Tensor Finite Element Methods for the Fractional Laplacian

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Outline of the talk

1. Fractional Laplacian and the Caffarelli-Silvestre extension
2. Analytic regularity of solutions
3. Finite element discretization
4. Numerical results

Outline

Fractional Laplacian

In the following $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, is a bounded, convex, polytopal domain.

Spectral Fractional Laplacian $(-\Delta)^s$

Let $\{\lambda_k, \varphi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^+ \times H_0^1(\Omega)$ be the eigenpairs of the Dirichlet Laplacian such that $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H_0^1(\Omega)$. Then for $w \in C_0^\infty(\Omega)$,

$$(-\Delta)^s w = \sum_{k=1}^{\infty} \lambda_k^s w_k \varphi_k, \quad w_k = \int_\Omega w\varphi_k, \quad k \in \mathbb{N}, \quad s \in (0,1).$$

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- By density $(-\Delta)^s : \mathbb{H}^s(\Omega) \to \mathbb{H}^{-s}(\Omega)$ can be extended to $\mathbb{H}^s(\Omega) = [L^2(\Omega), H_0^1(\Omega)]_s$, where $\mathbb{H}^{-s}(\Omega)$ is the dual space.
- An alternative *integral* formulation is of equal interest.
- The operator is *non-local*.
- Generalisation to $\mathcal{L}w = -\mathrm{div}(A\nabla w) + cw$ possible (see our 2017 arXiv paper).

Motivation

In, e.g., high intensity focused ultrasound for therapeutic surgery, acoustic attenuation typically exhibits a frequency dependency:

Plane wave solutions $u = e^{i(kx-\omega t)}$ satisfy $\operatorname{Im} k \approx \alpha_0 |\omega|^y, \qquad y \in (0,2).$

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- Main models use fractional in time derivatives, but fractional in space also of interest to reduce memory requirements.
- Many other sources of motivation: Anomalous diffusion processes in various areas such as electromagnetic fluids, ground-water solute transport, biology, finance, human travel and predator search patterns.

Problem

Devise an efficient numerical method to compute the solution u of

$$(-\Delta)^s u = f, \qquad \text{in } \Omega,$$

where $(-\Delta)^s$ is the spectral fractional Laplacian.

There has been a flurry of activity in recent years: Bonito-Pasciak; Nochetto-Otárola-Salgado; Ainsworth-Glusa etc.

How to compute $(-\Delta)^s$

Main difficulty is due to non-locality.

- In principle, can use the spectral definition, but in general this is very expensive.
- Use a Dunford-Taylor integral

$$(-\Delta)^s w = \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{C}} z^{-s} (z+\Delta)^{-1} w dz;$$

  see [Bonito, Pasciak '15; Bonito, Nochetto, Otárola, Pasciak, Salgado '17]
- Or (in this talk) to solve

$$(-\Delta)^s u = f, \qquad \text{on } \Omega,$$

  use the Caffarelli-Silvestre extension on the semi–infinite cylinder $\mathcal{C} := \Omega \times (0, \infty)$, which gives rise to a *local* boundary value problem.
    - Can we truncate $\mathcal{C}$?
    - Can the number of degrees of freedom be as if we are working in d-dimensions and not $d+1$?

The Caffarelli-Silvestre extension

Let $\mathscr{U}$ solve

$$\begin{cases} \mathfrak{L}\mathscr{U} = -\mathrm{div}\,(y^\alpha \nabla \mathscr{U}) = 0 & \text{in } \mathcal{C}, \quad x = (x', y) \in \mathbb{R}^{d+1}, \\ \mathscr{U} = 0 & \text{on } \partial_L \mathcal{C} = \partial\Omega \times (0, \infty), \\ \partial_{\nu^\alpha} \mathscr{U} = d_s f & \text{on } \Omega \times \{0\}, \end{cases}$$

where $d_s := 2^{1-2s}\Gamma(1-s)/\Gamma(s)$, $\alpha = 1 - 2s \in (-1, 1)$, and

$$\partial_{\nu^\alpha} \mathscr{U} = -\lim_{y \to 0^+} y^\alpha \mathscr{U}_y.$$

The fractional powers of $(-\Delta)$ and the Dirichlet-to-Neumann operator of the above problem are related by

$$d_s(-\Delta)^s u = \partial_{\nu^\alpha} \mathscr{U} \quad \text{in } \Omega$$

and hence

$$u = \lim_{y \to 0^+} \mathscr{U} = \mathrm{tr}_\Omega \mathscr{U}.$$

Weighted spaces

If $D \subset \mathbb{R}^{d+1}$, $L^2(y^\alpha, D)$ is the Lebesgue space for the measure $|y|^\alpha \, \mathrm{d}x$ and

$$H^1(y^\alpha, D) = \left\{ w \in L^2(y^\alpha, D) : |\nabla w| \in L^2(y^\alpha, D) \right\}.$$

with the norm

$$\|w\|_{H^1(y^\alpha, D)} = \left(\|w\|^2_{L^2(y^\alpha, D)} + \|\nabla w\|^2_{L^2(y^\alpha, D)} \right)^{\frac{1}{2}}.$$

Further define

$$\mathring{H}^1(y^\alpha, \mathcal{C}) = \left\{ w \in H^1(y^\alpha, \mathcal{C}) : w = 0 \text{ on } \partial_L \mathcal{C} \right\}.$$

We have the Poincaré inequality

$$\|w\|_{L^2(y^\alpha, \mathcal{C})} \lesssim \|\nabla w\|_{L^2(y^\alpha, \mathcal{C})} \quad \forall w \in \mathring{H}^1(y^\alpha, \mathcal{C})$$

and for $w \in H^1(y^\alpha, \mathcal{C})$, $\operatorname{tr}_\Omega w$ denotes its trace onto $\Omega \times \{0\}$

$$\operatorname{tr}_\Omega \mathring{H}^1(y^\alpha, \mathcal{C}) = \mathbb{H}^s(\Omega), \qquad \|\operatorname{tr}_\Omega w\|_{\mathbb{H}^s(\Omega)} \leq C_{\operatorname{tr}_\Omega} \|w\|_{\mathring{H}^1(y^\alpha, \mathcal{C})}.$$

The extension problem

Define the bilinear form $a_{\mathcal{C}} : \mathring{H}^1(y^\alpha, \mathcal{C}) \times \mathring{H}^1(y^\alpha, \mathcal{C}) \to \mathbb{R}$ by

$$a_{\mathcal{C}}(v, w) = \int_{\mathcal{C}} y^\alpha \nabla v \cdot \nabla w \, \mathrm{d}x' \, \mathrm{d}y, \qquad x = (x', y) \in \mathbb{R}^{d+1}$$

and note that it is continuous and also coercive.

Weak formulation [Caffarelli,Silvestre '07, Stinga, Torrea '10]

Given $f \in \mathbb{H}^{-s}(\Omega)$ Let $u \in \mathbb{H}^s(\Omega)$ solve

$$(-\Delta)^s u = f.$$

If $\mathscr{U} \in \mathring{H}^1(y^\alpha, \mathcal{C})$ solves

$$a_{\mathcal{C}}(\mathscr{U}, v) = d_s \langle f, \operatorname{tr}_\Omega v \rangle \qquad \forall v \in \mathring{H}^1(y^\alpha, \mathcal{C}),$$

then

$$u = \operatorname{tr}_\Omega \mathscr{U}$$

.

Truncated problem

Let $\mathcal{C}_{\mathcal{Y}}$ denote the truncated cylinder $\mathcal{C}_{\mathcal{Y}} = \Omega \times (0, \mathcal{Y})$ and

$$a_{\mathcal{C}_{\mathcal{Y}}}(v, w) = \int_{\mathcal{C}_{Y}} y^{\alpha} \nabla v \cdot \nabla w \, \mathrm{d}x' \, \mathrm{d}y.$$

Let $\mathcal{U}$ be the solution of the corresponding weak formulation with homogeneous Dirichlet boundary condition on
Then ([Nochetto, Otárola, Salgado '15])

$$\|\nabla(\mathscr{U} - \mathcal{U})\|_{L^2(y^{\alpha}, \mathcal{C})} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/4} \|f\|_{\mathbb{H}^{-s}(\Omega)}.$$

Outline

y-dependence of $\mathscr{U}$

The unique solution $\mathscr{U}$ admits the representation [Nochetto et al. '15]

$$\mathscr{U}(x', y) = \sum_{k=1}^{\infty} u_k \varphi_k(x') \psi_k(y), \quad u_k := \lambda_k^{-s} f_k.$$

The functions ψ_k solve

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}y^2}\psi_k(y) + \frac{\alpha}{y}\frac{\mathrm{d}}{\mathrm{d}y}\psi_k(y) - \lambda_k \psi_k(y) = 0, & y \in (0,\infty), \\ \psi_k(0) = 1, & \lim_{y\to\infty} \psi_k(y) = 0. \end{cases}$$

Thus, if $s = \frac{1}{2}$, we have $\psi_k(y) = \exp(-\sqrt{\lambda_k} y)$ and if $s \in (0,1) \setminus \{\frac{1}{2}\}$, then

$$\psi_k(y) = c_s(\sqrt{\lambda_k} y)^s K_s(\sqrt{\lambda_k} y), \qquad c_s = 2^{1-s}/\Gamma(s).$$

Note:

$$\lim_{z\downarrow 0} \frac{K_\nu(z)}{\frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}z\right)^{-\nu}} = 1 \quad \text{and} \quad \lim_{z\to\infty} K_\nu(z)\sqrt{z}e^z = \sqrt{\frac{\pi}{2}}.$$

Global regularity of $\mathscr{U}$

Let

$$\omega_{\beta,\gamma}(y) = y^{\beta} e^{\gamma y}, \qquad 0 \le \gamma < 2\sqrt{\lambda_1},$$

and

$$\|v\|_{L^2(\omega_{\beta,\gamma},\mathcal{C})} := \left(\int_0^\infty \int_\Omega \omega_{\beta,\gamma}(y) |v(x',y)|^2 \,\mathrm{d}x' \,\mathrm{d}y \right)^{\frac{1}{2}}.$$

Theorem

Let $\mathscr{U} \in \mathring{H}^1(y^\alpha, \mathcal{C})$ be the solution of the extension problem and let $0 \le \tilde{\nu} < s$ and $0 \le \nu < 1 + s$. Then there exists $\kappa > 1$ such that

$$\|\partial_y^{\ell+1} \mathscr{U}\|_{L^2(\omega_{\alpha+2\ell-2\tilde{\nu},\gamma},\mathcal{C})} \lesssim \kappa^{\ell+1} (\ell+1)! \|f\|_{\mathbb{H}^{-s+\tilde{\nu}}(\Omega)},$$
$$\|\nabla_{x'} \partial_y^{\ell+1} \mathscr{U}\|_{L^2(\omega_{\alpha+2(\ell+1)-2\nu,\gamma},\mathcal{C})} \lesssim \kappa^{\ell+1} (\ell+1)! \|f\|_{\mathbb{H}^{-s+\nu}(\Omega)},$$
$$\|\Delta_{x'} \partial_y^{\ell+1} \mathscr{U}\|_{L^2(\omega_{\alpha+2(\ell+1)-2\nu,\gamma},\mathcal{C})} \lesssim \kappa^{\ell+1} (\ell+1)! \|f\|_{\mathbb{H}^{1-s+\nu}(\Omega)}.$$

Outline

Finite element space

Let

$$\mathcal{G}^M = \{I_m\}_{m=1}^M \text{ in } [0, \mathcal{Y}] \qquad I_m = [y_{m-1}, y_m], y_0 = 0 \text{ and } y_M = \mathcal{Y}$$

and $\boldsymbol{r} = (r_1, r_2, \ldots, r_M) \in \mathbb{N}^M$.
Then the finite element space is

$$S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}), \mathcal{G}^M) = \left\{ v \in C[0,\mathcal{Y}] : v(\mathcal{Y}) = 0, v|_{I_m} \in \mathbb{P}_{r_m}(I_m), I_m \in \mathcal{G}^M \right\}.$$

In Ω, we consider Lagrangian FEM of polynomial degree $q \geq 1$ based on shape-regular, simplicial triangulations $\mathcal{T}$:

$$S^q_0(\Omega, \mathcal{T}) = \left\{ v_h \in C(\bar{\Omega}) : v_h|_K \in \mathbb{P}_q(K) \quad \forall K \in \mathcal{T},\ v_h|_{\partial\Omega} = 0 \right\}.$$

Finally we introduce the *tensor product space*

$$\mathbb{V}^{q,\boldsymbol{r}}_{h,M}(\mathcal{T}, \mathcal{G}^M) := S^q_0(\Omega, \mathcal{T}) \otimes S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}), \mathcal{G}^M) \subset \mathring{H}^1(y^\alpha, \mathcal{C}).$$

Finite element error

Let the discrete solution $\mathscr{U}_{h,M} = \mathscr{U} \in \mathbb{V}_{h,M}$ satisfy

$$a_{\mathcal{C}_\mathcal{Y}}(\mathscr{U}_{h,M}, \phi) = d_s \langle f, \operatorname{tr}_\Omega \phi \rangle \quad \forall \phi \in \mathbb{V}_{h,M} \ .$$

Lemma (Céa and truncation)

We have

$$\|\nabla(\mathscr{U} - \mathscr{U}_{h,M})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim \min_{v_{h,M} \in \mathbb{V}_{h,M}} \|\nabla(\mathscr{U} - v_{h,M})\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} + \|\nabla \mathscr{U}\|_{L^2(y^\alpha,\mathcal{C}\setminus\mathcal{C}_\mathcal{Y})}.$$

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On regular, simplicial triangulations of Ω *let the quasi-interpolation operator* $\Pi^q_{x'}$ *be uniformly stable on* $L^2(\Omega)$ *and* $H^1(\Omega)$ *and* $\pi^{\mathbf{r}}_y : H^1(y^\alpha, (0,\mathscr{Y})) \to S^{\mathbf{r}}_{\{\mathscr{Y}\}}((0,\mathscr{Y}), \mathcal{G}^M)$ *be a linear projector. Then*

$$\min_{v_{h,M} \in \mathbb{V}_{h,M}} \|\nabla(\mathscr{U} - v_{h,M})\|_{L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}})} \lesssim \|\nabla(\mathscr{U} - \Pi^q_{x'}\mathscr{U})\|_{L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}})} + \|\nabla(\mathscr{U} - \pi^{\mathbf{r}}_y \mathscr{U})\|_{L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}})} \; ,$$

Geometric meshes and *hp*-FEM

Consider geometric meshes $\mathcal{G}^M_{geo,\sigma}$ on $[0, \mathcal{Y}]$ with $\sigma \in (0,1)$ and

- $I_1 = [0, \mathcal{Y}\sigma^{M-1}]$, $I_i = [\mathcal{Y}\sigma^{M-i+1}, \mathcal{Y}\sigma^{M-i}]$ for $i = 2, \ldots, M$
- a *linear degree vector* $\boldsymbol{r}$ with slope $\mathfrak{s}$

$$r_i := \max\{1, \lceil \mathfrak{s} i \rceil\}\ , \quad i = 1, 2, ..., M.$$

- Note that the corresponding 1D FEM space has $O(M^2)$ degrees of freedom.
- This leads to exponential convergence for analytic functions that may have a singularity at $y = 0$.
- The construction is essentially taken from the work by Babuška and collaborators.
- Note that, Nochetto et al. used graded meshes towards $y = 0$ with P_1-FEM.
- Recently, Meidner, Pfefferer, Schrholz, and Vexler, '17, also used *hp*-FEM in y.

Error estimate

Consider the finite element space $\mathbb{V}^{1,\boldsymbol{r}}_{h,M}(\mathcal{T}^\ell, \mathcal{G}^M_{geo,\sigma})$, with the geometric *hp*-FEM in y-direction and a P_1 FEM on a sequence of shape-regular, simplicial triangulations $\mathcal{T}^\ell$ with mesh-width h_ℓ.

Theorem

Let $u \in \mathbb{H}^s(\Omega)$ and $\mathscr{U} \in \mathring{H}^1(y^\alpha, \mathcal{C})$ be solutions of the problems with with $f \in \mathbb{H}^{1-s}(\Omega)$. Let $M \sim |\log h_\ell|$, $\mathscr{Y} \sim |\log h_\ell|$ and $\mathscr{U}_{h_\ell,M} \in \mathbb{V}^{1,\boldsymbol{r}}_{h,M}(\mathcal{T}^\ell, \mathcal{G}^M_{geo,\sigma})$ be the discrete solution. Then there exists a minimal $\mathfrak{s}_{min}$ such that

$$\|u - \operatorname{tr}_\Omega \mathscr{U}_{h,M}\|_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla(\mathscr{U} - \mathscr{U}_{h_\ell,M})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim h_\ell \|f\|_{\mathbb{H}^{1-s}(\Omega)}.$$

The total number of degrees of freedom behaves like

$$\dim \mathbb{V}^{1,\boldsymbol{r}}_{h,M}(\mathcal{T}^\ell, \mathcal{G}^M_{geo,\sigma}) \sim \mathcal{N}_{\Omega,\mathscr{Y}} \sim M^2 h_\ell^{-2} \sim h_\ell^{-2}(\log h_\ell)^2 \sim \mathcal{N}_\Omega \log \mathcal{N}_\Omega,$$

where $\mathcal{N}_\Omega = \#\mathcal{T}^\ell$.

A y-semidiscrete eigenvalue decomposition

An eigenvalue problem

Find $(v,\mu) \in S^{\mathbf{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^M) \setminus \{0\} \times \mathbb{R}$ such that

$$\mu \int_0^{\mathcal{Y}} y^\alpha v'(y) w'(y) \,\mathrm{d}y = \int_0^{\mathcal{Y}} y^\alpha v(y) w(y) \,\mathrm{d}y \qquad \forall w \in S^{\mathbf{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^M).$$

- All μ are positive, and $S^{\mathbf{r}}_{\{\mathcal{Y}\}}((0,\mathcal{Y}),\mathcal{G}^M)$ has eigenbasis $(v_i)_{i=1}^{\mathcal{M}}$ such that,

$$\int_0^{\mathcal{Y}} y^\alpha v_i'(y) v_j'(y) \,\mathrm{d}y = \delta_{i,j}, \qquad \int_0^{\mathcal{Y}} y^\alpha v_i(y) v_j(y) \,\mathrm{d}y = \mu_i \delta_{i,j}.$$

- If $\mathcal{G}^M = \mathcal{G}^M_{geo,\sigma}$ and $c_1 M \le \mathcal{Y} \le c_2 M$, then there are constants C, b depending only on σ such that

$$\|v_i\|_{L^\infty(0,\mathcal{Y})} \le C M^{(1-\alpha)/2}, \qquad C^{-1} \mathfrak{s}^{-2} M^{-1} \sigma^M \le \mu_i \le C M^2.$$

Diagonalization and y-semidiscretization

y-semidiscrete problem

Find $\mathscr{U}_M \in \mathbb{V}^{\mathbf{r}}_M(\mathcal{C}_{\mathscr{Y}}) = H^1_0(\Omega) \otimes S^{\mathbf{r}}_{\{\mathscr{Y}\}}((0,\mathscr{Y}),\mathcal{G}^M)$ such that

$$a_{\mathcal{C}}(\mathscr{U}_M,\phi) = d_s\langle f, \operatorname{tr}_\Omega \phi\rangle \qquad \forall \phi \in \mathbb{V}^{\mathbf{r}}_M(\mathcal{C}_{\mathscr{Y}}).$$

- Write $\mathscr{U}_M(x',y) := \sum_{j=1}^{\mathcal{M}} U_j(x')v_j(y)$.
- Consider $\phi(x',y) = V(x')v_i(y)$, with $V \in H^1_0(\Omega)$ as a test function.
- This results in *decoupled* problems

$$\mu_i \int_\Omega (\nabla U_i, \nabla V) + \int_\Omega U_i V \,\mathrm{d}x' = d_s v_i(0)\langle f, V\rangle \qquad \forall V \in H^1_0(\Omega).$$

Importance of diagonalization

- The diagonalization shows that upto exponentially small error the solution $\mathscr{U}$ can be written as a sum of singularly perturbed problems.
- It can also be used in a fully discrete setting.
- One option is to discretize each singularly perturbed problem using an optimised FEM in Ω.
- We choose to use the same FEM in Ω for all the M problems:
  - We arrive at M decoupled linear systems with the same mass and stiffness matrices that can be solved in parallel.
  - Robust multigrid methods are available.
  - In this case the computational cost is (almost) optimal:

$$\text{computational cost} = O(M^3) + O(Mh^{-d}) = O(\mathcal{N}_\Omega \log \mathcal{N}_\Omega)$$

for the discretization $\mathbb{V}^{1,\boldsymbol{r}}_{h,M}(\mathcal{T}^\ell, \mathcal{G}^M_{geo,\sigma})$ with $M \sim \mathscr{Y} \sim \log h^{-1}$.

Outline

Test cases

We let Ω be the L-shape domain in 2D with vertices

$$\{(0,0),(1,0),(1,1),(-1,1),(-1,-1),(0,-1)\}.$$

We will consider two test cases

1. The following smooth exact solution:

$$u(x_1,x_2) = \sin \pi x_1 \sin \pi x_2, \quad f(x_1,x_2) = (2\pi^2)^s \sin \pi x_1 \sin \pi x_2.$$

2. Further we also consider the solution with the right-hand side

$$f(x_1,x_2) \equiv 1 \,.$$

   Notice that, in this case, f is analytic in $\overline{\Omega}$ but $f \in \mathbb{H}^{1-s}(\Omega)$ only for $s > 1/2$.

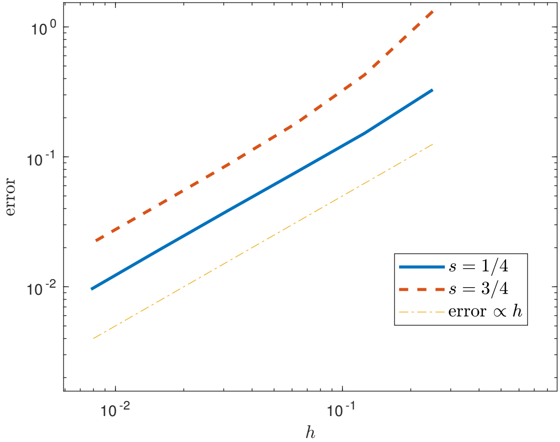
Implementation

- We use Netgen/NGSolve for the FEM in Ω.
- The *hp*-FEM in y implemented separately.
- The error measure will always be the energy norm

$$\|u - \mathrm{tr}_\Omega \mathscr{U}_{h,M}\|^2_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla(\mathscr{U} - \mathscr{U}_{h,M})\|^2_{L^2(y^\alpha,\mathcal{C})} = d_s \int_\Omega f(u - \mathrm{tr}_\Omega \mathscr{U}_{h,M}),$$

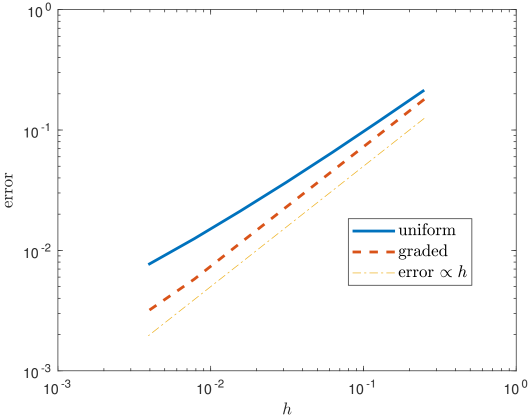
where $\mathscr{U}_{h,M}$ denotes the discrete solution in $\mathcal{C}_\mathcal{Y}$.

Smooth solution



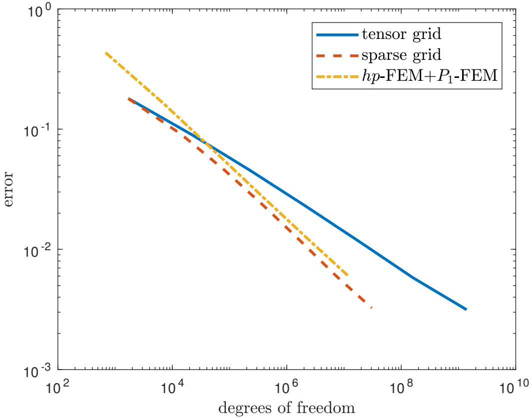
Convergence of the error in the energy norm versus the meshwidth in Ω for the smooths solution for two different values of s. A P_1-FEM on uniformly refined meshes in Ω and *hp*-FEM in $(0, \mathcal{Y})$ is used.

Non-smooth solution, $f \equiv 1$



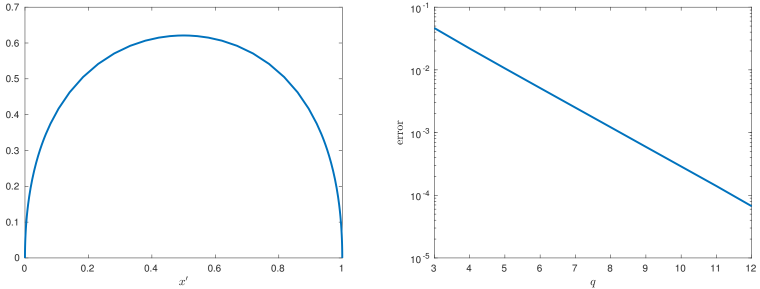
- Here $f \equiv 1$ and $s = 3/4$, leading to a solution with singular behavior near the re-entrant corner $(0, 0)$. Error graphs are shown for a P_1-FEM on uniformly refined meshes in Ω and on meshes refined towards the corner.
- This case also analyzed in our arXiv '17 paper.

Convergence against number of degrees of freedom



Convergence of the error versus the number of degrees of freedom with $f \equiv 1$ and $s = 3/4$. We compare hp-FEM in $(0, \mathcal{Y})$ with tensor grid and sparse grids, the latter two employing radical meshes and P_1-FEM in $(0, \mathcal{Y})$.

hp-FEM in $\Omega \times (0, \mathcal{Y})$ in 1D



Solution on $\Omega = (0,1)$ with algebraic boundary singularity. Convergence of error in energy norm of the hp-FEM on $\Omega \times (0, \mathcal{Y})$ against polynomial order q for $s = 0.25$ and $f \equiv 1$.

Conclusions

- We have developed and analyzed an almost optimal complexity algorithm for the (spectral) fractional Laplacian using *hp*-FEM in the extended and P_1-FEM in smooth Ω.
- For polygons we have proved that first order convergence is obtained if refinement towards corners is used and $f \in \mathbb{H}^{1-s}(\Omega)$.
- A sparse tensor product FEM based on multilevel P_1-FEM in Ω and P_1-FEM on radical meshes in y also achieves (almost) optimal complexity.
- Finally, we prove that if the data f is analytic in $\overline{\Omega}$, but not compatible, *hp*-FEM in full domain with anisotropic geometric meshes towards Ω result in exponential rates of convergence. Here Ω is smooth in 1D or 2D.

Some of this we touched upon in the talk, the details are in 2017, arXiv:1707.07367