1] = z, \\ \langle x^\sigma, u^\sigma \rangle u &= x_2 [0, 1, 0] = x. \end{aligned}$$

We observe that in the latter expansion, the shift in the term  $x^\sigma$  in the inner product is what allows the value of  $x$  at the point  $2 = \rho(b)$  to influence the value of the inner product—without this shift the value of  $x$  at the point 2 would not contribute to the series, and so the series could not sum to this value in general.

Clearly,  $L^2(\mathbb{T})$  cannot be spanned by the eigenfunctions, since  $\dim L^2(\mathbb{T}) = 2$  and there is only one eigenfunction. Furthermore,  $I^\sigma [1, 0, 0] = [0, 0, 0]$ , which illustrates Lemma 2.4, and shows that  $I^\sigma$  is not injective on  $C^0(\mathbb{T})$ .

The next example shows the necessity of the condition (2.3).

**Example 3.16.** We consider the same problem as in Example 3.15, except that we now take the boundary conditions to be

$$u(2) = u(3) = 0,$$

that is, a Dirichlet condition at  $b = 3$ , but a condition of the form (2.2) at  $a$  with  $a_0 = 0$ , that is (2.3) does not hold. In this case Eq. (1.1) becomes

$$-u^{\Delta\Delta}(1) = u(1) = \lambda u(2) = 0,$$

which shows that the boundary value problem (2.6) now has no eigenvalues, even though  $N = 1$ . This is consistent with Remark 3.6 since  $N(I_{XZ}^\sigma) = \{\mathbf{1}_1\}$ , that is  $\dim N(I_{XZ}^\sigma) = 1$ , so we expect to obtain  $N - \dim N(I_{XZ}^\sigma) = 0$  eigenvalues.

**Example 3.17.** Consider the time scale  $\mathbb{T} := [0, 1] \cup \{2\}$ , so that  $\mathbb{T}^\kappa = \mathbb{T}^{\kappa^2} = [0, 1]$ , and consider the Dirichlet problem

$$\begin{aligned} -u^{\Delta\Delta}(t) &= \lambda u^\sigma(t), \quad t \in \mathbb{T}^{\kappa^2}, \\ u(0) &= u(2) = 0. \end{aligned}$$

For each  $k \geq 1$  there is exactly one eigenvalue  $\lambda_k$  in the interval  $((k - 1)\pi, k\pi)$  satisfying

$$\tan \lambda_k^{1/2} = -\lambda_k^{1/2},$$

with corresponding eigenfunction given by

$$v_k(t) = \begin{cases} A_k \sin(\lambda_k^{1/2}t), & t \in [0, 1], \\ 0, & t = 2. \end{cases}$$

Now suppose that  $z \in C^2[0, 1]$ , with  $z(0) = 0$ ,  $z(1) = 1$ ,  $z'(1) = -1$  and define  $x$  to equal  $z$  on  $[0, 1]$  and  $x(2) = 0$ . Then  $z \in Z$  and  $x \in X$ . Also, the coefficients in the expansions (3.6), (3.14), are

$$\langle x^\sigma, v_k^\sigma \rangle = \langle z, v_k^\sigma \rangle = A_k \int_0^1 z(t) \sin(\lambda_k^{1/2}t) dt$$

(integration in the usual sense here). The series (3.6) then converges in the usual  $L^2$  sense on  $[0, 1]$ , but clearly does not converge at the point 1, since the functions in the expansion are  $v_k^\sigma$ , and  $v_k^\sigma(1) = 0$ . On the other hand, the expansion (3.14) does converge at the point 1, and this is possible because the expansion is in terms of the functions  $v_k$ .

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