Global stability of a Lotka–Volterra type predator–prey model with stage structure and time delay

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Abstract

A delayed Lotka–Volterra type predator–prey model with stage structure for predator is investigated. It is assumed in the model that the individuals in the predator population may belong to one of two classes: the immatures and the matures, the age to maturity is presented by a time delay, and that the immature predators do not have the ability to prey. By analyzing characteristic equations and using an iterative technique, a set of easily verifiable sufficient conditions are derived for the local and global stability of the nonnegative equilibria of the model. Numerical simulations are carried out to illustrate the validity of our results.

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1. Introduction

Stage-structured models have received great attention in recent years. The pioneering work of Aiello and Freedman [1] on a single species growth model with stage structure represents a mathematically more careful and biologically meaningful formulation approach. In [1], a model of single species population growth incorporating stage structure as a reasonable generalization of the
classical logistic model was formulated and discussed. This model assumes an average age to maturity which appears as a constant time delay reflecting a delayed birth of immatures and a reduced survival of immatures to their maturity. The model takes the form

\[
\begin{align*}
  \dot{x}_i(t) &= ax_m(t) - \gamma x_i(t) - \alpha e^{-\tau t} x_m(t - \tau), \\
  \dot{x}_m(t) &= \alpha e^{-\tau t} x_m(t - \tau) - \beta x_m^2(t), \quad t > \tau,
\end{align*}
\]

where \(x_i(t)\) denotes the immature population density, \(x_m(t)\) represents the mature population density, \(a > 0\) represents the birth rate, \(\gamma > 0\) is the immature death rate, \(\beta > 0\) is the mature death and overcrowding rate, \(\tau\) is the time to maturity. The term \(\alpha e^{-\tau t} x_m(t - \tau)\) represents the immatures who were born at time \(t - \tau\) and survive at time \(t\) (with the immature death rate \(\gamma\)), and therefore represents the transformation of immatures to matures. Following the way of Aiello and Freedman, many authors studied different kinds of stage-structured models and some significant work was carried out (see, for example, [2–12]).

Lotka–Volterra type predator–prey systems are very important in the models of multi-species populations interactions and have been studied by many authors (see, for example, [13–15]). We note that it is assumed in the classical predator–prey model that each individual predator admits the same ability to prey. This assumption seems not to be realistic for many animals. In the natural world, there are many species whose individuals have a life history that takes them through two stages, immature and mature, where immature predators are raised by their parents, and the rate they attack at prey and the reproductive rate can be ignored. Recently, Wang and Chen [9] and Magnusson [10] proposed and investigated predator–prey models with stage structure to analyze the effect of a stage structure for predator on the dynamics of predator–prey models. But these models ignore the duration time of immature predators. In [11], a predator–prey model with stage structure for predator was derived and discussed by Wang et al. to show the effect of the duration time of immature predator on the global dynamics of predator–prey system. Sufficient conditions were derived in [11] for the permanence and global stability of a positive equilibrium of the proposed model.

Motivated by the recent works of Aiello and Freedman [1] and Wang et al. [11], in this paper, we are concerned with the effect of stage structure for predator on Lotka–Volterra type predator–prey system. To do so, we study the following delayed differential system

\[
\begin{align*}
  \dot{x}(t) &= x(t)(r_1 - a_1 x(t) - a_{12} y_2(t)), \\
  \dot{y}_1(t) &= ax(t) y_2(t) - \gamma y_1(t) - \alpha e^{-\tau t} x(t - \tau) y_2(t - \tau), \\
  \dot{y}_2(t) &= \alpha e^{-\tau t} x(t - \tau) y_2(t - \tau) - r_2 y_2(t) - a_{22} y_2^2(t),
\end{align*}
\]

where \(x_i(t)\) denotes the immature population density, \(x_m(t)\) represents the mature population density, \(a > 0\) represents the birth rate, \(\gamma > 0\) is the immature death rate, \(\beta > 0\) is the mature death and overcrowding rate, \(\tau\) is the time to maturity. The term \(\alpha e^{-\tau t} x_m(t - \tau)\) represents the immatures who were born at time \(t - \tau\) and survive at time \(t\) (with the immature death rate \(\gamma\)), and therefore represents the transformation of immatures to matures. Following the way of Aiello and Freedman, many authors studied different kinds of stage-structured models and some significant work was carried out (see, for example, [2–12]).

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where \( x(t) \) denotes the density of prey population at time \( t \), \( y_1(t) \) and \( y_2(t) \) represent the densities of immature and mature individual predators at time \( t \), respectively. \( a_{11}, a_{12}, a_{22}, r_1, r_2, \alpha \) and \( \gamma \) are positive constants, \( \tau \) is a nonnegative constant. The model is derived under the following assumptions.

(A1) The prey population: the growth of the species is of Lotka–Volterra nature. \( r_1 \) is the intrinsic growth rate, \( a_{11} \) is the intra-specific competition rate.

(A2) The predator population: \( a_{12} \) is the capturing rate of mature predator, \( a_2 = a_{12} \) is the conversion rate of nutrients into the reproduction of the mature predator, \( a_{22} \) is the death and overcrowding rate of mature predators, the death rate of the immature population is proportional to the existing immature population with a proportionality \( \gamma \). The term \( \alpha e^{-\tau t} x(t-\tau) y_2(t-\tau) \) represents the number of immature predators that were born at time \( t-\tau \) which still survive at time \( t \) and are transferred from the immature stage to the mature stage at time \( t \). It is assumed in (1.2) that immature individual predators do not feed on prey and do not have the ability to reproduce.

The initial conditions for system (1.2) take the form of

\[
\begin{align*}
\phi(0) > 0, & \quad \psi_1(0) > 0, & \quad \psi_2(0) > 0, \\
x(\theta) = \phi(\theta), & \quad y_1(\theta) = \psi_1(\theta), & \quad y_2(\theta) = \psi_2(\theta),
\end{align*}
\]

where \( (\phi(\theta), \psi_1(\theta), \psi_2(\theta)) \in C([-\tau, 0], \mathbb{R}_{+0}^3) \), the Banach space of continuous functions mapping the interval \([-\tau, 0]\) into \( \mathbb{R}_{+0}^3 \), where \( \mathbb{R}_{+0}^3 = \{(x_1, x_2, x_3) : x_i \geq 0, i = 1, 2, 3\}\).

For continuity of the initial conditions, we further require

\[
\begin{align*}
y_1(0) &= \int_{-\tau}^{0} \alpha \phi(s) \psi_2(s) e^{\alpha s} \, ds.
\end{align*}
\]

The paper is organized as follows. In the next section, we will discuss the positivity and boundedness of solutions to system (1.2). In Section 3, we are concerned with the local stability of the nonnegative equilibria of system (1.2). In Section 4, by using an iterative technique, a set of easily verifiable sufficient conditions are derived for the global stability of the feasible equilibria of system (1.2). Numerical simulations are presented to illustrate the feasibility of our main results. A brief discussion is given in Section 5 to conclude this work.

2. Positivity and boundedness of solutions

In this section, we are concerned with the positivity and boundedness of solutions to system (1.2) with initial conditions (1.3) and (1.4).

**Theorem 2.1.** Solutions of system (1.2) with initial conditions (1.3) and (1.4) are positive for all \( t \geq 0 \).
Proof. Let \((x(t), y_1(t), y_2(t))\) be a solution of system (1.2) with initial conditions (1.3) and (1.4). Let us first consider \(y_2(t)\) for \(t \in [0, \tau]\). Noting that \(\psi_1(\sigma) > 0\), \(\psi_2(\sigma) > 0\) for \(\sigma \in [-\tau, 0]\), we derive from the third equation of system (1.2) that

\[
\dot{y}_2(t) = z e^{-\tau} \phi(t - \tau) \psi_2(t - \tau) - r_2 y_2(t) - a_{22} y_2^3(t) \geq - r_2 y_2(t) - a_{22} y_2^3(t).
\]

(2.1)

Therefore, a standard comparison argument shows that

\[
y_2(t) \geq \frac{r_2 y_2(0)}{a_{22} y_2(0) (e^{\tau} - 1) + r_2} > 0.
\]

It follows from the first equation of system (1.2) that

\[
x(t) = x(0) \exp \left\{ \int_0^t \left[ r_1 - a_{11} x(s) - a_{12} y_2(s) \right] ds \right\} > 0 \quad \text{for} \ t \in [0, \tau].
\]

By (1.4) and the second equation of system (1.2), we derive

\[
y_1(t) = \int_{t-\tau}^t z e^{-\tau} x(s) y_2(s) ds > 0 \quad \text{for} \ t \in [0, \tau].
\]

(2.2)

Continuing this process, in a similar way we treat the intervals \([\tau, 2\tau], \ldots, [n\tau, (n+1)\tau], \ n \in N\). Thus, \(x(t) > 0, y_1(t) > 0\) and \(y_2(t) > 0\) for all \(t \geq 0\). The proof is complete. □

Theorem 2.2. Positive solutions of system (1.2) with initial conditions (1.3) and (1.4) are ultimately bounded.

Proof. Suppose \(z(t) = (x(t), y_1(t), y_2(t))\) is any positive solution of system (1.2) with initial conditions (1.3) and (1.4).

Define

\[
\rho(t) = z x(t) + a_{12} y_1(t) + a_{12} y_2(t).
\]

Calculating the derivative of \(\rho(t)\) along positive solutions of (1.2), it follows

\[
\dot{\rho}(t) = r_1 z x(t) - z a_{11} x^2(t) - \gamma a_{12} y_1(t) - a_{12} y_2 y_3(t) - a_{12} a_{22} y_2^3(t)
\]

\[
\leq - a_{12} A \rho(t) + (r_1 x + a_{12} A) x(t) - z a_{11} x^2(t)
\]

\[
\leq - a_{12} A \rho(t) + \frac{(r_1 x + a_{12} A)^2}{4 z a_{11}},
\]

where \(A = \min\{r_2, \gamma\}\). Therefore, we have

\[
\limsup_{t \to +\infty} \rho(t) \leq \frac{(r_1 x + a_{12} A)^2}{4 a_{11} a_{12} x A}.
\]

This completes the proof. □
3. Local asymptotic stability

By setting \( \dot{x} = y_1 = y_2 = 0 \) in system (1.2), it is easy to see that system (1.2) has at least two nonnegative equilibria: \( E_0(0, 0, 0) \), \( E_1(r_1/a_{11}, 0, 0) \). Further, if the following holds

(H1) \( r_1 x e^{-r_1 t} > r_2 a_{11} \),

then system (1.2) has a unique positive equilibrium \( E^*(x^*, y_1^*, y_2^*) \), where

\[
    x^* = \frac{r_1 a_{22} + r_2 a_{12}}{a_{11} a_{22} + a_{12} x e^{-r_1 t}}, \quad y_1^* = \frac{\lambda x^* y_2^*}{\gamma} \left(1 - e^{-r_1 t}\right), \quad y_2^* = \frac{r_1 x e^{-r_1 t} - r_2 a_{11}}{a_{11} a_{22} + a_{12} x e^{-r_1 t}}.
\]

(3.1)

**Theorem 3.1.** The nonnegative equilibrium \( E_0(0, 0, 0) \) is always unstable. If \( r_1 x e^{-r_1 t} < r_2 a_{11} \), the nonnegative equilibrium \( E_1(r_1/a_{11}, 0, 0) \) is locally asymptotically stable; if \( r_1 x e^{-r_1 t} > r_2 a_{11} \), \( E_1 \) is locally unstable.

**Proof.** The characteristic equation of the equilibrium \( E_0(0, 0, 0) \) takes the form

\[
    (\lambda - r_1)(\lambda + r_2)(\lambda + \gamma) = 0.
\]

Clearly, \( \lambda = r_1 \) is always a positive eigenvalue, \( \lambda = -r_2, \lambda = -\gamma \) are always two negative eigenvalues. Therefore, the equilibrium \( E_0(0, 0, 0) \) is always unstable.

The characteristic equation of the equilibrium \( E_1(r_1/a_{11}, 0, 0) \) is of the form

\[
    (\lambda + r_1)(\lambda + \gamma)\left(\lambda + r_2 - \frac{r_1 x}{a_{11}} e^{-r_1 t} e^{-\lambda t}\right) = 0.
\]

Clearly, \( \lambda = -r_1 \) and \( \lambda = -\gamma \) are always two negative eigenvalues. All other eigenvalues are given by the solutions of equation

\[
    \lambda + r_2 - \left(\frac{r_1 x}{a_{11}}\right) e^{-r_1 t} e^{-\lambda t} = 0. \tag{3.2}
\]

If \( r_1 x e^{-r_1 t} > r_2 a_{11} \), we will show that there exists a positive eigenvalue. Let \( f(\lambda) = \lambda + r_2 - \left(\frac{r_1 x}{a_{11}}\right) e^{-r_1 t} e^{-\lambda t} \). Then it follows that

\[
    f(0) = r_2 - \left(\frac{r_1 x}{a_{11}}\right) e^{-r_1 t} < 0, \quad \lim_{t \to +\infty} f(\lambda) = +\infty, \quad f'(\lambda) = 1 + \frac{r_1 x}{a_{11}} e^{-r_1 t} e^{-\lambda t} > 0.
\]

Hence, \( f(\lambda) = 0 \) has a positive root. Accordingly, the equilibrium \( E_1 \) is unstable.

If \( r_1 x e^{-r_1 t} < r_2 a_{11} \), we now claim that the solutions of \( f(\lambda) = 0 \) have only negative real parts. Suppose that \( \Re \lambda \geq 0 \). Then it follows from (3.2) that
Lemma 3.1. We need the following results from [4].

By applying Lemma 3.1, we have \( \text{Re} \lambda < 0 \), a contradiction. Thus we have \( \text{Re} \lambda < 0 \). Hence, if \( r_1 \alpha e^{-\gamma \tau} < r_2 a_{11} \), the equilibrium \( E_1(r_1/a_{11}, 0, 0) \) is locally asymptotically stable. This completes the proof. \( \square \)

In the following we will discuss the local asymptotic stability of the positive equilibrium \( E^*(x^*, y_1^*, y_2^*) \) of system (1.2). The characteristic equation of the positive equilibrium \( E^* \) takes the form

\[
(\lambda + \gamma) \left\{ \lambda^2 + a_1 \lambda + a_2 + (b_1 \lambda + b_2) e^{-\lambda \tau} \right\} = 0,
\]

where

\[
\begin{align*}
a_1 &= a_{11} x^* + x e^{-\gamma \tau} x^* + a_{22} y_2^*, \\
b_1 &= -x e^{-\gamma \tau} x^*, \\
a_2 &= a_{11} x^*(x e^{-\gamma \tau} x^* + a_{22} y_2^*), \\
b_2 &= x e^{-\gamma \tau} x^*(a_{12} y_2^* - a_{11} x^*).
\end{align*}
\]

Clearly, \( \lambda = -\gamma \) is always a negative eigenvalue. All other eigenvalues are given by solutions of the following equation

\[
\lambda^2 + a_1 \lambda + a_2 + (b_1 \lambda + b_2) e^{-\lambda \tau} = 0,
\]

where \( a_1, a_2, b_1 \) and \( b_2 \) are defined in (3.4).

In order to discuss the distribution of characteristic roots of Eq. (3.5), we need the following results from [4].

**Lemma 3.1.** In (3.5), assume that \( a_2 + b_2 \neq 0, a_1^2 + b_1^2 + b_2^2 \neq 0 \). The number of different positive (negative) imaginary roots of (3.5) can be zero, one, or two only.

1. If \( a_1^2 > b_2^2 \) and \( b_1^2 + 2a_2 - a_1^2 < 2 \sqrt{a_1^2 - b_2^2} \), then (3.5) (for \( \tau > 0 \)) has the same stability or instability as when \( \tau = 0 \).
2. If \( a_1^2 \leq b_2^2 \), and if (3.5) is unstable when \( \tau = 0 \), then it remains unstable for \( \tau > 0 \). If (3.5) is stable when \( \tau = 0 \), then it remains stable until \( \tau = \tau_0 \) and becomes unstable afterwards, where \( \tau_0 = \theta_1/\omega_+, \theta_1 \in [0, 2\pi], \omega_+ > 0 \),

\[
\omega_+^2 = \frac{1}{2} \left\{ b_1^2 + 2a_2 - a_1^2 + [(b_1^2 + 2a_2 - a_1^2)^2 - 4(a_2^2 - b_2^2)]^{1/2} \right\},
\]

\[
\cos \theta_1 = -\frac{a_1 b_1 \omega_+^2 + b_2 (a_2 - \omega_+^2)}{b_2^2 + b_1^2 \omega_+^2}.
\]

We now discuss the local stability of the positive equilibrium \( E^* \) of system (1.2) by applying Lemma 3.1.
On substituting $s = 0$ into (3.5), we derive that

$$
\lambda^2 + (a_{11}x^* + a_{22}y^*_2)\lambda + x^* y^*_2 (a_{11}a_{22} + a_{12} x e^{-\tau^*}) = 0.
$$

(3.7)

Therefore, we see that the positive equilibrium $E^*$ of (1.2) is locally asymptotically stable when $\tau = 0$.

It is easy to verify that

$$
b_1^2 + 2a_2 - a_1^2 = -a_{11}^2 x^*- a_{22}^2 y_2^* - 2a_{22} x e^{-\tau^*} x^* y^*_2 < 0,
$$

and

$$
a_2^2 - b_2^2 = x^* y_2^*[2a_{11} x e^{-\tau^*} (r_1 a_{22} + r_2 a_{12})
+ (r_1 x e^{-\tau^*} - r_2 a_{11})(a_{11} a_{22} - a_{12} x e^{-\tau^*})].
$$

By Lemma 3.1, we therefore obtain the following results on the stability of the positive equilibrium $E^*$.

**Theorem 3.2.** Let (H1) hold. Denote

$$
H(\tau) = 2a_{11} x e^{-\tau^*} (r_1 a_{22} + r_2 a_{12}) + (r_1 x e^{-\tau^*} - r_2 a_{11})(a_{11} a_{22} - a_{12} x e^{-\tau^*}).
$$

(3.8)

(i) If $H(\tau) > 0$, then the positive equilibrium $E^*$ of system (1.2) is locally asymptotically stable.

(ii) If $H(\tau) \leq 0$, then there exists a positive number $\tau_0$ such that the positive equilibrium $E^*$ of system (1.2) is locally asymptotically stable if $0 \leq \tau < \tau_0$ and is locally unstable if $\tau > \tau_0$.

**4. Global stability of nonnegative equilibria**

In this section, we are concerned with the global stability of the nonnegative equilibria $E^*$ and $E_1$ of system (1.2) by using an iterative technique developed by some authors (see, for example, [7,8,11,12]). In what follows, we need the following results from [7].

**Lemma 4.1.** Consider the following equation

$$
\dot{x}(t) = ax(t - \tau) - bx(t) - cx^2(t),
$$

where $a, b, c$ and $\tau$ are positive constants, $x(t) > 0$ for $t \in [-\tau, 0]$. We have

(i) If $a > b$, then $\lim_{t \to +\infty} x(t) = (a - b)/c$;

(ii) If $a < b$, then $\lim_{t \to +\infty} x(t) = 0$. 
We are now able to state and prove our result on the global stability of the positive equilibrium $E^*$ of system (1.2).

**Theorem 4.1.** Let (H1) hold. Assume further that $a_{11}a_{22} > a_{12}xe^{-\tau}$, then the positive equilibrium $E^*(x^*, y_1^*, y_2^*)$ of system (1.2) is globally asymptotically stable.

**Proof.** We note that if $a_{11}a_{22} > a_{12}xe^{-\tau}$, then it follows that $H(\tau) > 0$. Therefore, by Theorem 3.1 we see that the positive equilibrium $E^*$ is locally asymptotically stable. Thus, it suffices to verify that $E^*$ is globally attractive.

Let

$$U_1 = \limsup_{t \to +\infty} x(t), \quad V_1 = \liminf_{t \to +\infty} x(t),$$

$$U_2 = \limsup_{t \to +\infty} y_2(t), \quad V_2 = \liminf_{t \to +\infty} y_2(t).$$

In the following we will claim that $U_1 = V_1 = x^*$, $U_2 = V_2 = y_2^*$.

It follows from the first equation of system (1.2) that

$$\dot{x}(t) \leq x(t)(r_1 - a_{11}x(t)).$$

A standard comparison argument shows that

$$U_1 = \limsup_{t \to +\infty} x(t) \leq \frac{r_1}{a_{11}} := N_1^x.$$

Then for $\varepsilon > 0$ sufficiently small there exists a $T_{11} > 0$ such that if $t > T_{11}$, $x(t) \leq N_1^x + \varepsilon$. We therefore derive from the third equation of system (1.2) that for $t > T_{11} + \tau$,

$$\dot{y}_2(t) \leq xe^{-\tau}(N_1^x + \varepsilon)y_2(t - \tau) - r_2y_2(t) - a_{22}y_2^2(t).$$

Consider the following auxiliary equation

$$\dot{v}(t) = xe^{-\tau}(N_1^x + \varepsilon)v(t - \tau) - r_2v(t) - a_{22}v^2(t). \quad (4.1)$$

By Lemma 4.1 it follows from (4.1) that

$$\lim_{t \to +\infty} v(t) = \frac{xe^{-\tau}(N_1^x + \varepsilon) - r_2}{a_{22}}.$$

By comparison we obtain that

$$U_2 = \limsup_{t \to +\infty} y_2(t) \leq \frac{xe^{-\tau}(N_1^x + \varepsilon) - r_2}{a_{22}}.$$

Since this is true for any $\varepsilon > 0$ sufficiently small, it follows that $U_2 \leq N_1^y$, where

$$N_1^y = \frac{xe^{-\tau}N_1^x - r_2}{a_{22}}.$$
Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_{12} > T_{11} + \tau$ such that if $t > T_{12},$

\[ y_2(t) \leq N_1^\varepsilon + \varepsilon. \]

We derive from the first equation of system (1.2) that for $t > T_{12},$

\[ \dot{x}(t) \geq x(t)[r_1 - a_{11}x(t) - a_{12}(N_1^\varepsilon + \varepsilon)]. \]

A standard comparison argument shows that

\[ V_1 = \lim_{t \to +\infty} \inf x(t) \geq \frac{r_1 - a_{12}(N_1^\varepsilon + \varepsilon)}{a_{11}}. \]

Since this is true for any $\varepsilon > 0$ sufficiently small, it follows that $V_1 \geq M_1^\varepsilon,$ where

\[ M_1^\varepsilon = \frac{r_1 - a_{12}N_1^\varepsilon}{a_{11}}. \]

Therefore, for $\varepsilon > 0$ sufficiently small, there is a $T_{21} > T_{12}$ such that if $t > T_{21},$ $x(t) \geq M_1^\varepsilon - \varepsilon.$ We obtain from the third equation of system (1.2) that for $t > T_{21} + \tau,$

\[ \dot{y}_2(t) \geq \alpha e^{\gamma t}(M_1^\varepsilon - \varepsilon)y_2(t - \tau) - r_2y_2(t) - a_{22}y_2^2(t). \]

Consider the following auxiliary equation

\[ \dot{v}(t) = \alpha e^{\gamma t}(M_1^\varepsilon - \varepsilon)v(t - \tau) - r_2v(t) - a_{22}v^2(t). \]

By Lemma 4.1, it follows that

\[ \lim_{t \to +\infty} v(t) = \frac{\alpha e^{\gamma t}(M_1^\varepsilon - \varepsilon) - r_2}{a_{22}}. \]

By comparison we derive that

\[ V_2 = \lim_{t \to +\infty} y_2(t) \geq \frac{\alpha e^{\gamma t}(M_1^\varepsilon - \varepsilon) - r_2}{a_{22}}. \]

Since this is true for any $\varepsilon > 0$ sufficiently small, it follows that $V_2 \geq M_1^\varepsilon,$ where

\[ M_1^\varepsilon = \frac{\alpha e^{\gamma t}M_1^\varepsilon - r_2}{a_{22}}. \]

Therefore, for $\varepsilon > 0$ sufficiently small, there exists a $T_{22} > T_{21} + \tau$ such that if $t > T_{22},$ $y_2(t) \geq M_1^\varepsilon - \varepsilon.$

We derive from the first equation of system (1.2) that for $t > T_{22},$

\[ \dot{x}(t) \leq x(t)[r_1 - a_{11}x(t) - a_{12}(M_1^\varepsilon - \varepsilon)]. \]

A standard comparison argument shows that

\[ U_1 = \limsup_{t \to +\infty} x(t) \leq \frac{r_1 - a_{12}(M_1^\varepsilon - \varepsilon)}{a_{11}}. \]
Since this is true for any \( \varepsilon > 0 \) sufficiently small, it follows that \( U_1 \leq N_1^y \), where

\[
N_1^y = \frac{r_1 - a_{12}M_1^y}{a_{11}}.
\]

Hence, for \( \varepsilon > 0 \) sufficiently small, there is a \( T_{31} \gg T_{22} \) such that if \( t > T_{31} \), \( x(t) \leq N_1^y + \varepsilon \). We therefore obtain from the third equation of system (1.2) that for \( t > T_{31} + \tau \),

\[
\dot{y}_2(t) \leq \alpha e^{-\gamma t}(N_2^x + \varepsilon)y_2(t) - r_2y_2(t) - a_{22}y_2^2(t).
\]

Consider the following auxiliary equation

\[
\dot{v}(t) = \alpha e^{-\gamma t}(N_2^x + \varepsilon)v(t) - r_2v(t) - a_{22}v^2(t).
\]

By Lemma 4.1 we get

\[
\lim_{t \to +\infty} v(t) = \frac{\alpha e^{-\gamma t}(N_2^x + \varepsilon) - r_2}{a_{22}}.
\]

By comparison, it follows that

\[
U_2 = \limsup_{t \to +\infty} y_2(t) \leq \frac{\alpha e^{-\gamma t}(N_2^x + \varepsilon) - r_2}{a_{22}}.
\]

Therefore, for \( \varepsilon > 0 \) sufficiently small, there exists a \( T_{32} \gg T_{31} + \tau \) such that if \( t > T_{32} \), \( y_2(t) \leq N_2^y + \varepsilon \).

It follows from the first equation of system (1.2) that for \( t > T_{32} \),

\[
\dot{x}(t) \geq x(t)[r_1 - a_{11}x(t) - a_{12}(N_2^x + \varepsilon)].
\]

A comparison argument yields

\[
V_1 = \liminf_{t \to +\infty} x(t) \geq \frac{r_1 - a_{12}(N_2^x + \varepsilon)}{a_{11}}.
\]

Since this is true for any \( \varepsilon > 0 \) sufficiently small, it follows that \( V_1 \geq M_2^x \), where

\[
M_2^x = \frac{r_1 - a_{12}N_2^x}{a_{11}}.
\]

Therefore, for any \( \varepsilon > 0 \) sufficiently small, there is a \( T_{41} \gg T_{32} \) such that if \( t > T_{41} \), \( x(t) \geq M_2^x - \varepsilon \).

It follows from the third equation of system (1.2) that for \( t > T_{41} + \tau \)

\[
\dot{y}_2(t) \geq \alpha e^{-\gamma t}(M_2^x - \varepsilon)y_2(t - \tau) - r_2y_2(t) - a_{22}y_2^2(t).
\]

Consider the following auxiliary equation

\[
\dot{v}(t) = \alpha e^{-\gamma t}(M_2^x - \varepsilon)v(t - \tau) - r_2v(t) - a_{22}v^2(t).
\]
By Lemma 4.1 we have
\[ \lim_{t \to +\infty} v(t) = \frac{\alpha e^{-\tau t} (M_2^* - \varepsilon) - r_2}{a_{22}}. \]

By comparison, it follows that
\[ V_2 = \liminf_{t \to +\infty} y_2(t) \geq \frac{\alpha e^{-\tau t} (M_2^* - \varepsilon) - r_2}{a_{22}}. \]

Since this is true for any \( \varepsilon > 0 \) sufficiently small, it follows that \( V_2 \geq M_2^* \), where
\[ M_2^* = \frac{\alpha e^{-\tau t} M_2^* - r_2}{a_{22}}. \]

Continuing this process, we obtain four sequences \( M_n^x, M_n^y, N_n^x, N_n^y, n = 1, 2, \ldots \), such that for \( n \geq 2 \),
\[ N_n^x = \frac{r_1 - a_{12} M_{n-1}^x}{a_{11}}, \]
\[ N_n^y = \frac{\alpha e^{-\tau t} N_n^x - r_2}{a_{22}}, \]
\[ M_n^x = \frac{r_1 - a_{12} N_n^y}{a_{11}}, \]
\[ M_n^y = \frac{\alpha e^{-\tau t} M_n^x - r_2}{a_{22}}. \] (4.2)

Clearly we have
\[ M_n^x \leq V_1 \leq U_1 \leq N_n^x, \quad M_n^y \leq V_2 \leq U_2 \leq N_n^y. \] (4.3)

It follows from (4.2) that
\[ N_{n+1}^x = \frac{(a_{11} a_{22} - a_{12} \alpha e^{-\tau t})(r_1 a_{22} + r_2 a_{12})}{a_{11}^2 a_{22}^2} + \frac{a_{12}^2 x^2 e^{-2\tau t}}{a_{11}^2 a_{22}^2} N_n^x \] (4.4)

Noting that \( N_n^x \geq x^* \) and \( a_{11} a_{22} > a_{12} \alpha e^{-\tau t} \), we derive from (4.4) that
\[ N_{n+1}^x - N_n^x = \frac{(a_{11}^2 a_{22}^2 - a_{12}^2 x^2 e^{-2\tau t})x^*}{a_{11}^2 a_{22}^2} \left( \frac{a_{12}^2 x^2 e^{-2\tau t}}{a_{11}^2 a_{22}^2} - 1 \right) N_n^x \]
\[ \leq \frac{(a_{11}^2 a_{22}^2 - a_{12}^2 x^2 e^{-2\tau t})x^*}{a_{11}^2 a_{22}^2} \left( \frac{a_{12}^2 x^2 e^{-2\tau t}}{a_{11}^2 a_{22}^2} - 1 \right) x^* = 0. \]

Thus, the sequence \( N_n^x \) is monotonically decreasing. Therefore we see that \( \lim_{n \to +\infty} N_n^x \) exists. Taking \( n \to +\infty \), it follows from (4.4) that
\[ \lim_{n \to +\infty} N_n^x = \frac{r_1 a_{22} + r_2 a_{12}}{a_{11} a_{22} + a_{12} \alpha e^{-\tau t}} = x^*. \] (4.5)
We therefore derive from (4.2) and (4.5) that
\[
\lim_{n \to +\infty} N_n^x = y_1^*, \quad \lim_{n \to +\infty} M_n^x = x^*, \quad \lim_{n \to +\infty} M_n^y = y_2^*.
\] (4.6)

It follows from (4.5) and (4.6) that
\[U_1 = V_1 = x^*, \quad U_2 = V_2 = y_2^*.
\]

We therefore have
\[
\lim_{t \to +\infty} x(t) = x^*, \quad \lim_{t \to +\infty} y_2(t) = y_2^*.
\]

We note from (2.2) that
\[
y_1(t) = \frac{\alpha \int_0^t e^{\gamma t} x(s) y_2(s) \, ds}{e^{\gamma t}}.
\] (4.7)

By using L’Hospital’s rule, it follows from (4.7) that
\[
\lim_{t \to +\infty} y_1(t) = \lim_{t \to +\infty} \frac{\alpha (e^{\gamma t} x(t) y_2(t) - e^{\gamma (t-\tau)} x(t-\tau) y_2(t-\tau))}{\gamma e^{\gamma t}}
\]
\[
= \frac{\alpha}{\gamma} \lim_{t \to +\infty} \{ x(t) y_2(t) - e^{-\gamma t} x(t-\tau) y_2(t-\tau) \}
\]
\[
= \frac{\alpha}{\gamma} (1 - e^{-\gamma t}) x^* y_2^* = y_1^*.
\] (4.8)

This completes the proof. \(\square\)

We are now in a position to discuss the global stability of the nonnegative equilibrium \(E_1(r_1/a_{11}, 0, 0)\) of system (1.2) when \(r_1 \alpha e^{-\gamma t} < r_2 a_{11}\).

**Theorem 4.2.** If \(r_1 \alpha e^{-\gamma t} < r_2 a_{11}\), the nonnegative equilibrium \(E_1(r_1/a_{11}, 0, 0)\) of system (1.2) is globally asymptotically stable. In this case, both of the immature and mature predator populations will go to extinction.

**Proof.** By Theorem 3.1 we see that the equilibrium \(E_1(r_1/a_{11}, 0, 0)\) is locally asymptotically stable if \(r_1 \alpha e^{-\gamma t} < r_2 a_{11}\). Hence, it is sufficient to show that the equilibrium \(E_1\) is globally attractive.

It follows from the first equation of system (1.2) that
\[
\dot{x}(t) \leq x(t) (r_1 - a_{11} x(t)).
\]

A standard comparison argument shows that
\[
\limsup_{t \to +\infty} x(t) \leq \frac{r_1}{a_{11}}. \quad (4.9)
\]

Therefore, for \(\varepsilon > 0\) sufficiently small satisfying \(\alpha e^{-\gamma t} (r_1/a_{11} + \varepsilon) < r\), there is a \(T_1 > 0\) such that if \(t > T_1\), \(x(t) \leq r_1/a_{11} + \varepsilon\).
We derive from the third equation of system (1.2) for \( t > T_1 + \tau \) that
\[
\dot{y}_2(t) \leq \alpha e^{-\tau \tau}(r_1/a_{11} + \varepsilon)y_2(t - \tau) - r_2y_2(t) - a_{22}y_2^2(t).
\]
Consider the following auxiliary equation
\[
\dot{v}(t) = \alpha e^{-\tau \tau}(r_1/a_{11} + \varepsilon)v(t - \tau) - r_2v(t) - a_{22}v^2(t).
\]
Noting that \( \alpha e^{-\tau \tau}(r_1/a_{11} + \varepsilon) < r \), by Lemma 4.1 it follows that
\[
\lim_{t \to +\infty} v(t) = 0.
\]
By comparison, we derive that
\[
\lim_{t \to +\infty} y_2(t) = 0.
\]
Hence, for any \( \varepsilon > 0 \) sufficiently small, there exists a \( T_2 > T_1 + \tau \) such that if
\( t > T_2, 0 < y_2(t) < \varepsilon \).

It follows from the first equation of system (1.2) that for \( t > T_2 \)
\[
\dot{x}(t) \geq x(t)(r_1 - a_{11}x(t) - a_{12} \varepsilon),
\]
which yields
\[
\liminf_{t \to +\infty} x(t) \geq \frac{r_1 - a_{12} \varepsilon}{a_{11}}.
\]
Since it is true for any \( \varepsilon > 0 \) sufficiently small, we have
\[
\liminf_{t \to +\infty} x(t) \geq \frac{r_1}{a_{11}},
\]
which, together with (4.9), leads to
\[
\lim_{t \to +\infty} x(t) = \frac{r_1}{a_{11}}.
\]
In a similar way in the proof of (4.8), it is easy to show that \( \lim_{t \to +\infty} y_1(t) = 0 \).

The proof is complete. \( \square \)

Finally, we give two examples to illustrate the feasibility of our main results in Theorems 3.2, 4.1 and 4.2.

**Example 1.** As an example, we consider the following system
\[
\begin{align*}
\dot{x}(t) & = x(t)(8 - 4x(t) - 5y_2(t)), \\
y_1(t) & = 6x(t)y_2(t) - y_1(t) - 6e^{-\tau}x(t - \tau)y_2(t - \tau), \\
y_2(t) & = 6e^{-\tau}x(t - \tau)y_2(t - \tau) - 1/4 y_2(t) - 5y_2^2(t),
\end{align*}
\]
(4.10)
where \( \tau > 0 \) is a constant time delay. It is easy to verify that if \( \tau < \ln 48 \), then system (4.10) has three nonnegative equilibria: \( E_0(0, 0, 0) \), \( E_1(2, 0, 0) \) and \( E^*(x^*, y_1^*, y_2^*) \), where

\[
x^* = \frac{33}{8(2 + 3e^{-\tau})}, \quad y_1^* = 6x^*y_2^*(1 - e^{-\tau}), \quad y_2^* = \frac{48e^{-\tau} - 1}{20 + 30e^{-\tau}}.
\]

By Theorem 4.1, we see that if \( \ln(3/2) < \tau < \ln 48 \), the positive equilibrium \( E^* \) of system (4.10) is globally asymptotically stable. By Theorem 4.2, we see that if \( \tau > \ln 48 \), then the nonnegative equilibrium \( E_1(2, 0, 0) \) of system (4.10) is globally asymptotically stable. Numerical integration can be carried out by using standard algorithms in MATLAB. Numerical simulations also suggest the facts above (see, Figs. 1 and 2).

**Example 2.** Consider the following system

\[
\begin{align*}
\dot{x}(t) &= x(t)(1 - x(t) - y_2(t)), \\
\dot{y}_1(t) &= 8x(t)y_2(t) - y_1(t) - 8e^{-\tau}x(t - \tau)y_2(t - \tau), \\
\dot{y}_2(t) &= 8e^{-\tau}x(t - \tau)y_2(t - \tau) - 0.3y_2(t) - \frac{1}{104}y_2^2(t),
\end{align*}
\]

where \( \tau > 0 \) is a constant time delay. It is easy to show that if \( \tau < \ln(80/3) \), system (4.11) admits three nonnegative equilibria: \( E_0(0, 0, 0) \), \( E_1(1, 0, 0) \) and \( E^*(x^*, y_1^*, y_2^*) \), where

\[
x^* = \frac{3001}{1 + 80,000e^{-\tau}}, \quad y_1^* = 8x^*y_2^*(1 - e^{-\tau}), \quad y_2^* = \frac{80,000e^{-\tau} - 3000}{1 + 80,000e^{-\tau}}.
\]

Fig. 1. The temporal solution found by numerical integration of system (4.10) with \( \tau = 2 \) and \( (\phi(0), \psi_1(0), \psi_2(0)) \equiv (0.6, 2.16(1 - e^{-2}), 0.6) \).
If we let \( a_{11}a_{22} - a_{12}x e^{-\tau} = 1/10^4 - 8 e^{-\tau} > 0 \), then \( \tau > \ln 80,000 \). Thus, we failed in applying Theorem 4.1. Denote

\[
\begin{align*}
a_1 &= x^* + 8 e^{-\tau} x^* + \frac{1}{10^4} y_2^*, \\
b_1 &= -8 e^{-\tau} x^*, \\
a_2 &= x^* \left( 8 e^{-\tau} x^* + \frac{1}{10^4} y_2^* \right), \\
b_2 &= 8 e^{-\tau} x^* (y_2^* - x^*), \\
S_0(\tau) &= \tau - \frac{\theta}{\omega_+}, \\
S_1(\tau) &= \tau - \frac{\theta + 2\pi}{\omega_+},
\end{align*}
\]

where \( \theta \) and \( \omega_+ \) are defined in (3.6). The plots of functions \( S_0(\tau), S_1(\tau) \) and \( H(\tau) \) are shown in Fig. 3. Denote \( \tau^* = \ln \{160,000/(9003 + \sqrt{81,024,009})\} \). It is easy to show that \( H(\tau) \leq 0 \) for \( \tau \in [0, \tau^*) \), and \( H(\tau) > 0 \) for \( \tau \in (\tau^*, \ln(80/3)) \). By the stability switch criteria developed in [16], we see that there exist \( \tau_0 \) and \( \tau_1 \), such that if \( 0 \leq \tau < \tau_0 \), the positive equilibrium \( E^* \) of (4.11) is locally asymptotically stable; if \( \tau_0 < \tau < \tau_1 \), \( E^* \) becomes locally unstable, if \( \tau > \tau_1 \), then \( E^* \) becomes stable again until \( \tau = \ln(80/3) \). Afterwards, \( E^* \) disappears, and the nonnegative equilibrium \( E_1(1,0,0) \) is globally asymptotically stable. Numerical simulations also confirm this phenomenon (see, Figs. 4–6, respectively).

5. Discussion

In this paper, motivated by the work of Aiello and Freedman [1], we incorporated stage structure for predator into a Lotka–Volterra type
predator–prey model. By analyzing the characteristic equations, we discussed the local stability of three nonnegative equilibria of system (1.2). By Theorems 3.1 and 3.2, we see that the equilibrium \(E_0(0, 0, 0)\) is always unstable; the equilibrium \(E_1(r_1/a_{11}, 0, 0)\) is locally stable if \(r_1x_1e^{-\tau} < r_2a_{11}\), and is unstable if \(r_1x_1e^{-\tau} > r_2a_{11}\). If \(H(\tau) > 0\), the positive equilibrium \(E^*\) of (1.2) is locally stable, and if \(H(\tau) \leq 0\), there is a \(\tau_0\), such that for \(0 \leq \tau < \tau_0\), \(E^*\) is locally asymptotically stable, if \(\tau > \tau_0\), \(E^*\) becomes unstable. Noting that \(\ln \left\{2r_1a_{12}x/\left(3a_{11}(r_1a_{22} + r_2a_{12}) + a_{11}\sqrt{9(r_1a_{22} + r_2a_{12})^2 - 4r_1r_2a_{12}a_{22}}\right)\right\} < \ln(r_1x/(r_2a_{11}))\),

\[
\begin{align*}
\text{Fig. 3. The plots of functions } S_0(\tau), S_1(\tau) \text{ and } H(\tau). \\
\text{Fig. 4. The temporal solution found by numerical integration of system (4.11) with } \tau = 0.1 \text{ and } (\phi(\theta), \psi_1(\theta), \psi_2(\theta)) \equiv (0.6, 2.88(1 - e^{-0.1}), 0.6).
\end{align*}
\]
there is a \( \tau_0 > \tau_0 \) such that if \( \tau > \tau_1 \), \( E^* \) becomes locally stable again until \( \tau = (1 / \gamma) \ln (r_1 \alpha / (r_2 a_{11})) \). By using iterative technique, we established sufficient conditions for the global asymptotic stability of the positive equilibrium of system (1.2). By Theorem 4.1, we know that if \( (1 / \gamma) \ln (a_{12} \alpha / (a_{11} a_{22})) < \tau < (1 / \gamma) \ln (r_1 \alpha / (r_2 a_{11})) \), the positive equilibrium \( E^* \) is globally asymptotically stable. By Theorem 4.2 we see that if \( \tau > (1 / \gamma) \ln (r_1 \alpha / (r_2 a_{11})) \), then the positive equilibrium \( E^* \) disappears and the nonnegative equilibrium

Fig. 5. The temporal solution found by numerical integration of system (4.11) with \( \tau = 0.8 \) and \( (\phi(\theta), \psi_1(\theta), \psi_2(\theta)) \equiv (0.6, 2.88(1 - e^{-0.5}), 0.6) \).

Fig. 6. The temporal solution found by numerical integration of system (4.11) with \( \tau = 2 \) and \( (\phi(\theta), \psi_1(\theta), \psi_2(\theta)) \equiv (0.6, 2.88(1 - e^{-2}), 0.6) \).
$E_1(r_1/a_{11}, 0, 0)$ is globally asymptotically stable. Therefore, longer delay due to maturity time may lead the predator population to extinction.

References