Persistence and attractivity in an $N$-species ratio-dependent predator–prey system with distributed time delays

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Abstract

An $N$-species ratio-dependent predator–prey food-chain model with distributed time delays is investigated. It is shown that the system is permanent under some appropriate conditions, and sufficient conditions are obtained for the global attractivity of the positive equilibrium of the system. © 2002 Elsevier Science Inc. All rights reserved.

Keywords: Distributed time delay; Uniform persistence; Global attractivity

1. Introduction

One of the most important and interesting topics in mathematical ecology concerns the survival of species in ecological systems. Continuous models, usually in the form of differential equations, have formed a large part of the traditional mathematical ecology literature. Recently, there is growing explicit biological and physiological evidence [1–4] that in many situations, especially when predators have to search for food (and therefore have to share or compete for food), a more suitable general predator–prey theory should be based on the so-called ratio-dependent theory, which can be roughly stated as that the per capita predator growth rate should be a function of the ratio of prey to...
predator abundance, and so should be the so-called predator functional response. This is strongly supported by numerous field and laboratory experiments and observations [2,3,5,6]. Specifically, the standard Lotka–Volterra type models, on which nearly all existing theory is built, assumes that the per capita rate of predation depends on the prey numbers only. An alternative assumption is that, as the number of predators change slowly (relative to prey change), there is often competition among the predators, and the per capita rate of predation depends on the numbers of both prey and predator, most likely and simply on their ratio. Generally, a ratio-dependent predator–prey model takes the form

\[
\begin{align*}
\dot{x} &= xf(x) - yp(x/y), \\
\dot{y} &= cyq(x/y) - dy.
\end{align*}
\] (1.1)

Here \(p(x)\) is so-called predator functional response. \(p(x), q(x)\) satisfy the usual properties such as being nonnegative and increasing, and equal to zero at zero.

Since the traditional Lotka–Volterra type predator–prey model with Michaelis–Menten type functional response received great attention among theoretical and mathematical biologists, we will focus our attention here on the ratio-dependent type predator–prey model with Michaelis–Menten type functional response, which takes the form

\[
\begin{align*}
\dot{x} &= ax(1 - x/K) - cxy/(my + x), \\
\dot{y} &= y(-d + fx/(my + x)).
\end{align*}
\] (1.2)

From a formal point of view, this model looks very similar to the much better known and applied Michaelis–Menten–Holling predator–prey model

\[
\begin{align*}
\dot{x} &= ax(1 - x/K) - cxy/(b + x), \\
\dot{y} &= y(-d + fx/(b + x)).
\end{align*}
\] (1.3)

Indeed, the only difference between models (1.2) and (1.3) is that the parameter \(b\) in (1.3) is replaced by \(my\) in (1.2). Such a term \((my\ or \ b)\) is proportional to the so-called searching time of the predator, namely, the time spent by each predator to find one prey. Thus, in the Michaelis–Menten–Holling model the searching time is assumed to be independent of predator density, while in the ratio-dependent Michaelis–Menten type model, the searching time is proportional to predator density (i.e., predators strongly interfere).

On the other hand, it is well known that time delays in ecological system can have a considerable influence on the qualitative behavior of these systems. It is generally recognized that some kinds of time delays are inevitable in population interactions and tend to be destabilizing in the sense that longer delays may destroy the stability of positive equilibrium (see [7,8] and the references cited therein). Time delay due to gestation is among them, because generally the consumption of prey by the predator throughout its past history governs
the present birth rate of the predator. The effect of this kind of delay on the asymptotic behavior of populations has been studied by a number of authors (see for example [9–12]).

In this paper, we incorporate distributed time delay due to gestation into the ratio-dependent predator–prey system. For the $N$-species ratio-dependent predator–prey food-chain model with Michaelis–Menten type functional response, this results in the following delayed system

$$
\begin{align*}
\dot{x}_1(t) &= x_1(t) \left( a_1 - a_{11}x_1(t) - \frac{a_{12}x_2(t)}{m_{12}x_2(t) + x_1(t)} \right), \\
\dot{x}_2(t) &= x_2(t) \left( -a_2 + a_{21} \int_0^\infty K_1(s) \frac{x_1(t-s)}{m_{12}x_2(t-s) + x_1(t-s)} \, ds \right. \\
&\quad \left. - \frac{a_{23}x_3(t)}{m_{23}x_3(t) + x_2(t)} \right), \\
\vdots \\
\dot{x}_i(t) &= x_i(t) \left( -a_i + a_{i,i-1} \int_0^\infty K_{i-1}(s) \frac{x_{i-1}(t-s)}{m_{i-1,i-1}x_i(t-s) + x_{i-1}(t-s)} \, ds \right. \\
&\quad \left. - \frac{a_{i,i+1}x_{i+1}(t)}{m_{i,i+1}x_{i+1}(t) + x_i(t)} \right), \\
\vdots \\
\dot{x}_n(t) &= x_n(t) \left( -a_n + a_{n,n-1} \int_0^\infty K_{n-1}(s) \frac{x_{n-1}(t-s)}{m_{n-1,n}x_n(t-s) + x_{n-1}(t-s)} \, ds \right) 
\end{align*}
$$

with initial conditions

$$
\begin{equation}
\begin{aligned}
x_i(s) &= \phi_i(s), \quad s \in (-\infty, 0], \quad \phi_i(0) > 0 \ (i = 1, 2, \ldots, n), \\
\end{aligned}
\end{equation}
$$

where $x_i(t)$ represents the density of the $i$th population, respectively, $i = 1, 2, \ldots, n$. $\phi_i$ ($i = 1, 2, \ldots, n$) are continuous bounded functions; $a_i$, $a_{ij}$ ($i = 1, 2, \ldots, n$) and $m_{i,i+1}$ ($1, 2, \ldots, n-1$) are positive constants. $K_i(t): [0, \infty) \mapsto [0, +\infty)$ is piecewise continuous and normalized function such that

$$
\begin{equation}
\begin{aligned}
\int_0^\infty K_i(s) \, ds &= 1; \quad \sigma_i = \int_0^\infty sK_i(s) \, ds \leq \infty, \quad i = 1, 2, \ldots, n-1.
\end{aligned}
\end{equation}
$$

We adopt the following notations and concepts throughout this paper.

Let $R^n_+ = \{x \in R^n : x_i \geq 0, \ i = 1, 2, \ldots, n\}$. For ecological reasons, we consider system (1.4), only in $\text{Int} R^n_+$. 


Definition 1.1. System (1.4) is said to be uniformly persistent if there exists a compact region $D \subset \text{Int} \mathbb{R}^n_+$ such that every solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ of system (1.4) with initial conditions (1.5) eventually enters and remains in the region $D$.

The organization of this paper is as follows. In Section 2, we present permanence results for system (1.4). Section 3 provides sufficient conditions for the positive equilibrium of system (1.4) to be globally attractive. Finally, a suitable example is given to illustrate that the conditions of our theorems are feasible.

2. Uniform persistence

System (1.4) has a unique positive equilibrium if and only if the following conditions are true:

(H1) $a_{n,n-1} > a_n$;

(H2) $a_{n-1,n-2} > k_{n-1}$, where $k_{n-1} = a_{n-1} + \frac{a_{n-1,n} (a_{n,n-1} - a_n)}{a_{n,n-1} m_{n-1,n}}$;

(H3) $a_{i,i-1} > k_i$, where $k_i = a_i + \frac{a_{i,i+1} (a_{i+1,i} - k_{i+1})}{a_{i+1,i} m_{i+1,i}}$, $i = 2, 3, \ldots, n - 2$;

(H4) $a_1 > \frac{a_{12}}{a_{21}m_{12}}(a_{21} - k_2)$.

In the following, we always assume that such a positive equilibrium exists and denote it by $E^*(x_1^*, x_2^*, \ldots, x_n^*)$.

The following lemmas are elementary and are concerned with the qualitative nature of solutions of system (1.4).

Lemma 2.1. Solutions of system (1.4) corresponding to initial conditions (1.5) are defined on $[0, +\infty)$ and remain positive for all $t \geq 0$.

The proof of Lemma 2.1 is similar to that of Lemma 2.1 in Ref. [11], we omit it.

Lemma 2.2. Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ denote any positive solution of system (1.4) with initial conditions (1.5). Suppose that system (1.4) satisfies (H1)–(H3) and the following:

(H5) $\int_0^\infty K_i(s)e^{(a_{i+1,i} - a_{i+1})s} ds < \infty$, $i = 1, 2, \ldots, n - 1$.

Then there exists a $T > 0$ such that
where

\[ M_i > M_i^*, \quad i = 1, 2, \ldots, n; \]

\[ M_i^* = \frac{a_1}{a_{i1}}, \]

\[ M_i^* = \frac{M_{i-1}}{m_{i-1,a_i}} \left( a_{i,i-1} \int_0^\infty K_{i-1}(s)e^{(a_{i,j-1}-a_s)s}ds - a_i \right), \quad i = 2, 3, \ldots, n. \]  

(2.2)

**Proof.** It follows from the positivity of the solution of (1.4) that

\[ \dot{x}_1(t) \leq x_1(t)(a_1 - a_{11}x_1(t)). \]

A standard comparison argument shows that

\[ \limsup_{t \to +\infty} x_1(t) \leq \frac{a_1}{a_{11}} \equiv M_1^*. \]

Hence, there exists a \( T_1 > 0 \), if \( t \geq T_1 \), we have \( x_1(t) \leq M_1 > M_1^* \).

In addition, from the second equation of system (1.4), we can see that

\[ \dot{x}_2(t) \leq (a_{21} - a_2)x_2(t), \]

which implies that

\[ \ln \frac{x_2(t)}{x_2(t-s)} \leq (a_{21} - a_2)s \quad \text{for} \quad t \geq s \geq 0. \]

Thus, we have

\[ x_2(t-s) \geq x_2(t)e^{-(a_{21} - a_2)s} \quad \text{for} \quad t \geq s \geq 0. \]  

(2.3)

Notice that

\[ \lim_{t \to \infty} \int_{t-T_1}^\infty K_1(s) \frac{x_1(t-s)}{m_{12}x_2(t-s) + x_1(t-s)}ds = 0, \]

therefore, for any \( \varepsilon, 0 < \varepsilon < a_2 \), there exists a \( T_2 \geq T_1 \) such that for \( t \geq T_2 \)

\[ -\varepsilon < a_{21} \int_{t-T_1}^\infty K_1(s) \frac{x_1(t-s)}{m_{12}x_2(t-s) + x_1(t-s)}ds < \varepsilon. \]  

(2.4)
By (2.3), (2.4) and the second equation of (1.4), for all $t \geq T_2$, we obtain
\[
\begin{aligned}
\dot{x}_2(t) &\leq x_2(t) \left[ -a_2 + a_{21} \int_{t-T_1}^{t} K_1(s) \frac{x_1(t-s)}{m_{12}x_2(t-s) + x_1(t-s)} \, ds \\
&+ a_{21} \int_{t-T_1}^{\infty} K_1(s) \frac{x_1(t-s)}{m_{12}x_2(t-s) + x_1(t-s)} \, ds \right] \\
&\leq x_2(t) \left[ -a_2 + \varepsilon + a_{21} \int_{t-T_1}^{\infty} K_1(s) \frac{M_1}{m_{12}e^{(a_{21}-a_2)s}x_2(t) + M_1} \, ds \right] \\
&\leq x_2(t) \left[ -a_2 + \varepsilon + a_{21} \int_{t-T_1}^{\infty} K_1(s) \frac{M_1}{m_{12}e^{(a_{21}-a_2)s}x_2(t) + M_1} \, ds \right] \\
&\leq x_2(t) \left[ -a_2 + \varepsilon + a_{21} \int_{t-T_1}^{\infty} K_1(s) \frac{M_1e^{(a_{21}-a_2)s}}{m_{12}x_2(t) + M_1} \, ds \right] \\
&\leq \frac{x_2(t)}{m_{12}x_2(t) + M_1} \left[ M_1 \left( a_{21} \int_{t-T_1}^{\infty} K_1(s)e^{(a_{21}-a_2)s} \, ds - a_2 + \varepsilon \right) - m_{12}(a_2 - \varepsilon)x_2(t) \right]
\end{aligned}
\]

A standard comparison argument shows that
\[
\limsup_{t \to +\infty} x_2(t) \leq \frac{M_1}{m_{12}(a_2 - \varepsilon)} \left( a_{21} \int_{0}^{\infty} K_1(s)e^{(a_{21}-a_2)s} \, ds - a_2 + \varepsilon \right).
\]

Now let $\varepsilon \to 0$. Consequently, we have
\[
\limsup_{t \to +\infty} x_2(t) \leq \frac{M_1}{m_{12}a_2} \left( a_{21} \int_{0}^{\infty} K_1(s)e^{(a_{21}-a_2)s} \, ds - a_2 \right) \equiv M_2^*.
\]

Therefore, there exists a $T_2^* \geq T_2$ and an $M_2 > M_2^*$, if $t \geq T_2^*$, we have $x_2(t) \leq M_2$.

Similarly, under conditions (H1)--(H3) and (H5), one can prove that
\[
\limsup_{t \to +\infty} x_i(t) \leq \frac{M_{i-1}}{m_{i-1}a_i} \left( a_{i-1} \int_{0}^{\infty} K_{i-1}(s)e^{(a_{i-1}-a_i)s} \, ds - a_i \right) \equiv M_i^*, \quad i = 3, 4, \ldots, n.
\]

Hence, there exists a $T > T_2$, if $t \geq T$, we have $x_i(t) \leq M_i$ for some $M_i > M_i^*$, $i = 3, 4, \ldots, n$. The proof is complete. $\square$

**Theorem 2.1.** Suppose that system (1.4) satisfies (H5) and the following:

(H6) $a_1 > \frac{a_{12}}{m_{12}}$;

(H7) $a_{i,j-1} \int_{0}^{\infty} K_{i-1}(s) \exp \left\{ - \left( a_i + a_{j,i+1} \frac{a_{j,i+1}}{m_{i,j+1}} \right) s \right\} ds > \frac{a_i + a_{i,j+1}}{m_{i,j+1}}$,
\[
i = 2, 3, \ldots, n - 1;
\]

(H8) $a_{n,n-1} \int_{0}^{\infty} K_{n-1}(s) \exp \{-a_n s\} ds > a_n$.

Then system (1.4) is uniformly persistent.
Proof. Suppose \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) is a solution of system (1.4) which satisfies (1.5). According to the first equation of system (1.4), if (H6) holds, then

\[
\dot{x}_1(t) > x_1(t) \left[ a_1 - \frac{a_{12}}{m_{12}} - a_{11}x_1(t) \right],
\]

which implies that \( \liminf_{t \to +\infty} x_1(t) \geq (a_1 - (a_{12}/m_{12}))/a_{11} \equiv m_1 \). Hence, for large \( t \), \( x_1(t) > m_1/2 \)

It follows from (H7) and (H8) that there exists an \( A > 0 \) such that

\[
a_{i,i-1} \int_0^A K_{i-1}(s) e^{-(a_i + \frac{a_{i+1}}{m_{i+1}})s} ds > a_i + \frac{a_{i+1}}{m_{i+1}}, \quad i = 2, 3, \ldots, n - 1,
\]

\[
a_{n,n-1} \int_0^A K_{n-1}(s) e^{-a_n s} ds > a_n.
\]

Therefore, from the second equation in system (1.4), we have that for \( t \geq T^*_1 + A \)

\[
\dot{x}_2(t) > x_2(t) \left[ -a_2 - \frac{a_{23}}{m_{23}} + a_{21} \int_0^A K_1(s) \frac{m_1/2}{m_{12}x_2(t) + m_1/2} ds \right].
\]

From this it follows (in a similar manner to the derivation of (2.3) above) that

\[
x_2(t-s) \leq x_2(t) e^{\left( a_2 + \frac{a_{23}}{m_{23}} \right) s} \quad \text{for} \quad t \geq s \geq 0.
\]

Hence, for \( t \) sufficiently large, we have

\[
\dot{x}_2(t) > x_2(t) \left[ -a_2 - \frac{a_{23}}{m_{23}} + a_{21} \int_0^A K_1(s) \frac{m_1/2}{m_{12}x_2(t) + m_1/2} ds \right] \geq x_2(t) \left[ -a_2 - \frac{a_{23}}{m_{23}} + a_{21} \int_0^A K_1(s) \frac{m_1/2}{m_{12}x_2(t) + m_1/2} ds \right] = \frac{x_2(t)}{m_{12}x_2(t) + m_1/2} \left[ \frac{1}{2} m_1 \left( a_{21} \int_0^A K_1(s) \exp \left\{- \left( a_2 + \frac{a_{23}}{m_{23}} \right) s \right\} ds \right. \right. \]

\[
- \left. a_2 - \frac{a_{23}}{m_{23}} \right) - m_{12} \left( a_2 + \frac{a_{23}}{m_{23}} \right) x_2(t) \right],
\]

(2.6)
which yields

$$\liminf_{t \to +\infty} x_2(t) \geq \frac{m_1}{2m_1 \left( a_2 + \frac{a_{23}}{m_{23}} \right)} \left( a_{21} \int_0^A K_1(s) \exp \left\{ - \left( a_2 + \frac{a_{23}}{m_{23}} \right) s \right\} ds - a_2 - \frac{a_{23}}{m_{23}} \right) \equiv m_2.$$  

Therefore, for large $t$, we have $x_2(t) > m_2/2$.

Similarly, under assumption (H7), we can prove

$$\liminf_{t \to +\infty} x_i(t) \geq \frac{m_{i-1}}{2m_{i-1} \left( a_i + \frac{a_{i,j+1}}{m_{i,j+1}} \right)} \times \left( a_{i,j-1} \int_0^A K_{i-1}(s) \exp \left\{ - \left( a_i + \frac{a_{i,j+1}}{m_{i,j+1}} \right) s \right\} ds - a_i - \frac{a_{i,j+1}}{m_{i,j+1}} \right) \equiv m_i,$$

$i = 3, 4, \ldots, n - 1$.

Hence, for large $t$, we have $x_i(t) > m_i/2$, $i = 3, 4, \ldots, n - 1$.

Finally, under assumption (H8), we get

$$\liminf_{t \to +\infty} x_n(t) \geq \frac{m_{n-1}}{2m_{n-1}a_n} \left( a_{n,n-1} \int_0^A K_{n-1}(s) \exp^{-a_n s} ds - a_n \right) \equiv m_n.$$  

Therefore, for large $t$, we have $x_n(t) > m_n/2$.

Now, we let

$$D = \{ (x_1, x_2, \ldots, x_n) \mid m_i/2 \leq x_i \leq M_i, \ i = 1, 2, \ldots, n \}.$$  

Then $D$ is a bounded compact region in $R^n_+$ which has positive distance from coordinate planes. From what has been discussed above, we obtain that there exists a $T^* > 0$, if $t > T^*$, then every positive solution of system (1.4) with initial conditions (1.5) eventually enters and remains in the region $D$. The proof is complete. $\square$

3. Global attractivity

We now provide conditions under which the positive equilibrium $E^*$ of system (1.4) is globally attractive. Our strategy in the proof is to construct a suitable Lyapunov functional.

Let $P_i(u)$ be defined by

$$P_i(u) = \frac{u}{m_{i,i+1} + u}, \ i = 1, 2, \ldots, n - 1.$$  

Then system (1.4) can be rewritten as
Then system (3.1) becomes

\[
\begin{align*}
\dot{x}_1 &= x_1(t) \left\{ -a_{11}(x_1 - x_i^+) + a_{12} \left[ \frac{x_i^+}{x_1} P_1 \left( \frac{x_i^+}{x_2} \right) - \frac{x_2}{x_1} P_1 \left( \frac{x_1}{x_2} \right) \right] \right\}, \\
\dot{x}_2 &= x_2(t) \left\{ a_{21} \int_0^\infty K_1(s) \left[ P_1 \left( \frac{x_1(t-s)}{x_2(t-s)} \right) - P_1 \left( \frac{x_1}{x_2} \right) \right] \, ds \\
&\quad + a_{23} \left[ \frac{x_1^+}{x_2^+} P_2 \left( \frac{x_1^+}{x_3^+} \right) - \frac{x_3}{x_2} P_2 \left( \frac{x_2}{x_3} \right) \right] \right\}, \\
&\vdots \\
\dot{x}_i &= x_i(t) \left\{ a_{i,i-1} \int_0^\infty K_{i-1}(s) \left[ P_{i-1} \left( \frac{x_{i-1}(t-s)}{x_i(t-s)} \right) - P_{i-1} \left( \frac{x_{i-1}}{x_i} \right) \right] \, ds \\
&\quad + a_{i,i+1} \left[ \frac{x_{i+1}^+}{x_i^+} P_i \left( \frac{x_{i+1}^+}{x_i^+} \right) - \frac{x_i^+}{x_{i+1}} P_i \left( \frac{x_i}{x_{i+1}} \right) \right] \right\}, \\
&\vdots \\
\dot{x}_n &= a_{n,n-1}x_n(t) \int_0^\infty K_{n-1}(s) \left[ P_{n-1} \left( \frac{x_{n-1}(t-s)}{x_n(t-s)} \right) - P_{n-1} \left( \frac{x_{n-1}}{x_n} \right) \right] \, ds.
\end{align*}
\]

(3.1)

Define

\[
x_1 = x_1, \quad u_i = \frac{x_i}{x_{i+1}}, \quad u_i^+ = \frac{x_i^+}{x_{i+1}^+}, \quad i = 1, 2, \ldots, n - 1.
\]

Then system (3.1) becomes

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t) \left\{ -a_{11}(x_1 - x_i^+) + a_{12} \left[ \frac{P_1(u_i^+)}{u_i^+} - \frac{P_1(u_i)}{u_i} \right] \right\}, \\
\dot{u}_1(t) &= u_1(t) \left\{ -a_{11}(x_1 - x_i^+) + a_{12} \left[ \frac{P_1(u_i^+)}{u_i^+} - \frac{P_1(u_i)}{u_i} \right] \right\} \\
&\quad - a_{21} \int_0^\infty K_1(s) [P_1(u_1(t-s)) - P_1(u_1^+)] \, ds - a_{23} \left[ \frac{P_2(u_2^+)}{u_2^+} - \frac{P_2(u_2)}{u_2} \right], \\
\dot{u}_2(t) &= u_2(t) \left\{ a_{21} \int_0^\infty K_1(s) [P_1(u_1(t-s)) - P_1(u_1^+)] \, ds \\
&\quad + a_{23} \left[ \frac{P_2(u_2^+)}{u_2^+} - \frac{P_2(u_2)}{u_2} \right] \right\} \\
&\quad - a_{34} \left[ \frac{P_3(u_3^+)}{u_3^+} - \frac{P_3(u_3)}{u_3} \right], \\
&\vdots \\
\dot{u}_i(t) &= u_i(t) \left\{ a_{i,i-1} \int_0^\infty K_{i-1}(s) [P_{i-1}(u_{i-1}(t-s)) - P_{i-1}(u_{i-1}^+)] \\
&\quad + a_{i,i+1} \left[ \frac{P_i(u_i^+)}{u_i^+} - \frac{P_i(u_i)}{u_i} \right] \right\} \\
&\quad - a_{i+1,i} \int_0^\infty K_i(s) [P_i(u_i(t-s)) - P_i(u_i^+)] \, ds - a_{i+1,i+2} \left[ \frac{P_{i+1}(u_{i+1}^+)}{u_{i+1}^+} - \frac{P_{i+1}(u_{i+1})}{u_{i+1}} \right].
\end{align*}
\]
Define
\[ v_i(t) = x_i(t) - x_i^*, \quad v_{i+1}(t) = u_i(t) - u_i^*; \quad F_i(v_{i+1}) = P_i(u_i) - P_i(u_i^*), \quad i = 1, 2, \ldots, n - 1. \] (3.3)

It is easy to verify that
\[ F_i(v_{i+1}) = \frac{m_{i,i+1}v_{i+1}}{(m_{i,i+1} + u_i)(m_{i,i+1} + u_i^*)}; \]
\[ F'_i(v_{i+1})u_i = \frac{m_{i,i+1}u_i}{(m_{i,i+1} + u_i)^2} < 1; \]
\[ \frac{P_i(u_i^*)}{u_i^*} - \frac{P_i(u_i)}{u_i} = \frac{1}{m_{i,i+1}} F_i(v_{i+1}), \]
\[ i = 1, 2, \ldots, n - 1. \] (3.4)

Therefore, from (3.2)–(3.4), we finally obtain
\[ \dot{v}_1 = (v_1 + x_1^*) \left\{ - a_{11} v_1(t) + \frac{a_{12}}{m_{12}} F_1(v_2(t)) \right\}, \]
\[ \dot{v}_2 = (v_2 + u_1^*) \left\{ - a_{11} v_1(t) + \frac{a_{12}}{m_{12}} F_1(v_2(t)) - \frac{a_{21}}{m_{23}} F_2(v_3(t)) \right. \]
\[ - a_{21} \int_0^\infty K_1(s) F_1(v_2(t-s)) \, ds + \frac{a_{21}}{m_{23}} F_2(v_3(t)) \}
\[ \dot{v}_3 = (v_3 + u_2^*) \left\{ a_{21} \int_0^\infty K_1(s) F_1(v_2(t-s)) \, ds + \frac{a_{23}}{m_{23}} F_2(v_3(t)) \right. \]
\[ - a_{32} \int_0^\infty K_2(s) F_2(v_3(t-s)) \, ds - \frac{a_{32}}{m_{34}} F_3(v_4(t)) \}
\[ \vdots \]
\[ \dot{v}_i = (v_i + u_{i-1}^*) \left\{ a_{i-1,i-2} \int_0^\infty K_{i-2}(s) F_{i-2}(v_{i-1}(t-s)) \, ds + \frac{a_{i-1,i}}{m_{i-1,i}} F_{i-1}(v_i(t)) \right. \]
\[ - a_{i-1,i-1} \int_0^\infty K_{i-1}(s) F_{i-1}(v_i(t-s)) \, ds - \frac{a_{i-1,i}}{m_{i-1,i}} F_{i-1}(v_{i-1}(t)) \}
\[ \vdots \]
\[ \dot{v}_n = (v_n + u_{n-1}^*) \left\{ a_{n-1,n-2} \int_0^\infty K_{n-2}(s) F_{n-2}(v_{n-1}(t-s)) \, ds + \frac{a_{n-1,n}}{m_{n-1,n}} F_{n-1}(v_n(t)) \right. \]
\[ - a_{n,n-1} \int_0^\infty K_{n-1}(s) F_{n-1}(v_n(t-s)) \, ds \}. \] (3.5)
We now formulate a result on the global attractivity of the equilibrium \( E^* \) of (1.4) as follows.

**Theorem 3.1.** Suppose that system (1.4) satisfies (H1)–(H5) and the following:

(H9) \( A_i > 0, \quad i = 1, 2, \ldots, n, \)

where

\[
A_1 = c_1 a_{11} - \frac{1}{2} \sigma_1 a_{21} a_{11} (c_2 + c_3);
\]

\[
A_2 = c_2 \left[ a_{21} - \frac{a_{12}}{m_{12}} - \frac{1}{2} \sigma_1 a_{21} \left( a_{11} + 2 \frac{a_{12}}{m_{12}} + \frac{a_{23}}{m_{23}} + 2a_{21} \right) \right]
- \frac{1}{2} \sigma_1 a_{21} c_3 \left( \frac{a_{12}}{m_{12}} + a_{21} \right) - \frac{1}{2} a_{21} a_{32} \sigma_2 (c_3 + c_4);
\]

\[
A_3 = c_3 \left[ a_{32} - \frac{a_{23}}{m_{23}} - \frac{1}{2} a_{21} \sigma_1 \left( a_{11} + \frac{a_{12}}{m_{12}} + \frac{a_{23}}{m_{23}} + 2a_{21} \right) \right]
- \frac{1}{2} c_3 a_{32} \sigma_2 \left( a_{21} + 2 \frac{a_{23}}{m_{23}} + 2a_{32} + \frac{a_{34}}{m_{34}} \right)
- \frac{1}{2} a_{43} a_{32} \sigma_3 (c_4 + c_5) - \frac{1}{2} c_4 a_{32} \sigma_2 \left( \frac{a_{23}}{m_{23}} + a_{32} \right);
\]

\[\vdots\]

\[
A_{i+1} = c_{i+1} \left[ a_{i+1,i} - \frac{a_{i+1,i+1}}{m_{i+1,i+1}} - \frac{1}{2} a_{i+1,i} \sigma_i \left( a_{i,i-1} + 2 \frac{a_{i,i+1}}{m_{i,i+1}} + 2a_{i+1,i} + \frac{a_{i+1,i+1}}{m_{i+1,i+1}} \right) \right]
- \frac{1}{2} a_{i+2,i+1} a_{i+1,i} \sigma_i \left( c_i + c_{i+1} \right) - \frac{1}{2} c_{i+2} a_{i+1,i} \sigma_i \left( \frac{a_{i+1,i+1}}{m_{i+1,i+1}} + a_{i+1,i} \right)
- \frac{1}{2} a_{i,i-1} \sigma_{i-1} \left( c_i + c_{i+1} \right) \frac{a_{i,i+1}}{m_{i,i+1}}, \quad i = 3, 4, \ldots, n - 2;
\]

\[
A_n = a_{n,n-1} - \frac{a_{n-1,n}}{m_{n-1,n}} - \frac{1}{2} a_{n-1,n-2} \sigma_{n-2} \left( a_{n-1,n-2} + \frac{a_{n-2,n-1}}{m_{n-2,n-1}} + \frac{a_{n-1,n}}{m_{n-1,n}} + a_{n-2,n-3} \right)
- \frac{1}{2} a_{n-1,n-2} \sigma_{n-2} \left( c_{n-1} + 1 \right) \frac{a_{n-1,n}}{m_{n-1,n}}
- \frac{1}{2} c_{n,n-1} \sigma_{n-1} \left( 2 \frac{a_{n-1,n}}{m_{n-1,n}} + a_{n-1,n-2} + 2a_{n,n-1} \right),
\]

where \( c_n = 1, \ c_{n+1} = 0, \ c_i (i = 1, 2, \ldots, n - 1) \) are determined by the following:

\[
c_1 \frac{a_{12}}{m_{12}} = c_2 a_{11}, \quad c_i \frac{a_{i,i+1}}{m_{i,i+1}} = c_{i+1} a_{i,i-1}, \quad i = 2, 3, \ldots, n - 1.
\]
Then the positive equilibrium $E^*(x_1^*, x_2^*, \ldots, x_n^*)$ of system (1.4) is globally attractive.

**Proof.** To prove that the global attractive of the positive equilibrium $E^*$ of system (1.4) is equivalent to that of the trivial solution of (3.5).

Let

$$V_1(t) = V_1(v(t)) = c_1 \left( v_1 - x_1^* \ln \frac{v_1 + x_1^*}{x_1^*} \right) + \sum_{i=1}^{n-1} c_{i+1} \int_{a_i^*}^{a_i} \frac{P_i(v) - P_i(u_i^*)}{v} \, dv,$$

where $c_i \ (i = 1, 2, \ldots, n)$ are defined by (H9).

Calculating the derivative of $V_1(t)$ along the solution of (3.5), we have

$$\frac{d}{dt} V_1(t) = c_1 \frac{v_1}{v_1 + x_1^*} \dot{v}_1(t) + \sum_{i=1}^{n-1} c_{i+1} \frac{F_i(v_{i+1})}{u_i} \dot{v}_{i+1}(t)$$

$$= -c_1 a_{11} v_1^2(t) + \sum_{i=1}^{n-1} c_{i+1} \left( \frac{a_{i,i+1}}{m_{i,i+1}} F_i^2(v_{i+1}(t)) - \sum_{i=2}^{n-1} c_i \frac{a_{i,i+1}}{m_{i,i+1}} F_i(v_i) F_i(v_{i+1}) \right)$$

$$- \sum_{i=2}^{n-1} c_i a_{i,i-1} F_i(v_{i+1}) \int_0^\infty K_{i-1}(s) F_{i-1}(v_i(t - s)) \, ds$$

$$+ \sum_{i=2}^{n-1} c_{i+1} a_{i,i-1} F_i(v_{i+1}) \int_0^\infty K_{i-1}(s) F_{i-1}(v_i(t - s)) \, ds$$

$$= -c_1 a_{11} v_1^2(t) + \sum_{i=1}^{n-1} c_{i+1} \left( \frac{a_{i,i+1}}{m_{i,i+1}} - a_{i+1,i} \right) F_i^2(v_{i+1}(t))$$

$$+ \sum_{i=2}^{n-1} c_i a_{i,i-1} F_i(v_{i+1}) \int_0^t \int_{t-s}^t K_{i-1}(s) F_{i-1}(v_i(l)) v_i'(l) \, dl \, ds$$

$$- \sum_{i=2}^{n-1} c_{i+1} a_{i,i-1} F_i(v_{i+1}) \int_0^t \int_{t-s}^t K_{i-1}(s) F_{i-1}(v_i(l)) v_i'(l) \, dl \, ds + B(t)$$

$$= -c_1 a_{11} v_1^2(t) + \sum_{i=1}^{n-1} c_{i+1} \left( \frac{a_{i,i+1}}{m_{i,i+1}} - a_{i+1,i} \right) F_i^2(v_{i+1}(t))$$

$$+ \sum_{i=2}^{n-1} a_{i,i-1} (c_i F_i(v_i) - c_{i+1} F_i(v_{i+1}))$$

$$\times \int_0^t \int_{t-s}^t K_{i-1}(s) F_{i-1}^*(v_i(l)) v_i'(l) \, dl \, ds$$

$$+ c_n a_{n,n-1} F_{n-1}(v_n) \int_0^t \int_{t-s}^t K_{n-1}(s) F_{n-1}^*(v_n(l)) v_n'(l) \, dl \, ds + B(t)$$
\[\begin{align*}
&= -c_1 a_{11} v_1^2(t) + \sum_{i=1}^{n-1} c_{i+1} \left( a_{i,i+1} \frac{a_{i+1,i}}{m_{i,i+1}} - a_{i+1,i} \right) F_i^2(v_{i+1}(t)) \\
&+ a_{21} (c_2 F_1(v_2(t)) - c_3 F_2(v_3(t))) \int_0^t \int_{t-s}^t K_1(s) F_i'(v_2(l)) u_1(l) \\
&\times \left[ -a_{11} v_1(l) + \frac{a_{12}}{m_{12}} F_1(v_2(l)) - \frac{a_{23}}{m_{23}} F_2(v_3(l)) \\
&- a_{21} \int_0^\infty K_1(r) F_1(v_2(l - r)) \, dr \right] \, dl \, ds \\
&+ \sum_{i=3}^{n-1} a_{i-1} \left( c_i F_{i-1}(v_i) - c_{i+1} F_i(v_{i+1}) \right) \int_0^t \int_{t-s}^t K_{i-1}(s) F'_{i-1}(v_i(l)) u_{i-1}(l) \\
&\times \left[ a_{i-1,i-2} \int_0^\infty K_{i-2}(r) F_{i-2}(v_{i-1}(l - r)) \, dr + \frac{a_{i-1,i}}{m_{i-1,i}} F_{i-1}(v_i(l)) \\
&- a_{i,i-1} \int_0^\infty K_{i-1}(r) F_{i-1}(v_{i-1}(l - r)) \, dr - \frac{a_{i,i+1}}{m_{i,i+1}} F_i(v_{i+1}(l)) \right] \, dl \, ds \\
&+ c_n a_{n,n-1} F_{n-1}(v_n) \int_0^t \int_{t-s}^t K_{n-1}(s) F'_{n-1}(v_n(l)) u_{n-1}(l) \\
&\times \left[ \frac{a_{n-1,n}}{m_{n-1,n}} F_{n-1}(v_n(l)) + a_{n-1,n-2} \int_0^\infty K_{n-2}(r) F_{n-2}(v_{n-1}(l - r)) \, dr \\
&- a_{n,n-1} \int_0^\infty K_{n-1}(r) F_{n-1}(v_{n-1}(l - r)) \, dr \right] \, dl \, ds + B(t) \\
&\leq -c_1 a_{11} v_1^2(t) + \sum_{i=1}^{n-1} c_{i+1} \left( a_{i,i+1} \frac{a_{i+1,i}}{m_{i,i+1}} - a_{i+1,i} \right) F_i^2(v_{i+1}(t)) \\
&+ a_{21} (c_2 |F_1(v_2(t))| + c_3 |F_2(v_3(t))|) \int_0^t \int_{t-s}^t K_1(s) F_i'(v_2(l)) u_1(l) \\
&\times \left[ a_{11} |v_1(l)| + \frac{a_{12}}{m_{12}} |F_1(v_2(l))| + \frac{a_{23}}{m_{23}} |F_2(v_3(l))| \\
&+ a_{21} \int_0^\infty K_1(r) |F_1(v_2(l - r))| \, dr \right] \, dl \, ds \\
&+ \sum_{i=3}^{n-1} a_{i-1} \left( c_i |F_{i-1}(v_i)| + c_{i+1} |F_i(v_{i+1})| \right) \int_0^t \int_{t-s}^t K_{i-1}(s) F'_{i-1}(v_i(l)) u_{i-1}(l) \\
&\times \left[ a_{i-1,i-2} \int_0^\infty K_{i-2}(r) F_{i-2}(v_{i-1}(l - r)) \, dr + \frac{a_{i-1,i}}{m_{i-1,i}} |F_{i-1}(v_i(l))| \right] \end{align*}\]
\[ + a_{i,j-1} \int_0^\infty K_{i-1}(r) |F_{i-1}(v_i(l - r))| \, dr + \frac{a_{i,i+1}}{m_{i,i+1}} |F_i(v_{i+1}(l))| \] \[ + c_n a_{n,n-1} |F_{n-1}(v_n)| \int_0^\infty \int_{t-s}^t K_{n-1}(s) F'_{n-1}(v_n(l)) u_{n-1}(l) \] \[ \times \left[ \frac{a_{n-1,n}}{m_{n-1,n}} |F_{n-1}(v_n(l))| + a_{n-1,n-2} \int_0^\infty K_{n-2}(r) |F_{n-2}(v_{n-1}(l - r))| \, dr \right. \] \[ + a_{n,n-1} \int_0^\infty K_{n-1}(r) |F_{n-1}(v_{n-1}(l - r))| \, dr \left] \right. \] \[ \, dl \, ds + B(t), \] 

\[(3.7)\]

where

\[ B(t) = \sum_{i=2}^n c_i a_{i-1,i} F_{i-1}(v_i) \int_t^\infty K_{i-1}(s) \left[ F_{i-1}(v_i(t)) - F_{i-1}(v_i(t - s)) \right] \, ds \] \[ - \sum_{i=2}^{n-1} c_{i+1} a_{i,i-1} F_i(v_{i+1}) \int_t^\infty K_{i-1}(s) \left[ F_{i-1}(v_i(t)) - F_{i-1}(v_i(t - s)) \right] \, ds. \]

Notice the fact in (3.4), we get

\[ \frac{d}{dt} V_1(t) \leq -c_1 a_{11} v_1^2(t) + \sum_{i=1}^{n-1} c_{i+1} \left( \frac{a_{i,i+1}}{m_{i,i+1}} - a_{i+1,i} \right) F_i^2(v_{i+1}(t)) \] \[ + a_{21} \left( c_2 |F_1(v_2(t))| + c_3 |F_2(v_3(t))| \right) \] \[ \times \int_0^\infty \int_{t-s}^t K_{1}(s) \left[ a_{11} v_1(l) + \frac{a_{12}}{m_{12}} |F_1(v_2(l))| + \frac{a_{23}}{m_{23}} |F_2(v_3(l))| \right] \] \[ + a_{21} \int_0^\infty K_{1}(r) |F_1(v_2(l - r))| \, dr \] \[ \, d\ell \, ds \] \[ + \sum_{i=2}^{n-1} a_{i,i-1} c_i |F_{i-1}(v_i)| + c_{i+1} |F_i(v_{i+1})| \int_0^\infty \int_{t-s}^t K_{i-1}(s) \] \[ \times \left[ a_{i-1,i-2} \int_0^\infty K_{i-2}(r) |F_{i-2}(v_{i-1}(l - r))| \, dr + \frac{a_{i-1,i}}{m_{i-1,i}} |F_{i-1}(v_i(l))| \right] \] \[ + a_{i,i-1} \int_0^\infty K_{i-1}(r) |F_{i-1}(v_i(l - r))| \, dr + \frac{a_{i,i+1}}{m_{i,i+1}} |F_i(v_{i+1}(l))| \right] \] \[ \, dl \, ds \] \[ + c_n a_{n,n-1} |F_{n-1}(v_n)| \int_0^\infty \int_{t-s}^t K_{n-1}(s) \] \[ \times \left[ \frac{a_{n-1,n}}{m_{n-1,n}} |F_{n-1}(v_n(l))| + a_{n-1,n-2} \int_0^\infty K_{n-2}(r) |F_{n-2}(v_{n-1}(l - r))| \, dr \right. \] \[ + a_{n,n-1} \int_0^\infty K_{n-1}(r) |F_{n-1}(v_{n-1}(l - r))| \, dr \left] \right. \] \[ \, dl \, ds + B(t). \] 

\[(3.8)\]

Using the inequality \( a^2 + b^2 \geq 2ab \), we derive for \( t \geq T \) that
\[
\frac{d}{dt} V_1(t) \leq -c_1 a_{11} V_1^2(t) + \sum_{i=1}^{n-1} c_{i+1} \left( \frac{a_{i+1}}{m_{i+1}} - a_{i+1,i} \right) F_i^2(v_{i+1}(t)) \\
+ \frac{1}{2} a_{21} \sigma_1 \left( a_{11} + \frac{a_{12}}{m_{12}} + \frac{a_{23}}{m_{23}} + a_{21} \right) \\
\times (c_2 F_1^2(v_2(t)) + c_3 F_2^2(v_3(t))) + \frac{1}{2} a_{21}(c_2 + c_3) \\
\times \int_0^\infty \int_{t-s}^t K_1(s) \left[ a_{11} v_1^2(l) + \frac{a_{12}}{m_{12}} F_1^2(v_2(l)) + \frac{a_{23}}{m_{23}} F_2^2(v_3(l)) \right] \\
+ a_{21} \int_0^\infty K_1(r) F_1^2(v_2(l-r)) \, dr \, ds \\
+ \frac{1}{2} \sum_{i=3}^{n-1} a_{i,i-1} \sigma_{i-1} \left( a_{i-1,i-2} + \frac{a_{i-1,i}}{m_{i-1,i}} + a_{i,i-1} + \frac{a_{i+1,i}}{m_{i+1,i}} \right) \\
\times (c_i F_{i-1}^2(v_i(t)) + c_{i+1} F_{i+1}^2(v_{i+1}(t))) + \frac{1}{2} \sum_{i=3}^{n-1} a_{i,i-1}(c_i + c_{i+1}) \\
\times \int_0^\infty \int_{t-s}^t K_{i-1}(s) \left[ a_{i-1,i-2} \int_0^\infty K_{i-2}(r) F_{i-2}^2(v_{i-1}(l-r)) \, dr + \frac{a_{i-1,i}}{m_{i-1,i}} F_{i-1}^2(v_i(l)) \right] \\
+ a_{i,i-1} \int_0^\infty K_{i-1}(r) F_{i-1}^2(v_i(l-r)) \, dr + \frac{a_{i+1,i}}{m_{i+1,i}} F_{i}^2(v_{i+1}(l)) \right] \, dl \, ds \\
+ \frac{1}{2} c_{a} a_{n-1} a_{n-1} \left( \frac{a_{n-1,n}}{m_{n-1,n}} + a_{n-1,n-2} + a_{n,n-1} \right) F_{n-1}^2(v_n(t)) \\
+ \frac{1}{2} c_{a} a_{n,n-1} \int_0^\infty \int_{t-s}^t K_{n-1}(s) \left[ a_{n-1,n} F_{n-1}^2(v_n(l)) \right] \\
+ a_{n-1,n-2} \int_0^\infty K_{n-2}(r) F_{n-2}^2(v_{n-1}(l-r)) \, dr \\
+ a_{n,n-1} \int_0^\infty K_{n-1}(r) F_{n-1}^2(v_n(l-r)) \, dr \] \, dl \, ds + B(t). \tag{3.9}
\]

Define

\[
V_2(t) = V_1(t) + \frac{1}{2} a_{21}(c_2 + c_3) \int_0^\infty \int_{t-s}^t K_1(s) \left[ a_{11} v_1^2(l) + \frac{a_{12}}{m_{12}} F_1^2(v_2(l)) + \frac{a_{23}}{m_{23}} F_2^2(v_3(l)) \right] \\
+ a_{21} \int_0^\infty K_1(r) F_1^2(v_2(l-r)) \, dr \, ds \\
+ \frac{1}{2} \sum_{i=3}^{n-1} a_{i,i-1}(c_i + c_{i+1}) \int_0^\infty \int_{t-s}^t K_{i-1}(s) \right]
\]
We now define a Lyapunov functional

\[
F_2(t) \leq -c_1 a_{11} v_1^2(t) + \sum_{i=1}^{n-1} c_{i+1} \left( a_{i+1,i} - a_i \right) F_i^2(v_i(t)) + \frac{1}{2} a_{21} \sigma_1 \left( a_{11} + \frac{a_{12}}{m_{12}} + \frac{a_{23}}{m_{23}} + a_{21} \right) \left( c_2 F_1^2(v_2(t)) + c_3 F_2^2(v_3(t)) + \frac{1}{2} \sigma_1 \left( a_{11} + \frac{a_{12}}{m_{12}} + \frac{a_{23}}{m_{23}} + a_{21} \right) \left( F_1^2(v_2(t)) + a_{23} F_2^2(v_3(t)) \right)
\]

Then we have from (3.6), (3.9) and (3.10) that for \( t \geq T \)

\[
\frac{d}{dt} F_2(t) \leq -c_1 a_{11} v_1^2(t) + \sum_{i=1}^{n-1} c_{i+1} \left( a_{i+1,i} - a_i \right) F_i^2(v_i(t)) + \frac{1}{2} a_{21} \sigma_1 \left( a_{11} + \frac{a_{12}}{m_{12}} + \frac{a_{23}}{m_{23}} + a_{21} \right) \left( c_2 F_1^2(v_2(t)) + c_3 F_2^2(v_3(t)) + \frac{1}{2} \sigma_1 \left( a_{11} + \frac{a_{12}}{m_{12}} + \frac{a_{23}}{m_{23}} + a_{21} \right) \left( F_1^2(v_2(t)) + a_{23} F_2^2(v_3(t)) \right)
\]

We now define a Lyapunov functional \( V(t) \) as
\[ V(t) = V_2(t) + \frac{1}{2} \sigma_1 a_{i_1}^2 (c_2 + c_3) \int_0^\infty \int_{t-r}^t K_1(r) F_i^2 \left( v_{i-1}^2 \right) \, dr \, dr \\
+ \frac{1}{2} \sum_{i=3}^{n-1} \sigma_{i-1} a_{i-1} \left( c_i + c_{i+1} \right) \left[ a_{i-1,i-2} \int_0^\infty \int_{t-r}^t K_{i-1}^2(r) F_{i-1}^2 \left( v_{i-1}^2 \right) \, dr \, dr \right] \\
+ a_{i,i-1} \int_0^\infty \int_{t-r}^t K_{i-1}^2(r) F_{i-1}^2 \left( v_{i-1}^2 \right) \, dr \, dr \\
+ a_{n,n-1} \int_0^\infty \int_{t-r}^t K_{n-1}^2(r) F_{n-1}^2 \left( v_n^2 \right) \, dr \, dr \] (3.12)

Then we have from (3.6) and (3.10)–(3.12) that for \( t \geq T \)

\[ \frac{d}{dt} V(t) \leq -c_1 a_{i_1} v_1^2(t) + \sum_{i=1}^{n-1} c_{i+1} \left( \frac{a_{i,i+1}}{m_{i,i+1}} - a_{i+1,i} \right) F_i^2(v_{i-1}(t)) \\
+ \frac{1}{2} a_{11} \sigma_1 \left( a_{11} + \frac{a_{12}}{m_{12}} + \frac{a_{23}}{m_{23}} + a_{21} \right) \left( c_2 F_1^2(v_2(t)) + c_3 F_2^2(v_3(t)) \right) \\
+ \frac{1}{2} \sigma_1 a_{21} \left( c_2 + c_3 \right) \left[ a_{11} v_1^2(t) + \frac{a_{12}}{m_{12}} F_1^2(v_2(t)) + \frac{a_{23}}{m_{23}} F_2^2(v_3(t)) \right] \\
+ a_{21} F_1^2(v_2(t)) \\
+ \frac{1}{2} \sum_{i=3}^{n-1} a_{i,i-1} \sigma_{i-1} \left( a_{i-1,i-2} + \frac{a_{i-1,i}}{m_{i-1,i}} + a_{i+1,i} \right) \left( c_i F_i^2(v_i(t)) \right) \\
+ c_{i+1} F_{i-1}^2(v_{i-1}(t))) \\
+ \frac{1}{2} \sum_{i=3}^{n-1} a_{i,i-1} \left( c_i + c_{i+1} \right) \left[ a_{i-1,i-2} F_{i-2}^2(v_{i-1}(t)) + \frac{a_{i-1,i}}{m_{i-1,i}} F_{i-1}^2(v_i(t)) \right] \\
+ a_{i,i-1} F_{i-1}^2(v_i(t)) + \frac{a_{i,i+1}}{m_{i+1,i}} F_i^2(v_{i+1}(t)) \\
+ \frac{1}{2} c_{n} \sigma_{n-1} a_{n,n-1} \left( \frac{a_{n-1,n}}{m_{n-1,n}} + a_{n-1,n-2} + a_{n,n-1} \right) F_{n-1}^2(v_n(t)) \\
+ \frac{1}{2} c_{n} \sigma_{n-1} a_{n,n-1} \left[ \frac{a_{n-1,n}}{m_{n-1,n}} F_{n-1}^2(v_n(t)) + a_{n-1,n-2} F_{n-2}^2(v_{n-1}(t)) \right] \\
+ a_{n,n-1} F_{n-1}^2(v_{n-1}(t)) \right] + B(t) \]
\begin{align*}
&\leq - A_1 v_1^2(t) - \sum_{i=1}^{n-1} A_{i+1} F_i^2(v_{i+1}(t)) \\
&\quad + \frac{3}{2} \sum_{i=2}^{n} c_i a_{i,i-1} F_i^2(v_i(t)) \int_t^\infty K_{i-1}(s) \, ds \\
&\quad + \frac{1}{2} \sum_{i=2}^{n} c_i a_{i,i-1} \int_t^\infty K_{i-1}(s) F_i^2(v_i(t-s)) \, ds \\
&\quad + \frac{1}{2} \sum_{i=2}^{n-1} c_{i+1} a_{i,i-1} F_i^2(v_{i+1}(t)) \int_t^\infty K_{i-1}(s) \, ds \\
&\quad + \frac{1}{2} \sum_{i=2}^{n-1} c_{i+1} a_{i,i-1} \int_t^\infty K_{i-1}(s) F_i^2(v_i(t-s)) \, ds \\
&\quad - A_1 v_1^2(t) - \sum_{i=1}^{n-1} A_{i+1} F_i^2(v_{i+1}(t)) \\
&\quad + \frac{3}{2} \sum_{i=2}^{n} c_i a_{i,i-1} F_i^2(v_i(t)) \int_t^\infty K_{i-1}(s) \, ds \\
&\quad + \frac{1}{2} \sum_{i=2}^{n} c_i a_{i,i-1} F_i^2(||\psi||) \int_t^\infty K_{i-1}(s) \, ds \\
&\quad + \frac{1}{2} \sum_{i=2}^{n-1} c_{i+1} a_{i,i-1} F_i^2(v_{i+1}(t)) \int_t^\infty K_{i-1}(s) \, ds \\
&\quad + \frac{1}{2} \sum_{i=2}^{n-1} c_{i+1} a_{i,i-1} \int_t^\infty K_{i-1}(s) F_i^2(||\psi||) \, ds \\
&\quad = - A_1 v_1^2(t) - \sum_{i=1}^{n-1} A_{i+1} F_i^2(v_{i+1}(t)) \\
&\quad + a_{21} \left( \frac{3}{2} c_2 + \frac{1}{2} c_3 \right) \int_t^\infty K_1(s) \, ds F_1^2(v_2(t)) \\
&\quad + \sum_{i=2}^{n-2} \left[ a_{i+1,i} \left( \frac{3}{2} c_{i+1} + \frac{1}{2} c_{i+2} \right) \int_t^\infty K_i(s) \, ds + c_{i+1} a_{i,i-1} \int_t^\infty K_{i-1}(s) \, ds \right] \\
&\quad \times F_i^2(v_{i+1}(t)) + \left[ \frac{3}{2} c_n a_{n,n-1} \int_t^\infty K_{n-1}(s) \, ds + c_n a_{n-1,n-2} \int_t^\infty K_{n-2}(s) \, ds \right] \\
&\quad \times F_{n-1}^2(v_n(t)) + \frac{1}{2} \sum_{i=2}^{n} c_i a_{i,i-1} F_i^2(||\psi||) \int_t^\infty K_{i-1}(s) \, ds \\
&\quad + \frac{1}{2} \sum_{i=2}^{n-1} c_{i+1} a_{i,i-1} F_i^2(||\psi||) \int_t^\infty K_{i-1}(s) \, ds,
\end{align*}
where \( \psi_i \in BC(-\infty, 0] \) is the initial data of \( v_i(t) \), \( i = 1, 2, \ldots, n \), and \( A_i \) \( (i = 1, 2, \ldots, n) \) are defined by (H9). Therefore, we can find \( \varepsilon_i > 0 \) \( (i = 1, 2, \ldots, n - 1) \) such that

\[
a_{21} \left( \frac{3}{2} c_2 + \frac{1}{2} c_3 \right) \varepsilon_1 < A_2,
\]

\[
\left[ a_{i+1,i} \left( \frac{3}{2} c_{i+1} + \frac{1}{2} c_{i+2} \right) + c_{i+1} a_{i,i-1} \right] \varepsilon_i < A_{i+1},
\]

\( i = 2, \ldots, n - 2, \)

\[
\left( \frac{3}{2} c_n a_{n,n-1} + c_n a_{n-1,n-2} \right) \varepsilon_{n-1} < A_n.
\]

Let \( T = T(\varepsilon_1, \ldots, \varepsilon_{n-1}) > 0 \) be such that \( \int_0^T K_i(s) \, ds < \min_{i=1,2,\ldots,n-1} \{ \varepsilon_i \} \) for all \( t \geq T, i = 1, \ldots, n - 1 \). It then follows from (3.13) that for all \( t \geq T \).

\[
\frac{d}{dt} V(t) \leq -A_1 v_1^2(t) - \left[ A_2 - a_{21} \left( \frac{3}{2} c_2 + \frac{1}{2} c_3 \right) \varepsilon_1 \right] F_2^2(v_2(t))
\]

\[
- \sum_{i=2}^{n-2} \left[ A_{i+1} - a_{i+1,i} \left( \frac{3}{2} c_{i+1} + \frac{1}{2} c_{i+2} \right) \varepsilon_i - c_{i+1} a_{i,i-1} \varepsilon_i \right] F_2^2(v_{i+1}(t))
\]

\[
- \left( A_n - \frac{3}{2} c_n a_{n,n-1} \varepsilon_{n-1} - c_n a_{n-1,n-2} \varepsilon_{n-2} \right) F_2^2(v_n(t))
\]

\[
+ \frac{1}{2} \sum_{i=2}^{n-1} c_i a_{i,i-1} F_{i-1}^2(\|\psi_i\|) \int_t^\infty K_{i-1}(s) \, ds
\]

\[
+ \frac{1}{2} \sum_{i=2}^{n-1} c_i a_{i,i-1} F_{i-1}^2(\|\psi_i\|) \int_t^\infty K_{i-1}(s) \, ds.
\]

(3.14)

Integrating both sides of (3.14) from \( T \) to \( t \geq T \), we get

\[
V_1(t) \leq V(t) + A_1 \int_T^t v_1^2(s) \, ds + \left[ A_2 - a_{21} \left( \frac{3}{2} c_2 + \frac{1}{2} c_3 \right) \varepsilon_1 \right] \int_T^t F_2^2(v_2(s)) \, ds
\]

\[
+ \sum_{i=2}^{n-2} \left[ A_{i+1} - a_{i+1,i} \left( \frac{3}{2} c_{i+1} + \frac{1}{2} c_{i+2} \right) \varepsilon_i - c_{i+1} a_{i,i-1} \varepsilon_i \right] \int_T^t F_2^2(v_{i+1}(s)) \, ds
\]

\[
+ \left( A_n - \frac{3}{2} c_n a_{n,n-1} \varepsilon_{n-1} - c_n a_{n-1,n-2} \varepsilon_{n-2} \right) \int_T^t F_2^2(v_n(s)) \, ds
\]

\[
\leq V(T) + \frac{1}{2} \sum_{i=2}^{n-1} c_i a_{i,i-1} F_{i-1}^2(\|\psi_i\|) \int_T^\infty \int_s^\infty K_{i-1}(u) \, du \, ds
\]

\[
+ \frac{1}{2} \sum_{i=2}^{n-1} c_i a_{i,i-1} F_{i-1}^2(\|\psi_i\|) \int_T^\infty \int_s^\infty K_{i-1}(u) \, du \, ds
\]
changing the variables

lim \( \frac{1}{n} \sum_{i=2}^{n} c_i a_i v_i \to 1 \) \( \int_0^{\infty} \frac{1}{s} sK_{i-1}(s) ds \)

This implies that \( v_i(t) (i = 1, 2, \ldots, n) \) are bounded, and \( v_i^2(t), F_i^2(v_{i+1}(t)) \in L^1[0, +\infty), i = 1, \ldots, n - 1 \). If follows from the equations in system (3.5) and the boundedness of \( v_i(t) \) that \( \dot{v}_i(t) (i = 1, \ldots, n) \) are also bounded. Therefore, \( v_i(t) (i = 1, \ldots, n) \) are uniformly continuous on \([0, \infty)\). If follows that \( F_i(v_{i+1}(t)) (i = 1, \ldots, n - 1) \) are also uniformly continuous. Thus, by the Babalat's Lemma (see [13]), we have \( v_i(t) \to 0, F_i(v_{i+1}(t)) \to 0 \) as \( t \to \infty \). This leads to \( \lim_{t \to \infty} v_i(t) = 0, i = 1, 2, \ldots, n \). Therefore, we have \( \lim_{t \to \infty} x_i(t) = x_i^* \) for every solution \((x_1(t), x_2(t), \ldots, x_n(t))\) of system (1.4), this completes the proof. \( \square \)

We remark that for the instantaneous (without time delay) model (1.4), after changing the variables \((x_1, u_1, \ldots, u_{n-1}), u_i = x_i/x_{i+1}, i = 1, 2, \ldots, n - 1, \) and centering the equation on the positive equilibrium \( E^* \), we have

\[
\begin{align*}
\dot{v}_1 &= (v_1 + x_1^*) \left\{ -a_{11} v_1(t) + \frac{a_{12}}{m_{12}} F_1(v_2(t)) \right\}, \\
\dot{v}_2 &= (v_2 + x_1^*) \left\{ -a_{11} v_1(t) + \left( \frac{a_{12}}{m_{12}} - a_{21} \right) F_1(v_2(t)) - \frac{a_{23}}{m_{23}} F_2(v_3(t)) \right\}, \\
\dot{v}_3 &= (v_3 + x_2^*) \left\{ a_{21} F_1(v_2(t)) + \left( \frac{a_{23}}{m_{23}} - a_{32} \right) F_2(v_3(t)) - \frac{a_{34}}{m_{34}} F_3(v_4(t)) \right\}, \\
&\vdots \\
\dot{v}_i &= (v_i + x_{i-1}^*) \left\{ a_{i-1, i-2} F_{i-2}(v_{i-3}(t)) + \left( \frac{a_{i-1, i}}{m_{i-1, i}} - a_{i, i-1} \right) F_{i-1}(v_i(t)) - \frac{a_{i, i+1}}{m_{i, i+1}} F_i(v_{i+1}(t)) \right\}, \\
&\vdots \\
\dot{v}_n &= (v_n + x_{n-1}^*) \left\{ a_{n-1, n-2} F_{n-2}(v_{n-3}(t)) + \left( \frac{a_{n-1, n}}{m_{n-1, n}} - a_{n, n-1} \right) F_{n-1}(v_n(t)) \right\}
\end{align*}
\]

(3.15)

for which we have the following result.

**Corollary.** Suppose that system (3.15) satisfies (H1)--(H4) and the following:

(H10) \( a_{i+1, i} > \frac{a_{i, i+1}}{m_{i, i+1}}, \quad i = 1, 2, \ldots, n - 1. \)

Then the trivial solution of (3.15) is globally asymptotically stable.
Proof. Consider a Lyapunov function $V_1(t)$ defined by (3.6). Then along the solution of (3.15), we have

$$\frac{d}{dt} V_1(t) \leq - c_1 a_1 v_1^2(t) + \sum_{i=1}^{n-1} c_{i+1} \left( \frac{a_{i+1}}{m_{i+1}} - a_{i+1,i} \right) F_i^2(v_{i+1}(t)),\)$$

which is negative definite provided that (H10) holds, where $c_i$ ($i = 1, 2, \ldots, n$) are defined by (H9). This completes the proof. □

From what has been discussed above, we can conclude that, whenever (H1)–(H4) and (H10) hold, the positive equilibrium of the instantaneous model is globally attractive and the same is true for the positive equilibrium of the delayed model provided that mean time delays $\sigma_i$ ($i = 1, 2, \ldots, n - 1$) are small enough to satisfy the conditions in (H9).

Finally, we give a suitable example to illustrate the feasibility of the conditions of Theorems 2.1 and 3.1.

Example. We consider the following system:

\[
\begin{align*}
\dot{x}_1 &= x_1(t) \left( 2 - 2x_1(t) - \frac{x_2(t)}{x_2(t) + x_1(t)} \right), \\
\dot{x}_2 &= x_2(t) \left( -1 + 3 \int_0^\infty x_1 e^{-x_1s} \frac{x_1(t - s)}{x_2(t - s) + x_1(t - s)} ds - \frac{x_3(t)}{x_3(t) + x_2(t)} \right), \\
\dot{x}_3 &= x_3(t) \left( -1 + 2 \int_0^\infty x_2 e^{-x_2s} \frac{x_2(t - s)}{x_3(t - s) + x_2(t - s)} ds \right). 
\end{align*}
\]

(3.16)

System (3.16) has a unique positive equilibrium $E^* \left( \frac{3}{4}, \frac{3}{4}, \frac{3}{4} \right)$. It is easy to verify that system (3.16) satisfies all the assumptions in (H1)–(H8) when $x_1 > 2$ and $x_2 > 1$. Using Theorem 2.1, we know that system (3.16) is uniformly persistent. Using Theorem 3.1, we know that the positive equilibrium $E^* \left( \frac{3}{4}, \frac{3}{4}, \frac{3}{4} \right)$ is globally attractive provided that $x_1 > 21$ and $x_2 > 18$.

Acknowledgements

The first author wishes to thank the Department of Mathematics, University of Dundee for the hospitality, support, and the excellent working conditions provided to him during his visit in Dundee.
References