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Global Asymptotic Stability in a Nonautonomous n-Species Lotka–Volterra Predator–Prey System with Infinite Delays

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In this paper, a delayed n-species nonautonomous Lotka–Volterra type food-chain system without dominating instantaneous negative feedback is investigated. By means of a Lyapunov functional, sufficient conditions are derived for the global asymptotic stability of the positive solution of the system. As a corollary, it is shown that the global asymptotic stability of the positive solution is maintained provided that the delayed negative feedbacks dominate other interspecific interaction effects with delays and the mean delays are sufficiently small.

Keywords: Infinite delay; Global asymptotic stability; Lyapunov functional

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1. INTRODUCTION

The classical Lotka–Volterra type systems are very important in models of multi-species population dynamics. There are considerable works on the study of global asymptotic stability of Lotka–Volterra type systems with time delays that have been developed in Refs. [1–3,7,8,10–15]. In addition to these, the books of Gopalsamy [4] and Kuang [9] are good sources for the topic of Lotka–Volterra type systems with time delays. Recently, by constructing appropriate Lyapunov functionals, Bereketoglu and Gyori [2] and Teng and Yu [13] have studied the global asymptotic stability of nonautonomous Lotka–Volterra type systems with finite and infinite delays, respectively. The systems they considered contain instantaneous delay-indenpendent terms which dominate other intraspecific and interspecific interaction effects with delays. However, when such delay-indenpendent terms are not present, the Lyapunov functionals in Bereketoglu and Gyori [2] and Teng and Yu [13] are not valid. In view of the fact that in real-life interactions, instantaneous responses are rare or weak relative to delayed responses, more realistic models should consist of delay differential systems without instantaneous (negative) feedbacks. Most of the known convergence results for delayed systems require dominating or strong instantaneous negative feedbacks or some restrictions on initial conditions. In order to justify the common belief that "small delays are negligible in some modelling process as far as stabilities are concerned", it is thus important to show under some reasonable assumptions that the global stability of a Lotka–Volterra system or other general systems persist when time delays are small enough. This is indeed a long standing question in the qualitative analysis of systems of delay differential equations. Kuang [7] presented a partial answer to this open question for Lotka–Volterra type systems with a saturated equilibrium.

Motivated by the works of Bereketoglu and Gyori [2], Teng and Yu [13] and Kuang [7], in this paper we consider delayed nonautonomous Lotka–Volterra type systems without saturated steady state solutions and without dominating instantaneous negative feedbacks. For n-species Lotka–Volterra food-chain system with infinite delays, this results in the following delayed system

$$\begin{cases} \dot{x}_1(t) = x_1(t)\Big(r_1(t) - a_{11}(t)\displaystyle\int_0^\infty K_{11}(s)x_1(t-s)\,ds - a_{12}(t) \\ \qquad\qquad \times \displaystyle\int_0^\infty K_{12}(s)x_2(t-s)\,ds\Big) \\ \dot{x}_i(t) = x_i(t)\Big(-r_i(t) + a_{i,\,i-1}(t)\displaystyle\int_0^\infty K_{i,\,i-1}(s)x_{i-1}(t-s)\,ds - a_{ii}(t) \\ \qquad\qquad \times \displaystyle\int_0^\infty K_{ii}(s)x_i(t-s)\,ds - a_{i,\,i+1}(t)\displaystyle\int_0^\infty K_{i,\,i+1}(s)x_{i+1}(t-s)\,ds\Big) \\ \qquad\qquad (i = 2,\ldots,n-1) \\ \dot{x}_n(t) = x_n(t)\Big(-r_n(t) + a_{n,\,n-1}(t)\displaystyle\int_0^\infty K_{n,\,n-1}(s)x_{n-1}(t-s)\,ds \\ \qquad -a_{nn}(t)\displaystyle\int_0^\infty K_{nn}(s)x_n(t-s)\,ds\Big), \end{cases} \tag{1.1}$$

where $x_i(t)$ denotes the density of the prey and predator population, respectively.

In this paper, for system (1.1) we always assume that for all $i,j = 1,2,\ldots,n$:

(H1) $r_i(t), a_{ij}(t)$ are continuous, bounded functions defined on $[0,+\infty)$, $r_i(t) > 0$, $a_{ij}(t) > 0$ for all $t \in [0,\infty)$;

(H2) $K_{ij}(t) : [0,\infty) \mapsto [0,\infty)$ are piecewise continuous and normalized functions such that

$$\int_0^\infty K_{ij}(s)\,ds = 1; \qquad \sigma_{ij} = \int_0^\infty sK_{ij}(s)\,ds < \infty. \tag{1.2}$$

Motivated by applications of system (1.1) to population dynamics, we assume that solutions of system (1.1) satisfy the initial conditions

$$x_i(\theta) = \phi_i(\theta), \quad \theta \in (-\infty,0], \quad \phi_i(0) > 0, \quad i = 1,2,\ldots,n, \tag{1.3}$$

where each ϕ_i is a given nonnegative and bounded continuous function on $(-\infty,0]$. It is well known that by the fundamental theory of functional differential equations [6], system (1.1) has a unique solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ satisfying the initial conditions (1.3). It is easy to verify that solutions of system (1.1) corresponding to initial conditions (1.3) are defined on $[0, +\infty)$ and remain positive for all $t \geq 0$. In this paper, the solution of system (1.1) satisfying the initial conditions (1.3) is said to be positive.

If $f(t)$ is a continuous bounded function defined on $[0, +\infty)$, we set

$$f^L = \inf_{t\geq 0} f(t), \qquad f^M = \sup_{t\geq 0} f(t). \tag{1.4}$$

In the following we say a positive solution of system (1.1) is globally asymptotically stable if it attracts all other positive solutions of the system.

The organization of this paper is as follows. In the next section, we prove the boundedness of the positive solutions of system (1.1) with initial conditions (1.3) under appropriate condition, and sufficient conditions are derived for the positive solutions of system (1.1) to be globally asymptotically stable.

2. MAIN RESULTS

LEMMA 2.1 *Let* $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ *denote any positive solution of system* (1.1) *with initial conditions* (1.3). *If system* (1.1) *satisfies* $(H1)$–$(H2)$ *and the following:*

(H3) $a^M_{n,n-1} R_{n-1} > a^L_{n-1,n-1} r^L_n$,

where

$$R_i = \frac{a^M_{i,i-1} R_{i-1}}{a^L_{i-1,i-1}} - r^L_i, \quad i = 3, 4, \ldots, n-1;$$

$$R_2 = \frac{a^M_{21} r^M_1}{a^L_{11}} - r^L_2,$$

then there exits a $T > 0$ *such that*

$$0 < x_i(t) \leq M_i \quad (i = 1, 2, \ldots, n), \text{ for } t \geq T, \tag{2.1}$$

where

$$M_1 > M_1^* = \frac{r_1^M}{a_{11}^L \int_0^\infty K_{11}(s) e^{-r_1^M s} ds},$$

$$M_2 > M_2^* = \frac{a_{21}^M M_1 - r_2^L}{a_{22}^L \int_0^\infty K_{22}(s) e^{-(a_{21}^M M_1 - r_2^L)s} ds},$$

$$M_i > M_i^* = \frac{a_{i,i-1}^M M_{i-1} - r_i^L}{a_{ii}^L \int_0^\infty K_{ii}(s) e^{-(a_{i,i-1}^M M_{i-1} - r_i^L)s} ds}, \quad i = 3, 4, \ldots, n. \tag{2.2}$$

□

The proof of Lemma 2.1 is similar to that of Theorem 2.1 in Gopalsamy and He [5] and is therefore omitted here.

In the following, by improving the method given in Refs. [2,13], we further discuss the global asymptotic stability of the positive solutions of system (1.1). Now we state our main result below.

THEOREM 2.1 *In addition to* (*H*1)–(*H*3), *assume further that there exist* $c_i > 0, i = 1, 2, \ldots, n$, *such that the following hold:*

(H4)
$$\liminf_{t\to\infty} \Bigg[c_1 a_{11}(t) - c_2 \int_0^\infty K_{21}(s) a_{21}(t+s)\, ds$$
$$- c_1 (r_1(t) + a_{11}(t) M_1 + a_{12}(t) M_2)$$
$$\times \int_0^\infty \int_{t-s}^t K_{11}(s) a_{11}(v+s)\, dv\, ds$$
$$- \sum_{k=1}^{2} c_k M_k \int_0^\infty \int_0^\infty \int_{t+r-s}^{t+r} K_{kk}(s) a_{kk}(v+s) K_{k1}(r)$$
$$\times a_{k1}(t+r)\, dv\, ds\, dr \Bigg] > 0;$$

(H5) $$\liminf_{t\to\infty}\Bigg[c_i a_{ii}(t) - c_{i-1}\int_0^\infty K_{i-1,\,i}(s)a_{i-1,\,i}(t+s)\,ds - c_{i+1} \times \int_0^\infty K_{i+1,\,i}(s)a_{i+1,\,i}(t+s)\,ds - c_i\left(r_i(t) + \sum_{k=i-1}^{i+1} a_{ik}(t)M_k\right)\int_0^\infty\int_{t-s}^{t} K_{ii}(s)a_{ii}(v+s)\,dv\,ds - \sum_{k=i-1}^{i+1} c_k M_k \int_0^\infty\int_0^\infty\int_{t+r-s}^{t+r} K_{kk}(s)a_{kk}(v+s)K_{ki}(r) \times a_{ki}(t+r)\,dv\,ds\,dr\Bigg] > 0, \quad i = 2, 3, \ldots, n-1;$$

(H6) $$\liminf_{t\to\infty}\Bigg[c_n a_{nn}(t) - c_{n-1}\int_0^\infty K_{n-1,\,n}(s)a_{n-1,\,n}(t+s)\,ds - c_n(r_n(t) + a_{n,\,n-1}(t)M_{n-1} + a_{nn}(t)M_n) \times \int_0^\infty\int_{t-s}^{t} K_{nn}(s)a_{nn}(v+s)\,dv\,ds - \sum_{k=n-1}^{n} c_k M_k \int_0^\infty\int_0^\infty\int_{t+r-s}^{t+r} K_{kk}(s)a_{kk}(v+s) \times K_{kn}(r)a_{kn}(t+r)\,dv\,ds\,dr\Bigg] > 0.$$

Then for any positive solutions $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ and $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))$ of system (1.1), one has

$$\lim_{t\to\infty} |x_i(t) - y_i(t)| = 0, \quad i = 1, 2, \ldots, n. \tag{2.3}$$

□

Proof For any two nontrivial solutions $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ and $y(t) = (y_1(t),\ y_2(t), \ldots, y_n(t))$ of system (1.1), we have from

Lemma 2.1 that there exist positive constants T and M_i ($i = 1, 2, \dots, n$), such that for all $t \geq T$,

$$0 < x_i(t) \leq M_i, \quad 0 < y_i(t) \leq M_i, \qquad i = 1, 2, \dots, n. \tag{2.4}$$

Let

$$V_{11}(t) = |\ln x_1(t) - \ln y_1(t)|. \tag{2.5}$$

Calculating the upper right derivative of $V_{11}(t)$ along the solutions of (1.1), it follows that

$$\begin{aligned} D^+V_{11}(t) &= \left(\frac{\dot{x}_1(t)}{x_1(t)} - \frac{\dot{y}_1(t)}{y_1(t)}\right)\operatorname{sgn}(x_1(t)-y_1(t)) \\ &= \operatorname{sgn}(x_1(t)-y_1(t))\Big[-a_{11}(t)\int_0^\infty K_{11}(s)(x_1(t-s)-y_1(t-s))\,ds \\ &\qquad - a_{12}(t)\int_0^\infty K_{12}(s)(x_2(t-s)-y_2(t-s))\,ds\Big] \\ &= \operatorname{sgn}(x_1(t)-y_1(t))\Big[-a_{11}(t)(x_1(t)-y_1(t)) - a_{12}(t) \\ &\qquad \times \int_0^\infty K_{12}(s)(x_2(t-s)-y_2(t-s))\,ds + a_{11}(t) \\ &\qquad \times \int_0^t \int_{t-s}^t K_{11}(s)(\dot{x}_1(u)-\dot{y}_1(u))\,du\,ds + W_1(t)\Big], \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} W_1(t) = a_{11}(t)&\int_t^\infty K_{11}(s)(x_1(t) - x_1(t-s))\,ds \\ &- a_{11}(t)\int_t^\infty K_{11}(s)(y_1(t) - y_1(t-s))\,ds. \end{aligned}$$

On substituting (1.1) into (2.6), we derive

$$
\begin{aligned}
&D^+V_{11}(t)\\
&= \operatorname{sgn}(x_1(t)-y_1(t))[-a_{11}(t)(x_1(t)-y_1(t))\\
&\quad - a_{12}(t)\int_0^\infty K_{12}(s)(x_2(t-s)-y_2(t-s))\,ds\\
&\quad + a_{11}(t)\int_0^t\int_{t-s}^t K_{11}(s)\bigg\{x_1(u)\bigg[r_1(u)\\
&\quad - a_{11}(u)\int_0^\infty K_{11}(r)x_1(u-r)\,dr - a_{12}(u)\int_0^\infty K_{12}(r)x_2(u-r)\,dr\bigg]\\
&\quad - y_1(u)\bigg[r_1(u) - a_{11}(u)\int_0^\infty K_{11}(r)y_1(u-r)\,dr\\
&\quad - a_{12}(u)\int_0^\infty K_{12}(r)y_2(u-r)\,dr\bigg]\bigg\}du\,ds + W_1(t)\bigg]\\
&= \operatorname{sgn}(x_1(t)-y_1(t))\bigg[-a_{11}(t)(x_1(t)-y_1(t))\\
&\quad - a_{12}(t)\int_0^\infty K_{12}(s)(x_2(t-s)-y_2(t-s))\,ds\\
&\quad + a_{11}(t)\int_0^t\int_{t-s}^t K_{11}(s)\bigg\{(x_1(u)-y_1(u))\bigg[r_1(u)\\
&\quad - a_{11}(u)\int_0^\infty K_{11}(r)y_1(u-r)\,dr - a_{12}(u)\int_0^\infty K_{12}(r)y_2(u-r)\,dr\bigg]\\
&\quad + x_1(u)\bigg[-a_{11}(u)\int_0^\infty K_{11}(r)(x_1(u-r)-y_1(u-r))\,dr\\
&\quad - a_{12}(u)\int_0^\infty K_{12}(r)(x_2(u-r)-y_2(u-r))\,dr\bigg]\bigg\}du\,ds + W_1(t)\bigg]\\
&\le -a_{11}(t)|x_1(t)-y_1(t)|\\
&\quad + a_{12}(t)\int_0^\infty K_{12}(s)|x_2(t-s)-y_2(t-s)|ds\\
&\quad + a_{11}(t)\int_0^\infty\int_{t-s}^t K_{11}(s)\bigg\{|x_1(u)-y_1(u)|\bigg[r_1(u)\\
&\quad + a_{11}(u)\int_0^\infty K_{11}(r)y_1(u-r)\,dr + a_{12}(u)\int_0^\infty K_{12}(r)y_2(u-r)\,dr\bigg]\\
&\quad + x_1(u)\bigg[a_{11}(u)\int_0^\infty K_{11}(r)|x_1(u-r)-y_1(u-r)|dr\\
&\quad + a_{12}(u)\int_0^\infty K_{12}(r)|x_2(u-r)-y_2(u-r)|dr\bigg]\bigg\}du\,ds + |W_1(t)|. \qquad (2.7)
\end{aligned}
$$

Define

$$
\begin{aligned}
V_{12}(t) = & \int_0^\infty \int_{t-s}^t K_{12}(s) a_{12}(u+s) |x_2(u) - y_2(u)| \, du \, ds \\
& + \int_0^\infty \int_{t-s}^t \int_v^t K_{11}(s) a_{11}(v+s) \Big[r_1(u) + a_{11}(u) \int_0^\infty K_{11}(r) y_1(u-r) \, dr \\
& + a_{12}(u) \int_0^\infty K_{12}(r) y_2(u-r) \, dr \Big] |x_1(u) - y_1(u)| \, du \, dv \, ds \\
& + \sum_{k=1}^2 \int_0^\infty \int_{t-s}^t \int_v^t K_{11}(s) a_{11}(v+s) a_{1k}(u) x_1(u) \\
& \times \left[\int_0^\infty K_{1k}(r) |x_k(u-r) - y_k(u-r)| dr \right] du \, dv \, ds. \qquad (2.8)
\end{aligned}
$$

It then follows from (2.7) and (2.8) that

$$
\begin{aligned}
D^+ V_{11}(t) + \dot{V}_{12}(t) \le & -a_{11}(t) |x_1(t) - y_1(t)| \\
& + \int_0^\infty K_{12}(s) a_{12}(t+s) \, ds |x_2(t) - y_2(t)| \\
& + \int_0^\infty \int_{t-s}^t K_{11}(s) a_{11}(v+s) \, dv \, ds \Big[r_1(t) \\
& + a_{11}(t) \int_0^\infty K_{11}(r) y_1(t-r) \, dr \\
& + a_{12}(t) \int_0^\infty K_{12}(r) y_2(t-r) \, dr \Big] |x_1(t) - y_1(t)| \\
& + x_1(t) \sum_{k=1}^2 a_{1k}(t) \int_0^\infty \int_{t-s}^t K_{11}(s) a_{11}(v+s) \, dv \, ds \\
& \times \int_0^\infty K_{1k}(r) |x_k(t-r) - y_k(t-r)| dr + |W_1(t)|. \qquad (2.9)
\end{aligned}
$$

Define

$$
V_1(t) = V_{11}(t) + V_{12}(t) + V_{13}(t), \qquad (2.10)
$$

in which

$$V_{13}(t) = \sum_{k=1}^{2} \int_0^\infty \int_{t-r}^{t} \int_0^\infty \int_{u+r-s}^{u+r} K_{11}(s)a_{11}(v+s) \times K_{1k}(r)a_{1k}(u+r)x_1(u+r)|x_k(u) - y_k(u)|dv\,ds\,du\,dr. \qquad (2.11)$$

Then it follows from (2.9), (2.10) and (2.11) that

$$\begin{aligned} D^+V_1(t) \le & -a_{11}(t)|x_1(t) - y_1(t)| \\ & + \int_0^\infty K_{12}(s)a_{12}(t+s)\,ds|x_2(t) - y_2(t)| \\ & + \int_0^\infty \int_{t-s}^{t} K_{11}(s)a_{11}(v+s)\,dv\,ds\Big[r_1(t) \\ & + a_{11}(t)\int_0^\infty K_{11}(r)y_1(t-r)\,dr \\ & + a_{12}(t)\int_0^\infty K_{12}(r)y_2(t-r)\,dr\Big]|x_1(t) - y_1(t)| \\ & + \sum_{k=1}^{2} \int_0^\infty \int_0^\infty \int_{t+r-s}^{t+r} K_{11}(s)a_{11}(v+s) \\ & \times K_{1k}(r)a_{1k}(t+r)x_1(t+r)\,dv\,ds\,dr|x_k(t) - y_k(t)| + |W_1(t)| \\ = & -a_{11}(t)|x_1(t) - y_1(t)| \\ & + \int_0^\infty K_{12}(s)a_{12}(t+s)\,ds|x_2(t) - y_2(t)| \\ & + r_1(t)\int_0^\infty \int_{t-s}^{t} K_{11}(s)a_{11}(v+s)\,dv\,ds|x_1(t) - y_1(t)| \\ & + \int_0^\infty \int_{t-s}^{t} K_{11}(s)a_{11}(v+s)\,dv\,ds\Big\{a_{11}(t)\int_0^{t-T} K_{11}(r)y_1(t-r)\,dr \\ & + a_{12}(t)\int_0^{t-T} K_{12}(r)y_2(t-r)\,dr\Big\}|x_1(t) - y_1(t)| \\ & + \sum_{k=1}^{2} \int_0^\infty \int_0^\infty \int_{t+r-s}^{t+r} K_{11}(s)a_{11}(v+s)K_{1k}(r)a_{1k}(t+r) \\ & \times x_1(t+r)\,dv\,ds\,dr|x_k(t) - y_k(t)| + |W_1(t)| + B_1(t), \end{aligned} \qquad (2.12)$$

where

$$B_1(t) = \int_0^{\infty} \int_{t-s}^{t} K_{11}(s) a_{11}(v+s)\, dv\, ds \Bigg[a_{11}(t) \int_{t-T}^{\infty} K_{11}(r) y_1(t-r)\, dr + a_{12}(t) \int_{t-T}^{\infty} K_{12}(r) y_2(t-r)\, dr \Bigg] |x_1(t) - y_1(t)|.$$

By (2.4), $x_1(t) \le M_1, y_k(t) \le M_k (k = 1, 2)$, for $t \ge T$. This, together with (2.12), for $t \ge T$, leads to

$$\begin{aligned} D^+ V_1(t) \le &-a_{11}(t)|x_1(t) - y_1(t)| + \int_0^{\infty} K_{12}(s) a_{12}(t+s) ds |x_2(t) - y_2(t)| \\ &+ \left(r_1(t) + \sum_{k=1}^{2} a_{1k}(t) M_k \right) \int_0^{\infty} \int_{t-s}^{t} K_{11}(s) a_{11}(v+s)\, dv\, ds \\ &\times |x_1(t) - y_1(t)| + M_1 \sum_{k=1}^{2} \int_0^{\infty} \int_0^{\infty} \int_{t+r-s}^{t+r} K_{11}(s) a_{11}(v+s) K_{1k}(r) \\ &\times a_{1k}(t+r)\, dv\, ds\, dr \times |x_k(t) - y_k(t)| + |W_1(t)| + B_1(t). \end{aligned} \tag{2.13}$$

Next, for $i = 2, \ldots, n-1$, let

$$V_i(t) = V_{i1}(t) + V_{i2}(t) + V_{i3}(t), \tag{2.14}$$

where

$$V_{i1}(t) = |\ln x_i(t) - \ln y_i(t)|,$$

$$\begin{aligned} V_{i2}(t) = &\int_0^{\infty} \int_{t-s}^{t} K_{i,\,i-1}(s) a_{i,\,i-1}(u+s) |x_{i-1}(u) - y_{i-1}(u)|\, du\, ds \\ &+ \int_0^{\infty} \int_{t-s}^{t} K_{i,\,i+1}(s) a_{i,\,i+1}(u+s) |x_{i+1}(u) - y_{i+1}(u)|\, du\, ds \\ &+ \int_0^{\infty} \int_{t-s}^{t} \int_v^t K_{ii}(s) a_{ii}(v+s) \Bigg[r_i(u) \\ &+ \sum_{k=i-1}^{i+1} a_{ik}(u) \int_0^{\infty} K_{ik}(r) y_k(u-r)\, dr \Bigg] |x_i(u) - y_i(u)|\, du\, dv\, ds \end{aligned}$$

$$+\sum_{k=i-1}^{i+1}\int_0^\infty\int_{t-s}^{t}\int_v^t K_{ii}(s)a_{ii}(v+s)a_{ik}(u)x_i(u)$$
$$\times\left[\int_0^\infty K_{ik}(r)|x_k(u-r)-y_k(u-r)|dr\right]du\,dv\,ds, \qquad (2.15)$$
$$V_{i3}(t)=\sum_{k=i-1}^{i+1}\int_0^\infty\int_{t-r}^{t}\int_0^\infty\int_{u+r-s}^{u+r} K_{ii}(s)a_{ii}(v+s)$$
$$\times K_{ik}(r)a_{ik}(u+r)x_i(u+r)|x_k(u)-y_k(u)|dv\,ds\,du\,dr.$$

Then it follows from (1.1), (2.14) and (2.15) that for $t\geq T$

$$D^+V_i(t)\leq -a_{ii}(t)|x_i(t)-y_i(t)|+\int_0^\infty K_{i,i-1}(s)a_{i,i-1}(t+s)\,ds$$
$$\times|x_{i-1}(t)-y_{i-1}(t)|+\int_0^\infty K_{i,i+1}(s)a_{i,i+1}(t+s)\,ds|x_{i+1}(t)-y_{i+1}(t)|$$
$$+\left(r_i(t)+\sum_{k=i-1}^{i+1}a_{ik}(t)M_k\right)\int_0^\infty\int_{t-s}^{t}K_{ii}(s)a_{ii}(v+s)\,dv\,ds$$
$$\times|x_i(t)-y_i(t)|+M_i\sum_{k=i-1}^{i+1}\int_0^\infty\int_0^\infty\int_{t+r-s}^{t+r}K_{ii}(s)a_{ii}(v+s)$$
$$\times a_{ik}(t+r)K_{ik}(r)\,dv\,ds\,dr\times|x_k(t)-y_k(t)|+|W_i(t)|+B_i(t), \quad (2.16)$$

where

$$W_i(t)=a_{ii}(t)\int_t^\infty K_{ii}(s)(x_i(t)-x_i(t-s))\,ds$$
$$-\,a_{ii}(t)\int_t^\infty K_{ii}(s)(y_i(t)-y_i(t-s))\,ds,$$
$$B_i(t)=\int_0^\infty\int_{t-s}^{t}K_{ii}(s)a_{ii}(v+s)\,dv\,ds$$
$$\times\sum_{j=i-1}^{i+1}a_{ij}(t)\int_{t-T}^\infty K_{ij}(r)y_j(t-r)\,dr|x_i(t)-y_i(t)|.$$

Finally, we define

$$V_n(t)=V_{n1}(t)+V_{n2}(t)+V_{n3}(t), \qquad (2.17)$$

where

$$V_{n1}(t) = |\ln x_n(t) - \ln y_n(t)|,$$

$$\begin{aligned}
V_{n2}(t) = {} & \int_0^\infty \int_{t-s}^t K_{n,n-1}(s) a_{n,n-1}(u+s)|x_{n-1}(u) - y_{n-1}(u)|\,du\,ds \\
& + \int_0^\infty \int_{t-s}^t \int_v^t K_{nn}(s) a_{nn}(v+s)\Bigg[r_n(u) \\
& + \sum_{j=n-1}^n a_{nj}(u) \int_0^\infty K_{nj}(r) y_j(u-r)\,dr \Bigg] |x_n(u) - y_n(u)|\,du\,dv\,ds \\
& + \sum_{k=n-1}^n \int_0^\infty \int_{t-s}^t \int_v^t K_{nn}(s) a_{nn}(v+s) a_{nk}(u) x_n(u) \\
& \times \left[\int_0^\infty K_{nk}(r)|x_k(u-r) - y_k(u-r)|dr \right] du\,dv\,ds,
\end{aligned}$$

$$\begin{aligned}
V_{n3}(t) = {} & \sum_{k=n-1}^n \int_0^\infty \int_{t-r}^t \int_0^\infty \int_{u+r-s}^{u+r} K_{nn}(s) a_{nn}(v+s) \\
& \times K_{nk}(r) a_{nk}(u+r) x_n(u+r)|x_k(u) - y_k(u)|\,dv\,ds\,du\,dr. \qquad (2.18)
\end{aligned}$$

Then along the solution of (1.1), we have that for $t \geq T$

$$\begin{aligned}
D^+ V_n(t) \leq {} & -a_{nn}(t)|x_n(t) - y_n(t)| \\
& + \int_0^\infty K_{n,n-1}(s) a_{n,n-1}(t+s)\,ds |x_{n-1}(t) - y_{n-1}(t)| \\
& + \left(r_n(t) + \sum_{k=n-1}^n a_{nk}(t) M_k \right) \\
& \times \int_0^\infty \int_{t-s}^t K_{nn}(s) a_{nn}(v+s)\,dv\,ds |x_n(t) - y_n(t)| \\
& + M_n \sum_{k=n-1}^n \int_0^\infty \int_0^\infty \int_{t+r-s}^{t+r} K_{nn}(s) a_{nn}(v+s) \\
& \times a_{nk}(t+r) K_{nk}(r)\,dv\,ds\,dr \\
& \times |x_k(t) - y_k(t)| + |W_n(t)| + B_n(t), \qquad (2.19)
\end{aligned}$$

where

$$W_n(t) = a_{nn}(t)\int_t^{\infty} K_{nn}(s)(x_n(t) - x_n(t-s))\,ds$$
$$- a_{nn}(t)\int_t^{\infty} K_{nn}(s)(y_n(t) - y_n(t-s))\,ds,$$
$$B_n(t) = \int_0^{\infty}\int_{t-s}^{t} K_{nn}(s)a_{nn}(v+s)\,dv\,ds \sum_{j=n-1}^{n} a_{nj}(t)$$
$$\times \int_{t-T}^{\infty} K_{nj}(r)y_j(t-r)\,dr|x_n(t) - y_n(t)|.$$

Now we define a Lyapunov functional $V(t)$ by

$$V(t) = \sum_{i=1}^{n} c_i V_i(t). \tag{2.20}$$

Then it follows from (2.13), (2.16), (2.19) and (2.20) that for $t \geq T$

$$D^+V(t) \leq -\sum_{i=1}^{n} A_i(t)|x_i(t) - y_i(t)| + W(t) + B(t), \tag{2.21}$$

where

$$A_1(t) = c_1 a_{11}(t) - c_2\int_0^{\infty} K_{21}(s)a_{21}(t+s)\,ds$$
$$- c_1(r_1(t) + a_{11}(t)M_1 + a_{12}(t)M_2)$$
$$\times \int_0^{\infty}\int_{t-s}^{t} K_{11}(s)a_{11}(v+s)\,dv\,ds$$
$$- \sum_{k=1}^{2} c_k M_k \int_0^{\infty}\int_0^{\infty}\int_{t+r-s}^{t+r} K_{kk}(s)a_{kk}(v+s)$$
$$\times K_{k1}(r)a_{k1}(t+r)\,dv\,ds\,dr;$$

$$
\begin{aligned}
A_i(t) &= c_i a_{ii}(t) - c_{i-1}\int_0^\infty K_{i-1,i}(s)a_{i-1,i}(t+s)\,ds - c_{i+1} \\
&\quad \times \int_0^\infty K_{i+1,i}(s)a_{i+1,i}(t+s)\,ds \\
&\quad - c_i\left(r_i(t) + \sum_{k=i-1}^{i+1} a_{ik}(t)M_k\right)\int_0^\infty \int_{t-s}^t K_{ii}(s)a_{ii}(v+s)\,dv\,ds \\
&\quad - \sum_{k=i-1}^{i+1} c_k M_k \int_0^\infty \int_0^\infty \int_{t+r-s}^{t+r} K_{kk}(s)a_{kk}(v+s)K_{ki}(r)a_{ki}(t+r)\,dv\,ds\,dr, \\
&\qquad i = 2,3,\ldots,n-1; \\
A_n(t) &= c_n a_{nn}(t) - c_{n-1}\int_0^\infty K_{n-1,n}(s)a_{n-1,n}(t+s)\,ds \\
&\quad - c_n(r_n(t) + a_{n,n-1}(t)M_{n-1} + a_{nn}(t)M_n)\int_0^\infty \int_{t-s}^t K_{nn}(s)a_{nn}(v+s)\,dv\,ds \\
&\quad - \sum_{k=n-1}^{n} c_k M_k \int_0^\infty \int_0^\infty \int_{t+r-s}^{t+r} K_{kk}(s)a_{kk}(v+s)K_{kn}(r)a_{kn}(t+r)\,dv\,ds\,dr; \\
W(t) &= \sum_{i=1}^n c_i|W_i(t)|; \qquad B(t) = \sum_{i=1}^n c_i B_i(t).
\end{aligned}
$$

By hypotheses (H4)–(H6), there exist constants $\alpha_i > 0$ $(i = 1,2,\ldots,n)$ and $T_1^* \geq T$, such that

$$
A_i(t) \geq \alpha_i > 0, \quad \text{for } t \geq T_1^*. \tag{2.22}
$$

We Note that

$$
\begin{aligned}
W(t) &\leq \sum_{i=1}^n c_i a_{ii}^M \left[|x_i(t) - y_i(t)| + \sup_{-\infty<s\leq 0}|x_i(s) - y_i(s)|\right]\int_t^\infty K_{ii}(s)\,ds; \\
B_1(t) &\leq a_{11}^M \sigma_{11} \sum_{j=1}^2 a_{1j}^M \sup_{-\infty<s\leq T} y_j(s)|x_1(t) - y_1(t)|\int_{t-T}^\infty K_{1j}(s)\,ds; \\
B_i(t) &\leq a_{ii}^M \sigma_{ii} \sum_{j=i-1}^{i+1} a_{ij}^M \sup_{-\infty<s\leq T} y_j(s)|x_i(t) - y_i(t)|\int_{t-T}^\infty K_{ij}(s)\,ds, \\
&\qquad i = 2,3,\ldots,n-1; \\
B_n(t) &\leq a_{nn}^M \sigma_{nn} \sum_{j=n-1}^{n} a_{nj}^M \sup_{-\infty<s\leq T} y_j(s)|x_n(t) - y_n(t)|\int_{t-T}^\infty K_{nj}(s)\,ds.
\end{aligned} \tag{2.23}
$$

By assumption (H2), $\int_t^\infty K_{ij}(s)\,ds \to 0$ and $\int_{t-T}^\infty K_{ij}(s)\,ds \to 0$ as $t \to \infty$, $i,j = 1,2,\ldots,n$. It then follows from (2.23) that there exist $\delta_i, \eta_i, 0 < \delta_i < (1/2)\alpha_i, 0 < \eta_i < (1/2)\alpha_i$ and a $T^* \geq T_1^*$ such that for $t \geq T^*$

$$\begin{aligned} W(t) &\leq \sum_{i=1}^n \delta_i |x_i(t) - y_i(t)| + Z_1(t), \\ B(t) &\leq \sum_{i=1}^n \eta_i |x_i(t) - y_i(t)|, \end{aligned} \tag{2.24}$$

where

$$Z_1(t) = \sum_{i=1}^n c_i a_{ii}^M \sup_{-\infty < s \leq 0} |x_i(s) - y_i(s)| \int_t^\infty K_{ii}(s)\,ds.$$

Integrating both sides of (2.21) on the interval $[T^*, t]$,

$$\begin{aligned} V(t) + \sum_{i=1}^n \int_{T^*}^t A_i(s)|x_i(s) - y_i(s)|\,ds &\leq V(T^*) + \int_{T^*}^t W(s)\,ds \\ &+ \int_{T^*}^t B(s)\,ds, \quad \text{for } t \geq T^*. \end{aligned} \tag{2.25}$$

It follows from (2.22), (2.24) and (2.25) that

$$\begin{aligned} V(t) + \sum_{i=1}^n (\alpha_i - \delta_i - \eta_i) \int_{T^*}^t |x_i(s) - y_i(s)|\,ds &\leq V(T^*) \\ &+ \int_{T^*}^t Z_1(s)\,ds, \quad \text{for } t \geq T^*. \end{aligned}$$

Notice that for $t \geq T^*$

$$\begin{aligned} \int_{T^*}^t Z_1(s)\,ds &= \sum_{i=1}^n c_i a_{ii}^M \sup_{-\infty < s \leq 0} |x_i(s) - y_i(s)| \int_{T^*}^t \int_s^\infty K_{ii}(r)\,dr\,ds \\ &\leq \sum_{i=1}^n c_i a_{ii}^M \sup_{-\infty < s \leq 0} |x_i(s) - y_i(s)| \int_0^\infty sK_{ii}(s)\,ds. \end{aligned} \tag{2.27}$$

It therefore follows from (2.26) and (2.27) that for $t \geq T^*$

$$V(t) + \sum_{i=1}^{n} (\alpha_i - \delta_i - \eta_i) \int_{T^*}^{t} |x_i(s) - y_i(s)|\, ds \leq C, \tag{2.28}$$

for some constant $C > 0$. Therefore, $V(t)$ is bounded on $[T^*, \infty)$ and also

$$\int_{T^*}^{\infty} |x_i(s) - y_i(s)|\, ds < \infty, \quad i = 1, 2, \ldots, n. \tag{2.29}$$

By Lemma 2.1, $|x_i(t) - y_i(t)|$ is bounded on $[T^*, \infty), i = 1, 2, \ldots, n$.

On the other hand, it is easy to see that $\dot{x}_i(t)$ and $\dot{y}_i(t)$ $(i = 1, 2, \ldots, n)$ are bounded for $t \geq T^*$. Therefore, $|x_i(t) - y_i(t)|$ is uniformly continuous on $[T^*, \infty), i = 1, 2, \ldots, n$. By Barbalat's Lemma (see [4]), one can conclude that

$$\lim_{t \to \infty} |x_i(t) - y_i(t)| = 0, \quad i = 1, 2, \ldots, n. \tag{2.30}$$

□

Remark 1 By Theorem 2.1 and the notation in (1.4), one can get a set of easily verifiable sufficient conditions that guarantee the global asymptotic stability of the positive solution of system (1.1). The following result gives an estimate on the size of delays in terms of the coefficients and the upper bounds r_i^M, a_{ij}^M and M_i, $i, j = 1, 2, \ldots, n$.

COROLLARY 2.1 *Assume* $(H1)$–$(H3)$ *hold. Then the positive solution of system* (1.1) *with initial conditions* (1.3) *is globally asymptotically stable provided that*

(H7) *there exist* $c_i > 0$, $i = 1, 2, \ldots, n$, *such that the following hold:*

$$\liminf_{t \to \infty} \Big[c_1 a_{11}(t) - c_2 \int_0^{\infty} K_{21}(s) a_{21}(t+s)\, ds - c_1 a_{11}^M \sigma_{11} \big(r_1^M + a_{11}^M M_1 + a_{12}^M M_2 \big) - \sum_{k=1}^{2} c_k a_{kk}^M a_{k1}^M M_k \sigma_{kk} \Big] > 0;$$

$$\liminf_{t\to\infty}\Bigg[c_i a_{ii}(t) - c_{i-1}\int_0^\infty K_{i-1,i}(s)a_{i-1,i}(t+s)\,ds - c_{i+1} \times \int_0^\infty K_{i+1,i}(s)a_{i+1,i}(t+s)\,ds - c_i a_{ii}^M \sigma_{ii}\Bigg(r_i^M + \sum_{k=i-1}^{i+1} a_{ik}^M M_k\Bigg) - \sum_{k=i-1}^{i+1} c_k a_{kk}^M a_{ki}^M M_k \sigma_{kk}\Bigg] > 0,$$

$$i = 2, 3, \dots, n-1;$$

$$\liminf_{t\to\infty}\Bigg[c_n a_{nn}(t) - c_{n-1}\int_0^\infty K_{n-1,n}(s)a_{n-1,n}(t+s)\,ds - c_n a_{nn}^M \sigma_{nn}\Big(r_n^M + a_{n,n-1}^M M_{n-1} + a_{nn}^M M_n\Big) - \sum_{k=n-1}^{n} c_k a_{kk}^M a_{kn}^M M_k \sigma_{kk}\Bigg] > 0. \qquad \square$$

Remark 2 If $K_{ij}(s) \equiv \delta(s)$, the Kronecker delta function, $i, j = 1, 2, \dots, n$, then $\sigma_{ij} = 0$ and system (1.1) reduces to the instantaneous system i.e. one without delay

$$\begin{cases} \dot{x}_1 = x_1(t)(r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t)) \\ \dot{x}_i = x_i(t)(-r_i(t) + a_{i,i-1}(t)x_{i-1}(t) - a_{ii}(t)x_i(t) \\ \qquad -a_{i,i+1}(t)x_{i+1}(t)) \quad (i = 2, \dots, n-1) \\ \dot{x}_n = x_n(t)(-r_n(t) + a_{n,n-1}(t)x_{n-1}(t) - a_{nn}(t)x_n(t)). \end{cases} \tag{2.31}$$

On substituting $K_{ij}(s) = \delta(s)$ and $\sigma_{ij} = 0$ $(i, j = 1, 2, \dots, n)$ into (H7), Corollary 2.1 yields that the positive solution of system (2.31) is globally asymptotically stable provided that

(H8) $a_{n,n-1}^M M_{n-1} > r_n^L$,

where

$$M_1 = \frac{r_1^M}{a_{11}^L}, \qquad M_2 = \frac{a_{21}^M M_1 - r_2^L}{a_{22}^L},$$

$$M_i = \frac{a_{i,i-1}^M M_{i-1} - r_i^L}{a_{ii}^L}, \quad i = 3, 4, \dots, n-1.$$

(H9) There exist positive constants c_i $(i = 1, 2, \dots, n)$ such that

$$\liminf_{t\to\infty}[c_1 a_{11}(t) - c_2 a_{21}(t)] > 0,$$
$$\liminf_{t\to\infty}[c_i a_{ii}(t) - c_{i-1} a_{i-1,\,i}(t) - c_{i+1} a_{i+1,\,i}(t)] > 0,$$
$$\liminf_{t\to\infty}[c_n a_{nn}(t) - c_{n-1} a_{n-1,\,n}(t)] > 0.$$

We therefore have the following corollary to Corollary 2.1.

COROLLARY 2.2 *The positive solution of system* (1.1) *with initial conditions* (1.2) *is globally asymptotically stable if the positive solution of the corresponding instantaneous system* (*without time delay*) *is globally asymptotically stable and* σ_{ij} *is sufficiently small satisfying* (*H*7). □

In this paper, we have shown that *small delays are negligible* for the global asymptotic stability of the positive solution of the delayed nonautonomous n-species Lotka–Volterra food-chain systems provided that the delayed negative feedbacks dominate other interspecific interaction effects with delays. We note that Theorem 2.1 cannot be obtained from [2, Theorem 1].

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