The covariant transport of electromagnetic fields and its relation to magnetohydrodynamics

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Ohm's law in ideal magnetohydrodynamics (MHD) leads to an induction equation which can be interpreted in terms of magnetic flux being transported by the plasma flow. It is shown that this frozen-in condition is the non-relativistic limit of a corresponding relativistic condition for the electromagnetic field tensor. Several invariants for this type of transport are analyzed. The relativistic formulation also includes a broader class of transporting flows, which may differ from the plasma flow. A classification and interpretation of these transporting flows is given and it is shown that the corresponding evolutions of the electromagnetic field also includes cases of non-ideal MHD evolution. Thus it is possible to find invariants in non-ideal MHD similar to the magnetic flux for ideal plasma flows.

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I. INTRODUCTION

Ideal Ohm's law in magnetohydrodynamics (MHD) implies the conservation of magnetic flux as well as of magnetic lines of force in the plasma flow ([1], [2]). This leads to the intuitive picture of magnetic flux being transported in the flow of the plasma. The mathematical foundation of this picture is given by the curl of Ohm's law which has the form of a Lie-derivative of a divergence free vector field. This implies the conservation of an integral of this vector field over an arbitrary surface, i.e. the magnetic flux. It also implies the preservation of magnetic field lines which was the motivation for a broad investigation on topological properties of magnetic fields (e.g. contributions in [3]). Properties as magnetic helicity, linkage or knottedness of magnetic field lines are invariant for ideal dynamics and therefore characterize the class of fields which are accessible starting from a given state by ideal plasma dynamics ([4], [5]).

In a preceding paper ([2]) we investigated whether it is possible to extend the conservation of magnetic topology beyond ideal MHD to non-ideal plasma dynamics. We found several classes of non-ideal evolution which preserve magnetic topology, but we also found that structural stability of the notion of magnetic topology is a severe restriction in the non-ideal case. Especially in non-ideal plasmas the magnetic topology may depend on the frame of reference. (This effect does not appear for ideal plasmas.) The upshot is that the notion of magnetic topology is not invariant. This naturally raises the problem of finding a concept related to magnetic topology which is invariant under Lorentz transformation.

We will show that there is no way to extend the concept of magnetic field lines invariantly (section II). Instead the notion of magnetic flux does have a relativistic generalization leading to the covariant transport of electromagnetic flux (section III). We will discuss its properties in section IV. Covariant transport requires a transporting flow, examples of such transporting flows and a classification are given in section V.

II. MAGNETIC TOPOLOGY UNDER LORENTZ TRANSFORMATION

In a preceding paper we gave several examples that show that magnetic topology is not invariant under Lorentz transformations. This is a consequence of the fact that the magnetic field is not a vector field in Minkowski-space but a part of the electromagnetic field tensor. Magnetic and electric field components mix under Lorentz transformations and the result does not depend on the topology of the magnetic field alone, but also on the electric field. Nevertheless, it is instructive to see what a relativistic analogue of the magnetic topology conserving equation would look like. Topology conservation in non-relativistic MHD is governed by an equation of the type

$$\partial_t \mathbf{B} + \mathbf{v}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{v} = \lambda\mathbf{B}, \qquad (1)$$

where $\mathbf{v}$ is the velocity of an arbitrary flow, not necessarily the plasma flow (see [1,6,2]). It transports the magnetic field lines, such that they are frozen in the flow. The scalar function λ is only restricted by $\mathbf{B}\cdot\nabla(\lambda + \nabla\cdot\mathbf{v}) = 0$ and may depend on space and time. Mathematically the frozen-in condition is appropriately represented by the Lie-derivative. Applied to various quantities, such as scalars, vectors or tensors, the Lie-derivative yields the corresponding transport equations for the frozen-in condition (see Appendix A or [7]). The above equation can be written with the help of the Lie-derivative [cf. Eq. A14]

$$L_{\mathbf{v}}\mathbf{B} = \mathbf{v}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{v}$$

applied to the vector field $\mathbf{B}$ as

$$\partial_t \mathbf{B} + L_{\mathbf{v}}\mathbf{B} = \lambda\mathbf{B}. \qquad (2)$$

This equation is not covariant as pointed out in [2]. Now the question arises, whether there exists a suitably modified version of Eq. (2) which is covariant. The covariant form of Eq. (2) is the Lie-derivative in Minkowski-space ($I\!M^4$) applied to a hypothetic four-vector field $B^{(4)}$

which is transported in a flow given by the four-velocity $V = (V^0, \mathbf{V})$.

$$
\begin{aligned}
& L_V \mathbf{B}^\alpha = \lambda \mathbf{B}^\alpha \quad \alpha = 0,1,2,3 \\
\Leftrightarrow & \left\{ \begin{array}{l} V^0 \partial_0 B^0 + \mathbf{V} \cdot \nabla B^0 - B^0 \partial_0 V^0 - \mathbf{B} \cdot \nabla V^0 = \lambda B^0 \\ V^0 \partial_0 \mathbf{B} + \mathbf{V} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{V} - B^0 \partial_0 \mathbf{V} = \lambda \mathbf{B} \end{array} \right.
\end{aligned} \tag{3}
$$

(The superscript 0 denotes the time component, while the cartesian vectors are bold face. Also, from here on all velocities are normalized to c, the velocity of light.) However, the only four-vector field the space components of which reduce to $\mathbf{B}$ in the non-relativistic limit is the contraction of $V^{(4)}$ with the dual electromagnetic field tensor $\mathcal{F}_{\mu\nu}$, i.e.

$$
\mathbf{B}^{(4)} = V_\nu \mathcal{F}^{\mu\nu} = (\mathbf{V} \cdot \mathbf{B}, V^0 \mathbf{B} - \mathbf{V} \times \mathbf{E}).
$$

Unfortunately it involves already the transporting velocity $V^{(4)}$ and it is therefore not suitable to define a covariant "topology" of the magnetic field. Moreover, we see that in Eqs. (3) the equation for the space components of the hypothetic field $B^{(4)}$ involves the time component B^0 and vice versa. So the equations for time and space components are coupled which excludes the possibility of simply defining $\mathbf{B}^{(4)} = (0, \mathbf{B})$. An exception is the non-relativistic limit where the Eqs. (3) reduce to

$$
\text{Eqs. (3)} \overset{\gamma \rightarrow 1}{\Rightarrow} \begin{array}{l} \partial_0 B^0 + \mathbf{v} \cdot \nabla B^0 = \lambda B^0 \\ \partial_0 \mathbf{B} + \mathbf{v} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{v} = \lambda \mathbf{B}, \end{array} \tag{4}
$$

provided the time component of $B^{(4)}$ in that limit is of the order $\mathcal{O}(v)$, i.e. of the order of a non-relativistic velocity. Here we rediscover Eq. (1) as the space component of the non-relativistic limit of Eqs. (3). However, as we already mentioned, there is no magnetic four-vector and therefore no straight forward way to generalize the notion of topology conservation of magnetic fields in a covariant way. But we will return to Eqs. (3) in another context.

III. THE COVARIANT TRANSPORT OF ELECTROMAGNETIC FIELDS

A. Equations of Transport

Mathematically the electromagnetic field is a closed two-form in Minkovski space, i.e. an antisymmetric tensor of degree two. Taking this into consideration, the search for a covariant generalization of magnetic topology conservation leads us to the Lie-derivative of the electromagnetic field tensor $F_{\mu\nu}$ (or in differential forms the two-form ω_F^2). Although the notation in differential forms is more compact and concise, the notation in tensors or cartesian vectors is more common and so for the convenience of the reader we give here all three equivalent expressions.

$$
L_V \omega_F^2 = 0 \tag{5}
$$

$$
\Leftrightarrow \epsilon^{\alpha\beta\gamma\delta} \partial_\alpha V^\nu F_{\nu\beta} = 0 \tag{6}
$$

$$
\Leftrightarrow \left\{ \begin{array}{l} \partial_0 (V^0 \mathbf{E} + \mathbf{V} \times \mathbf{B}) + \nabla(\mathbf{E} \cdot \mathbf{V}) = 0 \\ V^0 \partial_0 \mathbf{B} - \nabla \times (\mathbf{V} \times \mathbf{B}) - \nabla V^0 \times \mathbf{E} = 0 \end{array} \right. \tag{7}
$$

One might suggest that we should allow the right hand side (RHS) of Eq. (5) to be $\lambda\, \omega_F^2$ instead of 0, analogous to Eq. (1) or (3). There the free function λ reflects the fact that the topology of field lines does not depend on the strength of the vector field. Here such a term is restricted by the condition of vanishing divergence resulting in

$$
\begin{aligned}
& d\ (\lambda \omega_F^2) = 0 \\
\Leftrightarrow & \epsilon^{\alpha\beta\gamma\delta} \partial_\gamma\ (\lambda F_{\beta\delta}) = 0 \\
\Leftrightarrow & \left\{ \begin{array}{l} \mathbf{B} \cdot \nabla \lambda = 0 \\ \mathbf{B} \partial_0 \lambda + \nabla \lambda \times \mathbf{E} = 0. \end{array} \right.
\end{aligned} \tag{8}
$$

Hence there are no non-trivial solutions ($\lambda \neq const.$) except if $\mathbf{E} \cdot \mathbf{B} = 0$. Despite the fact that this is a necessary condition for the ideal plasma flows, which is assumed to hold approximately for most of the astrophysical plasmas, the exact condition is an exceptional case, which in all realistic situations is present at most on sets of measure zero. These are the border surfaces where $\mathbf{E} \cdot \mathbf{B}$ changes from $\mathbf{E} \cdot \mathbf{B} > 0$ to $\mathbf{E} \cdot \mathbf{B} < 0$. At these surfaces Eq. (8) requires $\lambda = const.$ in both regions separated by the $\mathbf{E} \cdot \mathbf{B} = 0$-surface. Continuity of λ than requires $\lambda = const.$ everywhere and at every time. This leads us to the statement that $\lambda \neq const.$ requires $\mathbf{E} \cdot \mathbf{B} = 0$ for regions of finite measure. However, these are singular cases which can be excluded without much loss of generality. The same holds for solution which have $\lambda = const. \neq 0$ which correspond to everywhere exponentially growing or decaying fields, so that we can simply use $\lambda = 0$.

So Eq. (5) is a covariant set of equations which governs the transport of electromagnetic fields, i.e. both the electric and the magnetic fields, in a flow given by the four-vector field $V^{(4)}$. We therefore call this the covariant transport of electromagnetic fields. The equations (5),(6) denote the antisymmetric differentiation of a four-vector equation which is found by integration,

$$
\begin{aligned}
& i_V \omega_F^2 = d\Phi \\
\Leftrightarrow & V^\nu F_{\nu\beta} = \partial_\beta \Phi \\
\Leftrightarrow & \left\{ \begin{array}{l} \mathbf{E} \cdot \mathbf{V} = -\partial_0 \Phi \\ V^0 \mathbf{E} + \mathbf{V} \times \mathbf{B} = \nabla \Phi. \end{array} \right.
\end{aligned} \tag{9}
$$

While the space component of Eq. (9) is very similar to the magnetic flux conserving Ohm's law, we have an additional time component which determines the evolution of the potential Φ. This evolution is solely determined by the transporting field $V^{(4)}$ which is seen most clearly by using the second equation to express the left hand side in terms of $\nabla\Phi$. This results in the equivalent system

$$
V^0 \partial_0 \Phi + \mathbf{V} \cdot \nabla \Phi = 0 \tag{10}
$$

$$
V^0 \mathbf{E} + \mathbf{V} \times \mathbf{B} = \nabla \Phi. \tag{11}
$$

Due to their formulation in differential forms these equations are independent of the choice of the coordinate system. They are therefore also suitable for the description of general relativistic situations, as for instance the accretion of plasma onto a black hole. (See for instance Carter in [8]. He also showed the embedding of these equations in a more general Lagrange or Hamilton formulation of fluid flows.) However, it is not our primary concern to investigate the properties of general relativistic plasma flows. Instead, we want to show that these equations, although having not such a simple intuitive meaning as the ideal Ohm's law, have similar conservation properties as magnetic topology conserving flows.

B. The non-relativistic limit

The relation of these covariant equations to the usual induction equation of MHD is found by taking the non-relativistic limit of Eq. (7). This limit requires a non-relativistic velocity $\|v\| = \epsilon \ll 1$ as well as small phase velocities $\mathcal{O}(\partial_0/\nabla) = \mathcal{O}(\epsilon)$ and a relation between the order of $\mathbf{E}$ and $\mathbf{B}$. This latter relation is usually derived from Faraday's law in MHD and states that $\mathcal{O}(\mathbf{E}) = \epsilon\, \mathcal{O}(\mathbf{B})$. Taking this into account we can neglect the last term in the second of Eqs. (7) and replace $V^{(4)} = (V^0, \mathbf{V})$ by $V^{(4)} = (1, \mathbf{v})$.

$$\text{Eqs. (7) with} \quad \mathcal{O}(v) = \mathcal{O}(E/B) \ll 1$$

$$\Rightarrow \begin{cases} \partial_0(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + \nabla(\mathbf{E}\cdot\mathbf{v}) = 0 \\ \partial_0\mathbf{B} - \nabla\times(\mathbf{v}\times\mathbf{B}) = 0 \end{cases} \tag{12}$$

$$\Leftrightarrow \begin{cases} \partial_0\Phi + \mathbf{v}\cdot\nabla\Phi = 0 \\ \mathbf{E} + \mathbf{v}\times\mathbf{B} = \nabla\Phi \end{cases} \tag{13}$$

In this limit the equation for the evolution of the magnetic field is the usual induction equation of ideal MHD and it is now decoupled from the equation for the electric field. The latter is not considered in MHD because the electric field can be eliminated from all equations of MHD. However, for all non-ideal cases this equation forms a non-trivial condition to the evolution of the electric field and we will return to it in section IV. Integration of Eq. (12) yields Eq. (13), which reveals the ideal Ohm's law ($\Phi = 0$) as a special case of the more general magnetic flux conserving form $\Phi \neq 0$.

It is worth mentioning that the condition $\mathcal{O}(\mathbf{E}) = \epsilon\, \mathcal{O}(\mathbf{B})$ is not a strict consequence of Faraday's law. To be precise it should read $\mathcal{O}(\mathbf{E}_{rot}) = \epsilon\, \mathcal{O}(\mathbf{B})$, where $\mathbf{E}_{rot}$ denotes the divergence-free part of the electric field. Therefore, we can assume for instance $\mathcal{O}(\mathbf{B}) < \mathcal{O}(\mathbf{E})$, which is consistent with Maxwell's equations if the electric field is dominated by a curl-free part. The corresponding non-relativistic limit of Eqs. (7),

$$\mathbf{E} = \nabla\Phi \tag{14}$$

$$\partial_0\Phi + \mathbf{v}\cdot\nabla\Phi = 0, \tag{15}$$

represents an electrohydrodynamics with a frozen-in potential of the electric field.

IV. INVARIANTS DERIVED FROM COVARIANT TRANSPORT

A. Electromagnetic flux conservation

Now we look for the relativistic analogue of the conservation of magnetic flux. We integrate the electromagnetic field tensor F over a two-dimensional surface C in $\mathbb{M}^4$ (see Eq. A1 for the convention on the surface elements $d\mathbf{a}$ and $d\mathbf{a_0}$):

$$\int_C F dA = \int_C \mathbf{B}\cdot d\mathbf{a} + \int_C \mathbf{E}\cdot d\mathbf{a_0}$$

The first term on the right hand side is the well known magnetic flux penetrating the surface C. The second term is the electric flux and exists only if the surface has a non-zero extension along the time axis as shown in Figure 1. Together they form a Lorentz invariant measure of electromagnetic flux penetrating C.

If this surface is transported by a flow $V^{(4)}$ in $\mathbb{M}^4$ the change of flux through the moving area is given by the Lie-derivative theorem (see [7], p. 370) which states that the change is zero as long as the tensor of the electromagnetic field is Lie-transported by the flow $V^{(4)}$, or in other words, if it is covariantly transported (see Figure 2 for an illustration). Hence, we can formulate the following theorem as the generalization of Alfvén's theorem of the frozen-in magnetic flux. (The covariant form of which was first stated by Lichnerowicz [9], however without a closer analysis or interpretation.)

a. Theorem (Transported flux) The electromagnetic flux penetrating a two-dimensional surface comoving in a flow $V^{(4)}$ in Minkowski space ($\mathbb{M}^4$) is constant for a covariantly (Lie-) transported field tensor.

$$\int_C F dA = \int_C \mathbf{B}\cdot d\mathbf{a} + \int_C \mathbf{E}\cdot d\mathbf{a_0} = const.$$

Here the transport of the surface C is now parametrized by a parameter (s) which is not necessarily the time,

$$\left(\frac{dX^0}{ds}, \frac{d\mathbf{X}}{ds}\right) = (V^0, \mathbf{V}), \quad (X^0, \mathbf{X}) \in C. \tag{16}$$

Only for $V^0 = const.$ we can identify s with the time X^0. Therefore this theorem reduces to the usual conservation of magnetic flux in the non-relativistic limit $V^0 = (1 - v^2)^{-1/2} \to 1$ in a smooth manner. Due to $V^0 \equiv 1$ in this limit a surface which has no extension in time direction, in other words a space-like surface, stays in a $t = const.$ plane during the transport and the second integral over the electric field is zero throughout.

$$\int_C \mathbf{B}\cdot d\mathbf{a} = const.$$

(for $V^0 \equiv 1$ and a space-like C at t_0).

On the other hand, a surface for which only the second integral contributes, i.e. a surface which contains the time axis, may develop a projection onto the space axes during the transport due to a non-vanishing $\partial_0 \mathbf{V}$. This projection does not contribute to $\int \mathbf{B}\cdot d\mathbf{a}$ if the magnetic field is parallel to that plane. Therefore, if we start with a plane spanned by a magnetic field line and the time axis, or if the flow is stationary ($\partial_0 \mathbf{V} = 0$) we deduce,

$$\int \mathbf{E}\cdot d\mathbf{a_0} = \int \mathbf{E}\cdot d\mathbf{l}\; dx_0 = const. \quad \text{for } d\mathbf{l} \parallel \mathbf{B}, \quad (17)$$

$$\Rightarrow \int \mathbf{E}\cdot d\mathbf{l} = const.. \quad (18)$$

For the case of a magnetic field line the integral can be expressed via the potential Φ,

$$\int_1^2 \mathbf{E}\cdot d\mathbf{l} = \Phi(\mathbf{x}, t)|_1^2 = const..$$

Note that this is not a consequence of the non-relativistic Ohm's law alone but of both of Eqs. (13). Boozer [10] stressed the importance of this quantity evaluated along magnetic field lines for the non-ideal behavior of the magnetic field and showed that for certain field configurations one can always find a transporting flow such that both Eqs. (13) hold.

In the non-relativistic limit the conservation of flux implies the conservation of magnetic lines of force. For relativistic flows, this does not hold. So, concerning the topology of the magnetic field there is no smooth limit from relativistic to non-relativistic flows and one can easily construct examples which show no conservation of magnetic field lines for all values of V^0 with exception of the limit $V^0 \equiv 1$. Thus magnetic flux conservation is a much more robust property of plasmas in the sense that it has a relativistic generalization in contrast to the conservation of magnetic field lines.

B. The Transport of E·B

There are further invariants related to covariant transport of electromagnetic fields. These invariants can be found by using operations which commute with the Lie-derivative (see [11], p. 201). The most important invariants obtained this way are (the four-form) $\mathbf{E}\cdot\mathbf{B}$ and the (three-form) helicity $H^{(4)} = (\mathbf{A}\cdot\mathbf{B}, A^0\mathbf{B} + \mathbf{E}\times\mathbf{A})$. The equation of transport for $\mathbf{E}\cdot\mathbf{B}$ can be deduced directly from Eq. (7),

$$\partial_0(V^0\mathbf{E}\cdot\mathbf{B}) + \nabla\cdot(\mathbf{V}\;\mathbf{E}\cdot\mathbf{B}) = 0 \quad (19)$$

This is a relativistic equation of continuity for the scalar density $\mathbf{E}\cdot\mathbf{B}$. The Lie-derivative theorem (see Appendix Eq. (A8)) states that the corresponding invariant of transport is found by integration over a four-dimensional volume

$$\int_{C^4} \mathbf{E}\cdot\mathbf{B}\; dx^4 = const.$$

We can rewrite Eq. (19) in the form

$$V^0\partial_0\mathbf{E}\cdot\mathbf{B} + \mathbf{V}\cdot\nabla\mathbf{E}\cdot\mathbf{B} = -(\partial_0 V^0 + \nabla\cdot\mathbf{V})\;\mathbf{E}\cdot\mathbf{B},$$

where it is obvious that points where $\mathbf{E}\cdot\mathbf{B} = 0$ are transported in the flow of $V^{(4)}$ without any changes. There is no appearing of new points where $\mathbf{E}\cdot\mathbf{B} = 0$, nor disappearing or merging of such points. Hence the topology of regions in space in which $\mathbf{E}\cdot\mathbf{B} < 0, = 0$, and > 0, respectively, is preserved (Figure 3). If for instance a new region $\mathbf{E}\cdot\mathbf{B} < 0$ appears inside a $\mathbf{E}\cdot\mathbf{B} > 0$ region, this indicates that there is no transport velocity for this evolution of the electromagnetic field which satisfies Eqs. (7).

In the non-relativistic limit $V^0 \equiv 1$ the transport equation simplifies to a normal equation of continuity and therefore the invariant quantity is $\mathbf{E}\cdot\mathbf{B}$ integrated over a three-dimensional area in $I\!R^3$ instead of a four-dimensional volume in $I\!M^4$.

$$\int_{C^3} \mathbf{E}\cdot\mathbf{B}\; d^3x = const. \quad \text{for } V^0 \equiv 1$$

C. Transport of the vector potential

Before we analyze the properties of the helicity four-vector we first have to investigate the transport of the vector potential, which is used in the expression for the helicity. In terms of differential forms the vector potential of the electromagnetic field is a one-form and its Lie-transport is governed by the equations

$$L_V\omega_A^1 = 0 \quad (20)$$

$$\Leftrightarrow \begin{cases} \partial_0(V^0A^0) + \mathbf{V}\cdot\nabla A^0 - \mathbf{A}\partial_0\mathbf{V} = 0 \\ V^0\partial_0\mathbf{A} + \nabla(\mathbf{V}\cdot\mathbf{A}) - \mathbf{V}\times\nabla\times\mathbf{A} - A^0\nabla V^0 = 0. \end{cases}$$

It is easy to prove that these equations imply the covariant transport Eq. (7) by applying the covariant derivative ($\partial_0\;\mathbf{eq} + \nabla eq_0, \nabla\times\mathbf{eq}$) to the time eq_0 and space components $\mathbf{eq}$ of Eq. (20). But does the reverse also hold ? This is an interesting problem because if we can derive Eq. (20) from covariant transport we can also derive transport equations for the helicity and a conservation property of the helicity similar to the conservation of the electromagnetic flux as we will see in the next section. Thus we convert the equations for covariant transport with the help of

$$\mathbf{E} = -\partial_0\mathbf{A} - \nabla A^0, \quad \mathbf{B} = \nabla\times\mathbf{A}$$

into equations for the vector potential. They meet the form of Eq. (20) for the condition

$$\begin{aligned} & d\Phi = -di_V\omega_A^1 \\ \Leftrightarrow\; & \partial_\beta\Phi = -\partial_\beta(V^\mu A_\mu) \\ \Leftrightarrow\; & \begin{cases} \partial_0\Phi = -\partial_0(V^0A^0 - \mathbf{V}\cdot\mathbf{A}) \\ \nabla\Phi = -\nabla(V^0A^0 - \mathbf{V}\cdot\mathbf{A}) \end{cases} \\ \Leftrightarrow\; & \Phi = -(V^0A^0 - \mathbf{V}\cdot\mathbf{A}) + \phi_c, \end{aligned} \quad (21)$$

where ϕ_c is a constant of integration. This is a new condition for the form of the potential Φ and it will not be satisfied for arbitrary choices of Φ and A^μ. Thus the transport of the vector potential is not a direct consequence of the transport of the electromagnetic field for the general case. But we can satisfy this equation with the help of a gauge of the vector potential,

$$\left\{ \begin{array}{l} A^0 \to A^0 - \partial_0 \Psi \\ \mathbf{A} \to \mathbf{A} + \nabla\Psi \end{array} \right.$$

which turns the condition (21) into an equation determining the evolution of the gauge potential Ψ.

$$V^0 \partial_0 \Psi = V^0 A^0 - \mathbf{V}\cdot(\mathbf{A} + \nabla\Psi) + \Phi + \phi_c$$

Hence we can always determine a gauge function Ψ, which may be chosen to be identically zero at a starting time, so that the vector potential is transported according to equations (20). A similar gauge was used in [12–14] for a non-relativistic formulation.

b. Theorem (Transport of vector potential) *For a covariantly transported electromagnetic field under a flow $V^{(4)}$ with $V^0 > 0$ there exists an evolution of the vector potential according to Eqs. (20) so that the vector potential is a Lie-transported one-form.*

The transport of the vector potential again implies the conservation of an integral. The vector potential integrated along a comoving line in $I\!M^4$ is constant [cf. Eq. (A5)].

$$\int_{C^1} \omega_A^1 = \int_{C^1} A^0 \; dx^0 - \int_{C^1} \mathbf{A}\cdot d\mathbf{l} = const. \tag{22}$$

Because the vector potential is not a measurable quantity for itself, this property is of limited use for the investigation of electromagnetic fields. It is, however, important for the evolution of the helicity four-vector which, integrated over a three-dimensional volume, is an important measure for the complexity of the electromagnetic field.

D. Transport of the helicity

The transport of the vector potential enables us to calculate the transport equation of the helicity density which turns out to be a four-vector in the relativistic case, containing the usual helicity density $\mathbf{A}\cdot\mathbf{B}$ as the time component.

$$\begin{array}{l} \omega_H^3 = \omega_A^1 \wedge \omega_F^2 \\ \Leftrightarrow H^\nu = A_\mu \mathcal{F}^{\nu\mu} \\ \Leftrightarrow \left\{ \begin{array}{l} H^0 = \mathbf{A}\cdot\mathbf{B} \\ \mathbf{H} = A^0\mathbf{B} + \mathbf{E}\times\mathbf{A} \end{array} \right. \end{array}$$

(Here $\mathcal{F}^{\mu\nu}$ is the dual tensor of the electromagnetic field.)

With the help of the transport equations for the electromagnetic field Eqs. (7) and the vector potential Eqs. (20) we can derive a transport equation for the helicity four-vector.

$$\begin{array}{l} L_V \omega_H^3 = 0 \\ \Leftrightarrow \left\{ \begin{array}{l} V^0 \partial_0 H^0 + \nabla\cdot(\mathbf{V} H^0) - \mathbf{H}\cdot\nabla V^0 = 0 \\ \partial_0(V^0 \mathbf{H}) - \nabla\times(\mathbf{V}\times\mathbf{H}) + \mathbf{V}\nabla\cdot\mathbf{H} - H^0 \partial_0 \mathbf{V} = 0 \end{array} \right. \end{array} \tag{23}$$

This in turn is equivalent to the helicity Lie-transported as a three-form. The integration over a three-dimensional volume Λ yields a Lorentz scalar, the total helicity of the volume Λ (see Appendix A2 for the convention on the volume elements dV and $d\mathbf{V_0}$).

$$\int_\Lambda H dV = \int_\Lambda H^0 \; dV - \int_\Lambda \mathbf{H}\cdot d\mathbf{V_0} \tag{24}$$

The first term on the RHS is the well known non-relativistic integral of helicity, the second term arises from a possible extension of the volume along the time axis. Together they form a Lorentz scalar. Similar to the integral of the electromagnetic flux, the use of the Lie-derivative theorem yields the following theorem. (The non-covariant form of helicity conservation was first stated by Woltjer [15] and Moffatt [16], a relativistic version without the gauge given above was given by Carter in [8].)

c. Theorem (Transported helicity) *There exists a natural gauge of the vector potential for covariantly transported electromagnetic fields such that the helicity integrated over a three-dimensional volume transported in a flow $V^{(4)}$ with $V^0 > 0$ in $I\!M^4$ is constant.*

$$\int_\Lambda H \; d\Lambda = \int_\Lambda H^0 \; dV - \int_\Lambda \mathbf{H}\cdot d\mathbf{V_0} = const.$$

Comparing the transport equations for the helicity Eq. (23) with the relativistic equations for preservation of field lines Eq. (3) we see, using $\lambda = -(\partial_0 V^0 + \nabla\cdot\mathbf{V})$, that they preserve the helicity field lines. It is a ironic feature here that we were looking without success for a relativistic form of magnetic field line conservation, and instead find it for the helicity vector field. Unfortunately the Eqs. (23) have no smooth magnetohydrodynamic limit. In the non-relativistic limit of Eq. (23) reduces to

$$\partial_0 H^0 + \nabla\cdot(\mathbf{v} H^0) = 0 \tag{25}$$

$$\partial_0 \mathbf{H} - \nabla\times(\mathbf{v}\times\mathbf{H}) + \mathbf{v}\;\nabla\cdot\mathbf{H} - H^0 \partial_0 \mathbf{v} = 0. \tag{26}$$

Here the equation for the space components is not an independent equation and is still coupled to the time component in the non-relativistic limit. This is due to the fact that the non-relativistic limit of Maxwell's equations not only requires $v \ll 1$ but also that time and length scales of variation of the electric and magnetic fields are of the order of a non-relativistic velocity $\mathcal{O}(\partial_0/\nabla) = \mathcal{O}(v)$. Hence,

$$\mathcal{O}(||\mathbf{E}||/||\mathbf{B}||) = \mathcal{O}(A^0/||\mathbf{A}||) = \mathcal{O}(||\mathbf{H}||/H^0) = \mathcal{O}(v),$$

and the last term in Eq. (23) is of the same order as the first. This explains why the transport equation for the time component Eq. (25), i.e. the scalar magnetic helicity, is well known in magnetohydrodynamics, in contrast to the equation of the space components. Nevertheless we can derive

$$\int H^0 \; d^3x = const.$$

for the limit $V^0 \equiv 1$ or more general $V^0 = const.$, while

$$\int \mathbf{H} \cdot d\mathbf{a} = const.$$

only holds for $H^0 \partial_0 \mathbf{V} = 0$. In this case Eq. (26) has the form of a transport equation for a surface density or two-form in $I\!R^3$ [see Eq. (A12)] which immediately yields this invariant.

V. TRANSPORTING FLOWS

A. Examples

The results derived so far would be of limited use if they were restricted to cases where we have an Ohm's law which meets the form Eq. (9). The only major application would be the ideal Ohm's law and the derived equation are either trivial or well known for this case. The strength of the concept is that it can be generalized. This generalization has a double meaning. First we can show how the already known invariants (flux, topology, helicity, $\mathbf{E} \cdot \mathbf{B}$) of ideal ($\Phi = 0$, $V^0 \equiv 1$) or at least magnetic flux conserving ($\Phi \neq 0$, $V^0 \equiv 1$) systems are modified if the characteristic velocities of our system became relativistic. This is important for the application of MHD to relativistic situations. The other more important point is that we can use this covariant description to find invariants of non-ideal systems also regardless of whether they are relativistic or not.

The crucial point of the generalization is the interpretation of the transporting velocity $V^{(4)}$. For the derivation of the invariants we used at no time that V^μ is the plasma velocity nor that it is of the velocity type $V^{(4)} = \left(1/\sqrt{1-v^2}, \mathbf{v}/\sqrt{1-v^2}\right)$ of particles with finite mass. Only for the existence of the vector potential we used $V^0 > 0$. Hence, V^μ is a general vector field in the Minkowski space restricted only by $V^0 > 0$ and we can also use this to extend the realm of our theory beyond that of ideal MHD. The case of the electromagnetic fields being transported in the plasma flow is a special case of the more general situation where we have a transport velocity which may differ from the plasma velocity. A simple example of such a case is given by an Ohm's law of the form

$$\mathbf{E} + \mathbf{v_P} \times \mathbf{B} = \frac{1}{en} \mathbf{J} \times \mathbf{B},$$

($\mathbf{v}_p$ is the plasma velocity), which can be converted in the form for covariant transport Eq. (9) with the help of $V^{(4)} = (1, \mathbf{v_P} - \frac{1}{en}\mathbf{J})$ and $\Phi = 0$. In this example we can also include a term $\frac{1}{en}\nabla P_e$ on the RHS, if we assume for instance $T = const.$ and use $\Phi = \frac{kT}{e}\ln(n)$.

Unfortunately the case that we can deduce transporting flows in this way directly from Ohm's law is an exception, restricted to special forms of Ohm's law. For the important case of a resistive term $\eta\mathbf{J}$ on the right hand side of Ohm's law we cannot derive such a general transport velocity. Nevertheless in certain situations we may have transporting flows for this case as well. Consider for example an Ohm's law of the form

$$\mathbf{E} + \mathbf{v_P} \times \mathbf{B} = \eta \mathbf{J} - \frac{1}{en} \nabla P_e(n)$$

A simple stationary one-dimensional solution for a certain region of space is given in cartesian geometry by

$$\begin{aligned}
\mathbf{B} &= B_x \; \mathbf{e}_x + B_y(x) \; \mathbf{e}_y \\
\mathbf{J} &= J_z(x) \; \mathbf{e}_z \\
n &= n(x) \\
\mathbf{E} &= -\frac{1}{e \; n(x)} \frac{dP_e(n(x))}{dx} \mathbf{e}_x + E_z \; \mathbf{e}_z \\
\mathbf{v_P} &= (E_z - \eta J_z(x))/B_x \quad \mathbf{e}_y
\end{aligned}$$

Although there is a resistive term we can meet the form Eq. (9) with a "virtual" velocity

$$V^0 = 1, \mathbf{V} = E_z/B_x \; \mathbf{e}_y$$

and the potential Φ:

$$\Phi = -\int \frac{1}{e \; n(x)} \frac{dP_e(n(x))}{dx} dx.$$

This is, of course, not the only solution, we can add for instance to $\mathbf{V}$ a component parallel to $\mathbf{B}$ which still satisfies both equations Eq. (9). We can also choose another $V^0(x)$ and in this way modify $\Phi(x)$ which may be necessary to match this solution for instance to a solution of an external ideal region.

The examples show that there exist transporting flows in all cases of ideal plasma flows and also for cases of non-ideal evolutions. For certain cases we can derive the transporting flow *a priori* from the form of Ohm's law without knowing any details of the solution. However, in the general case we have to prove the existence of a transporting flow for every single solution. This means that we have to answer the question: Given the possibly time-dependent electromagnetic field $(\mathbf{E}(\mathbf{x}, t), \mathbf{B}(\mathbf{x}, t))$, is there a transporting flow $V = (V^0(\mathbf{x}, t), \mathbf{V}(\mathbf{x}, t))$ such that the equations Eq. (7) hold? Alternatively we can answer this question on the level of Ohm's law Eq. (9), where the equations are more simple, but require to find the potential Φ in addition.

As an example, we try to find a solution for the simple configuration of a magnetic flux tube. A solution for

$\{V^{(4)}, \Phi\}$ is found by integrating Φ from the parallel component of Ohm's law along magnetic field lines, starting from an arbitrary cross-section where we can prescribe Φ at will, for instance $\Phi = 0$.

$$\Phi(\mathbf{x}, t) := \int V^0 \; \mathbf{E}\cdot\mathbf{e}_B \; dl \qquad (27)$$

To satisfy the perpendicular component of Ohm's law and the additional time component we split the space component of the transport velocity $\mathbf{V}$ in a part perpendicular $\mathbf{V}_\perp$ and parallel $\mathbf{V}_\parallel$ to $\mathbf{B}$. The former part is defined by

$$\mathbf{V}_\perp := \frac{(V^0\mathbf{E} - \nabla\Phi)\times\mathbf{B}}{B^2}$$

the latter by

$$\mathbf{V}_\parallel := -\frac{\partial_0\Phi + \mathbf{E}\cdot\mathbf{V}_\perp}{\mathbf{E}\cdot\mathbf{e}_B}.$$

Note that the last equation requires that $\mathbf{E}\cdot\mathbf{B}$ has a uniform sign in the region under consideration. Then $\{V^0, \mathbf{V}_\perp, \mathbf{V}_\parallel, \Phi\}$ form a complete solution of our problem, although we have not defined V^0 yet which we can do by simply setting $V^0 \equiv 1$. This of course is a solution which holds only for a certain region of space where there are no magnetic nulls or closed field lines and where $\mathbf{E}\cdot\mathbf{B}$ has a uniform sign. These requirements may prevent us from extending the solution beyond the flux tube to a global one. Consider for instance a closed flux tube with $\mathbf{E}\cdot\mathbf{B} > 0$ inside the tube. The integration of Φ along a closed field line requires according to Eq. (27) either $\Phi = 0$, and therefore $V^0 = 0$ and $\mathbf{V} = 0$ (which is the trivial solution which always exists), or $V^0 < 0$ on a part of the field line. Both alternatives do not fit the interpretation of $V^{(4)}$ as a transport velocity, in which we up to now assumed that $V^0 > 0$. Here the reader might ask whether we should assume $||V^{(4)}|| = \sqrt{{V^0}^2 - \mathbf{V}^2} = 1$, to interpret $V^{(4)}$ as a velocity of a virtual flow of particles, or at least $||V^{(4)}|| > 0$, to ensure the positiveness of V^0 under Lorentz transformations. The answer to this question requires a closer analysis of our interpretation of the transporting velocity in Minkowski space, which we try to give in the following section.

B. Classification

The existence of a transporting flow, i.e. a Lie-invariance or Lie-symmetry, is the most general form of an invariance of a physical system. It includes all the symmetries which conserve the metric such as translation (constant velocity), rotation (constant angle velocity) or time invariance ($V^0 = const., \mathbf{V} = 0$) as well as those which do not conserve the metric and thus in general depend on space and time. With this interpretation in mind the transporting flow $V^{(4)}$ should satisfy ${V^0}^2 + \mathbf{V}^2 > 0$ with possible exception of sets of measure zero, to have a definite meaning for the integral invariants as given by the Lie-derivative theorem (see Appendix A).(Otherwise, i.e. for $V^{(4)} \equiv 0$ on a whole region of $\mathbb{M}^4$, the meaning of the constance of comoving integrals is lost in this region.)

Now there are three subsets of transporting flows which are invariant with respect to Lorentz transformations. First, the space-like symmetries with $||V^{(4)}||^2 = {V^0}^2 - \mathbf{V}^2 < 0$ everywhere. An example is the solution $V = (0, \mathbf{B})$ for ideal plasmas, or for instance the generating velocity for a rotational symmetry if it exists. Secondly, the time-like symmetries, i.e. $||V^{(4)}||^2 > 0$ everywhere. This subset contains the usual non-relativistic plasma flows $V = (1, \mathbf{v})$ as well as their relativistic form $V = (1/\sqrt{1-v^2}, \mathbf{v}/\sqrt{1-v^2})$. Also for this group V^0 has always a definite sign in $\mathbb{M}^4$, which we can assume to be positive, otherwise $(-V^0, -\mathbf{V})$ is also a transporting flow. So we can always interpret the four-velocity by a velocity $\mathbf{v}$ in $\mathbb{R}^3$

$$V^{(4)} = (V^0, \mathbf{V}) \rightarrow \mathbf{v} := \mathbf{V}/V^0$$

and $\mathbf{v}$ is always smaller than the velocity of light. However, note that we cannot turn around the arrow in the last equation, that is, different choices for $V^{(4)}$ belonging to the same $\mathbf{V}/V^0$ may lead to different evolutions of the electromagnetic field. This holds for all Lie-transported quantities listed in the Appendix except the proper scalars i.e. quantities which are transported according to Eq. (A4). (For proper scalars the covariant equation is equivalent to its non-relativistic form with $\mathbf{v} = \mathbf{V}/V^0$.) Therefore, all the flows with $||V^{(4)}||^2 > 0$ have their own right to exist and are not simply related to flows with $V^0 \equiv 1$ or $||V^{(4)}||^2 = 1$.

The third group is the complement of the first and second one in the set of all flows with a Lie-symmetry. These are the flows which have regions in space and time where $||V^{(4)}||^2 > 0$ as well as $||V^{(4)}||^2 < 0$. In a subsequent paper we will show that this set contains solution which correspond to reconnection processes and that the framework of covariant transport allows for a precise definition of reconnection.

This subdivision of transporting flows is at the same time a subdivision of electromagnetic systems. First note that if we allow for $||V^{(4)}|| = 0$, i.e. the most general form of a transporting flow, all electromagnetic fields have a transporting flow (at least the trivial solution $V^{(4)} = (0, \mathbf{0})$) and that this flow is not unique, i.e. there may be multiple solutions. Secondly, the space of solutions, i.e. transporting flows for a given electromagnetic field, is a vector space, meaning that every solution can be scaled by a global constant and also a linear combination of solutions is a solution by itself. Naturally we are most interested in time-like symmetries of electromagnetic systems because they allow us to make predictions about the evolution with the help of the invariants derived above. Therefore we distinguish those electromagnetic fields which have a time-like symmetry from those which have none. We have done this by distinguishing transporting flows with $||V^{(4)}||^2 > 0$, $||V^{(4)}||^2 < 0$ respectively. But it is tempting to relate other non-time-like systems also to the group $||V^{(4)}||^2 > 0$, to extend the set of systems for which we can make predictions. This is indeed possible for another large group which have $V^0 > 0$ and $||\mathbf{V}/V^0|| < \infty$ in at least one frame of reference. They allow to define a global constant q larger than the maximum of $\sqrt{\mathbf{V}^2/(V^0)^2}$ in the region under consideration (G).

$$q := \sup_G \sqrt{\mathbf{V}^2/(V^0)^2} + 1$$

This allows for a scaling of the time by

$$t \to tq$$
$$\Rightarrow \mathbf{E} \to \mathbf{E}/q \quad \text{and} \quad \widetilde{V^0} \to qV^0$$

such that the scaled system has a time-like transporting flow. We call these systems MHD-equivalent, meaning that we can always scale them with the help of large values of q, such that the assumptions for electromagnetic fields in MHD $\mathcal{O}(E/B) \ll 1$ and $\mathcal{O}(\mathbf{V}/V^0) \ll 1$ hold, i.e. a small electric field compared to $\mathbf{B}$ and small velocities compared to c. The resulting subdivision of electromagnetic fields with respect to their transporting flows is sketched in Figure 4.

VI. CONCLUSIONS

In this paper we investigated the covariant generalization of magnetic flux conservation. This proves to be a conservation of the electromagnetic field tensor which beyond an induction equation for the magnetic field includes a transport equation for the electric field as well. The usual conservation of magnetic flux is found in the non-relativistic limit of these equations. In this limit the topology of the magnetic field is preserved too. In general the covariant transport of the electromagnetic field does not imply the conservation of magnetic topology. Moreover we have shown that there is no covariant generalization of this property of ideal or magnetic flux conserving plasmas. It has, however, other invariants, the electromagnetic flux, the helicity, and $\mathbf{E}\cdot\mathbf{B}$, which are conserved in the non-relativistic as well as in the general case. Similar to the magnetic flux and magnetic topology in ideal MHD, these invariants restrict the evolution of the plasma and allow for predictions of possible final states of an evolution.

Analogous to a smooth plasma flow in ideal MHD, which leads to smooth deformation of the magnetic flux and hence excludes magnetic reconnection and similar processes, the covariant transport of the electromagnetic field in Minkowski space represents a smooth redistribution of the electromagnetic flux. The transporting flows do not need to have the form $V^{(4)} = (\gamma_v, \gamma_v \mathbf{v})$ and we gave examples that more generals flows, restricted only by $||V^{(4)}||^2 > 0$, can be related to non-ideal evolutions of magnetohydrodynamic systems. Thus there exists a large group of non-ideal evolutions of plasmas which have similar invariants as in the ideal case.

For time-like ($||V^{(4)}||^2 > 0$) flows the transporting flow in $I\!\!M^4$ has a smooth representation $\mathbf{v} = \mathbf{V}/V^0$ in $I\!\!R^3$ also. If the flow satisfies in addition $V^0 = const.$ the flow $\mathbf{v}$ is a transporting flow for the magnetic field in $I\!\!R^3$ and the invariants reduce to the well-known invariants of magnetic flux conserving transport, i.e. the corresponding spatial integrals of $\mathbf{B}$, $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{E}\cdot\mathbf{B}$. For this case we have in addition to these invariants also the preservation of magnetic topology. Thus we can subdivide all electromagnetic fields in those which allow for a time-like transporting flow, or at least can be scaled to such a field, and those which do not have this property. The former have frozen-in invariants of the above mentioned type. The latter represent more violent evolutions of the electromagnetic field. In a subsequent paper we will show that this group also contains evolutions which show magnetic reconnection, an example of a non-smooth redistribution of the electromagnetic field.

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APPENDIX A: LIE-DERIVATIVE THEOREM

For reasons of completeness and as a reference we give here an overview over the differential forms ω^n in Minkowski space, their corresponding Lie-derivative with respect to the flow $V^{(4)} = (V^0, \mathbf{V})$ denoted by L_V, and the equivalent expressions in terms of vector analysis. (A splendid representation of this formalism is found in [7].) Due to the dimension four of Minkowski space all forms of degree higher than four vanish and we have the following identification between forms in $I\!\!M^4$ and the scalars (a, b) and vector fields $(\mathbf{A}, \mathbf{B})$ in $I\!\!R^3$. Vectors $A = (A^0, \mathbf{A})$ in $I\!\!M^4$ are denoted by capital letters, their space components are bold face and time components are indicated by superscript 0. The signature of the metric is (1,-1,-1,-1).

$$\begin{aligned}
\omega_a^0 &= a \\
\omega_A^1 &= A^0 dx^0 - A^i dx^i \\
\omega_{(\mathbf{AB})}^2 &= A^i dx^0 \wedge dx^i + \epsilon_{ijk} B^i dx^j \wedge dx^k \\
\omega_A^3 &= A^0 dx^1 \wedge dx^2 \wedge dx^3 - \epsilon_{ijk} A^i dx^0 \wedge dx^j \wedge dx^k \\
\omega_b^4 &= b \ dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3
\end{aligned}$$

Each of them imply the conservation of an integral over the n-form due to the Lie-derivative theorem (see [7], p. 370). The corresponding area of integration C^n

is an n-dimensional volume comoving in the flow $V^{(4)}$. For the integration, the orientation of these volumes is important. For one-dimensional volumes this is trivial but surfaces in $\mathbb{M}$ are characterized by six components for which we use the convention

$$\begin{aligned} d\mathbf{a} &= a^1 dx^2 \wedge dx^3 + a^2 dx^3 \wedge dx^1 + a^3 dx^1 \wedge dx^2 \\ d\mathbf{a_0} &= a_0^1 dx^1 \wedge dx^0 + a_0^2 dx^2 \wedge dx^0 + a_0^3 dx^3 \wedge dx^0, \end{aligned} \quad \text{(A1)}$$

while for three-dimensional volumes we use

$$\begin{aligned} dV &= dx^1 \wedge dx^2 \wedge dx^2 \\ d\mathbf{V_0} &= V_0^1 dx^0 \wedge dx^2 \wedge dx^3 \\ &+ V_0^2 dx^0 \wedge dx^3 \wedge dx^1 \\ &+ V_0^3 dx^0 \wedge dx^1 \wedge dx^2, \end{aligned} \quad \text{(A2)}$$

and the four-dimensional volume element is given by

$$dV^{(4)} = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \quad \text{(A3)}$$

Lie-transport in $\mathbb{M}^4$ and conserved integrals:

$$\begin{aligned} & L_V \omega_a^0 = 0 \quad \text{(A4)} \\ \Leftrightarrow\ & V^0 \partial_0 a + \mathbf{V} \cdot \nabla a = 0 \\ \Rightarrow\ & \int_{C^0} \omega_a^0 = a = const. \end{aligned}$$

$$\begin{aligned} & L_V \omega_A^1 = 0 \quad \text{(A5)} \\ \Leftrightarrow\ & \begin{cases} \partial_0 (V^0 A^0) + \mathbf{V} \cdot \nabla A^0 - \mathbf{A} \partial_0 \mathbf{V} = 0 \\ V^0 \partial_0 \mathbf{A} + \nabla (\mathbf{V} \cdot \mathbf{A}) - \mathbf{V} \times \nabla \times \mathbf{A} \\ \qquad - A^0 \nabla V^0 = 0 \end{cases} \\ \Rightarrow\ & \int_{C^1} \omega_A^1 = \int_{C^1} A^0 dx^0 - \int_{C^1} \mathbf{A} \cdot d\mathbf{l} = const. \end{aligned}$$

$$\begin{aligned} & L_V \omega_{\mathbf{AB}}^2 = 0 \quad \text{(A6)} \\ \Leftrightarrow\ & \begin{cases} \partial_0 (V^0 \mathbf{A}) + \nabla (\mathbf{V} \cdot \mathbf{A}) - \mathbf{V} \times \nabla \times \mathbf{A} \\ \qquad - \partial_0 \mathbf{V} \times \mathbf{B} = 0 \\ V^0 \partial_0 \mathbf{B} - \nabla \times \mathbf{V} \times \mathbf{B} + \mathbf{V}\ \nabla \cdot \mathbf{B} \\ \qquad + \nabla V^0 \times \mathbf{A} = 0 \end{cases} \\ \Rightarrow\ & \int_{C^2} \omega_{\mathbf{AB}}^2 = \int_{C^2} \mathbf{B} \cdot d\mathbf{a} - \int_{C^2} \mathbf{A} \cdot d\mathbf{a_0} = const. \end{aligned}$$

$$\begin{aligned} & L_V \omega_A^3 = 0 \quad \text{(A7)} \\ \Leftrightarrow\ & \begin{cases} V^0 \partial_0 A^0 + \nabla \cdot (\mathbf{V} A^0) - \mathbf{A} \cdot \nabla V^0 = 0 \\ \partial_0 (V^0 \mathbf{A}) - \nabla \times \mathbf{V} \times \mathbf{A} + \mathbf{V}\ \nabla \cdot \mathbf{A} \\ \qquad - A^0 \partial_0 \mathbf{V} = 0 \end{cases} \\ \Rightarrow\ & \int_{C^3} \omega_A^3 = \int_{C^3} A^0 dV - \int_{C^3} \mathbf{A} \cdot d\mathbf{V_0} = const. \end{aligned}$$

$$\begin{aligned} & L_V \omega_a^4 = 0 \quad \text{(A8)} \\ \Leftrightarrow\ & \partial_0 (a V^0) + \nabla \cdot (a \mathbf{V}) = 0 \\ \Rightarrow\ & \int_{C^4} \omega_a^4 = \int_{C^4} a\ dV^{(4)} = const. \end{aligned}$$

Beside these equations for the Lie-derivative of n-forms, there exists a Lie-derivative for (contravariant) vector fields as well.

$$\begin{aligned} & L_V A = 0 \quad \text{(A9)} \\ \Leftrightarrow\ & \begin{cases} V^0 \partial_0 A^0 + \mathbf{V} \cdot \nabla A^0 - \mathbf{A} \cdot \nabla V^0 - A^0 \partial_0 V^0 = 0 \\ V^0 \partial_0 \mathbf{A} + \mathbf{V} \cdot \nabla \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{V} - A^0 \partial_0 \mathbf{V} = 0 \end{cases} \end{aligned}$$

$\Rightarrow$ Conservation of field lines in $\mathbb{M}^4$

It is also instructive to compare these equations with the corresponding transport of quantities in the usual $\mathbb{R} \times \mathbb{R}^3$:

$$\begin{aligned} & \partial_0 a + L_v \omega_a^0 = 0 \quad \text{(A10)} \\ \Leftrightarrow\ & \partial_0 a + \mathbf{v} \cdot \nabla a = 0 \\ \Rightarrow\ & \int_{C^0} \omega_a = a = const. \end{aligned}$$

$$\begin{aligned} & \partial_0 \omega_A^1 + L_v \omega_A^1 = 0 \quad \text{(A11)} \\ \Leftrightarrow\ & \partial_0 \mathbf{A} + \nabla (\mathbf{v} \cdot \mathbf{A}) - \mathbf{v} \times \nabla \times \mathbf{A} = 0 \\ \Rightarrow\ & \int_{C^1} \omega_A^1 = \int_{C^1} \mathbf{A} \cdot d\mathbf{l} = const. \end{aligned}$$

$$\begin{aligned} & \partial_0 \omega_A^2 + L_v \omega_A^2 = 0 \quad \text{(A12)} \\ \Leftrightarrow\ & \partial_0 \mathbf{A} - \nabla \times \mathbf{v} \times \mathbf{A} + \mathbf{v}\ \nabla \cdot \mathbf{A} = 0 \\ \Rightarrow\ & \int_{c^2} \omega_A^2 = \int_{C^2} \mathbf{A} \cdot d\mathbf{a} = const. \end{aligned}$$

$$\begin{aligned} & \partial_0 \omega_a^3 + L_v \omega_a^3 = 0 \quad \text{(A13)} \\ \Leftrightarrow\ & \partial_0 a + \nabla \cdot (a \mathbf{v}) = 0 \\ \Rightarrow\ & \int_{C^3} \omega_a^3 = \int_{C^3} a dx^3 = const. \end{aligned}$$

And analogous to A9 for vector fields,

$$\begin{aligned} & \partial_0 \mathbf{A} + L_v \mathbf{A} = 0 \quad \text{(A14)} \\ \Leftrightarrow\ & \partial_0 \mathbf{A} + \mathbf{V} \cdot \nabla \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{V} = 0 \end{aligned}$$

$\Rightarrow$ Conservation of field lines in $\mathbb{R}^3$.

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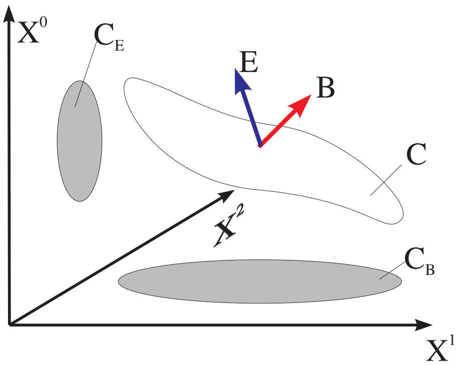
FIG. 1. The domains of integration of magnetic, C_B, and electric flux, C_E, for a two-dimensional surface in $I\!M^4$

FIG. 2. For covariant transport the electromagnetic flux is frozen in the flow $V^{(4)}$ in $I\!M^4$.

FIG. 3. The covariant transport conserves regions in space with $\mathbf{EB} < 0$ or > 0. Therefore a) corresponds to an evolution which is consistent with covariant transport for $V^0 > 0$, while b) is not.

FIG. 4. A subdivision of electromagnetic fields according to existence and norm of their transporting flows.

Figure 1 of 4, G. Hornig, PoP 22119



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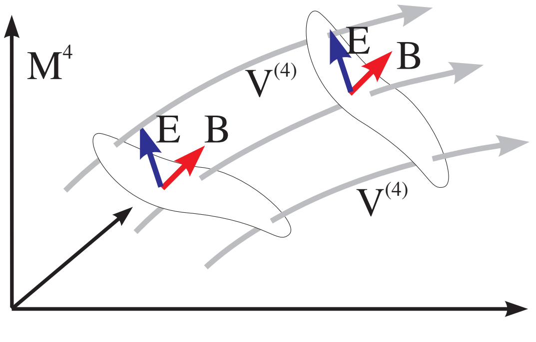

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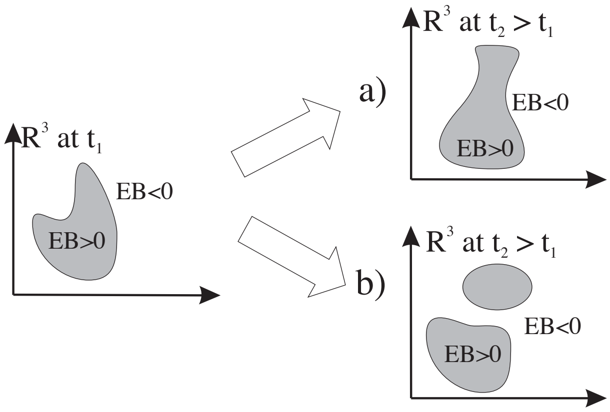

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