Exact Solutions for Magnetic Reconnective Annihilation in Curvilinear Geometry

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1 Introduction

Magnetic reconnection is a process which converts magnetic energy into kinetic and thermal energy of a plasma by means of a rearrangement of the topology of the magnetic field lines. This process is believed to be responsible for phenomena such as solar flares, geomagnetic substorms and sawtooth oscillations in tokamaks [1,2]. The aim of this paper is to present analytical solutions of the magnetohydrodynamcs (MHD) equations describing a type of reconnection denoted as magnetic reconnective annihilation.

For a steady incompressible plasma with uniform density and resistivity the dimensionless MHD equations consist of the curl of the momentum equation

$$(\mathbf{v}\cdot\nabla)\boldsymbol{\omega}-(\boldsymbol{\omega}\cdot\nabla)\mathbf{v}-(\mathbf{B}\cdot\nabla)\mathbf{j}+(\mathbf{j}\cdot\nabla)\mathbf{B}=0 \qquad (1)$$

and of the induction equation

$$\nabla\times(\mathbf{v}\times\mathbf{B})-\eta\nabla\times\mathbf{j}=0 \qquad (2)$$

where the magnetic field $\mathbf{B}$ and the velocity $\mathbf{v}$ satisfy the divergence-free conditions

$$\nabla\cdot\mathbf{v}=\nabla\cdot\mathbf{B}=0. \qquad (3)$$

In (1) and (2) $\boldsymbol{\omega}$ indicates the vorticity, $\mathbf{j}$ the current density and η the inverse magnetic Reynolds number.

2 Two-dimensional solutions

Given a cylindrical coordinate system (r,θ,z) it is assumed that the vector fields $\mathbf{B}$ and $\mathbf{v}$ lie in the plane r-θ and that they are not functions of z. Eqs. (1) and (2) can then be written as

$$[\psi,\nabla^2\psi]=[A,\nabla^2 A], \qquad (4)$$

$$E+\frac{1}{r}[\psi,A]=-\eta\nabla^2 A. \qquad (5)$$

where $[f,g]=\partial_r f\partial_\theta g-\partial_\theta f\partial_r g$. E is the electric field which is uniform and directed along z. The stream function ψ and the flux function A are defined in such a way that $\mathbf{v}=\nabla\psi\times\hat{\mathbf{z}}$ and $\mathbf{B}=\nabla A\times\hat{\mathbf{z}}$. If we substitute into (4) and (5) flux and stream functions of the form

$$A(r,\theta)=A_1(r)\theta+A_0(r), \qquad \psi(r,\theta)=\psi_1(r)\theta+\psi_0(r), \qquad (6)$$

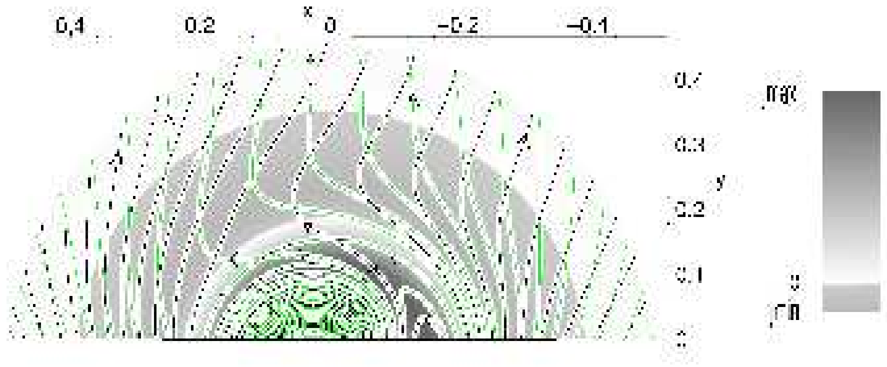

Figure 1: Example of magnetic field lines (black) and streamlines (green) configuration resulting from the resistive solutions. The plots of the field lines are superimposed to the distribution of the current density in gray half-tones. The critical radius is located at $r = 0.2$.

we obtain a system of four ordinary differential equations (ODEs) for the four unknown functions A_1, A_0, ψ_1 and ψ_0. Therefore the ansatz (6) is compatible with Eqs. (4) and (5). Since for astrophysical plasmas the value of the dimensionless resistivity η is very small (typically 10^{-12}), it is natural to analyze the system in the limit of the ideal Ohm's law, that is when $\eta = 0$. In this limit the functions ${A_0}'$ and ${\psi_0}'$ are given by

$$ {A_0}' = \frac{\alpha}{\alpha^2 - 1}\frac{Er}{A_1} + \frac{a}{\alpha}r + \frac{b}{\alpha r}, \qquad {\psi_0}' = \frac{1}{\alpha^2 - 1}\frac{Er}{A_1} + ar + \frac{b}{r}, \tag{7} $$

with constants α, a and b. From (7) it emerges that the the velocity field, the magnetic field (and in turn the current density) become singular in correspondance to the zeros of the function A_1. The presence of singularities indicates that at the points where A_1 vanishes the approximation of the ideal Ohm's law breaks down. These singularities can be removed considering a finite resistivity which smooths out the magnetic field gradients and prevent the accumulation of infinite magnetic flux at the zeros of A_1. Although A_1 in general vanishes for different values of r, most of the important features of the resistive solutions emerge even restricting the analysis to the neighbourhood of one zero of A_1, arbitrarily chosen, that from now on we denote as critical radius r_c. Explicit expressions for the exact resistive solutions and their detailed analysis can be found in [3-5].

An example of magnetic and velocity field configuration resulting from the exact resistive solutions is provided in Fig. 1. From the plot it can be seen that the topology of the magnetic field is characterized by the presence of an X-type null point located at $(r = r_c, \theta = 0)$. The null point is situated at the crossing of two separatrices lines, one of which is crossed by the plasma flow. Along the other separatrix, corresponding to the arc $r = r_c$, a curved current sheet is formed. This concentration of current density implies, in spite of the very small resistivity, a local violation of the frozen-in condition and allows for reconnection of magnetic field lines.

The magnetic configuration can be imagined as generated by three sources with alternating polarities lying on the plane $y = 0$. Therefore these solutions represent candidate models for the gradual phase of the wide class of solar flares generated by three sources with alternating polarities on the photospheric plane.

3 Solutions in $2\frac{1}{2}$D

The ansatz (6) presented in the previous section can be extended to provide three dimensional (3D) solutions with translational invariance along z. In fact seeking solutions of Eqs. (1-2) of the form

$$\mathbf{B} = \frac{1}{r}\frac{\partial A}{\partial \theta}\hat{\mathbf{r}} - \frac{\partial A}{\partial r}\hat{\boldsymbol{\theta}} + [H_1(r)\theta + H_0(r)]\hat{\mathbf{z}}, \quad \mathbf{v} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\mathbf{r}} - \frac{\partial \psi}{\partial r}\hat{\boldsymbol{\theta}} + [V_1(r)\theta + V_0(r)]\hat{\mathbf{z}} \quad (8)$$

with A and ψ as in (6), a system of eight ODEs for eight unknown functions is obtained, proving the compatibility of the ansatz (8). Moreover four of the eight equations of the resulting system coincide with the equations of the 2D case considered in Sec. 1. Therefore the radial and azimuthal components of $\mathbf{B}$ and $\mathbf{v}$ are the same as in the purely 2D case. Exact solutions derived using the ansatz (8) have been presented and discussed in [6]. Since in these solutions B_z can be finite everywhere, in general no three-dimensional null point is present in the resulting magnetic configuration. Therefore these solutions can represent a concrete case of magnetic reconnection in 3D at a site of finite magnetic field. Furthermore in the ideal limit these $2\frac{1}{2}$D solutions provide an example of solutions of the MHD equations with singularities due to the presence of singular magnetic field lines, that is of magnetic field lines with a finite component of the electric field parallel to them [7].

4 Solutions in 3D

The ansatz

$$\begin{aligned}\mathbf{B} &= \left[\frac{A_1(r)}{r} - \frac{\zeta}{2}r\right]\hat{\mathbf{r}} + [Y(r) - A_1'(r)\theta]\hat{\boldsymbol{\theta}} + [H_1(r)\theta + H_0(r) + \zeta z]\hat{\mathbf{z}},\\ \mathbf{v} &= \left[\frac{\psi_1(r)}{r} - \frac{\beta}{2}r\right]\hat{\mathbf{r}} + [W(r) - \psi_1'(r)\theta]\hat{\boldsymbol{\theta}} + [V_1(r)\theta + V_0(r) + \beta z]\hat{\mathbf{z}},\end{aligned} \quad (9)$$

with constants ζ and β, is compatible with the Eqs. (1-3) and provides a system of ODEs from which fully 3D solutions for magnetic reconnection can be found. Particular solutions of the system for A_1 and ψ_1 are given by

$$\psi_1 = \frac{\beta}{\zeta}A_1 = \frac{\beta}{\zeta}a_1 \ln r \quad (10)$$

with constant a_1. Furthermore if the additional constraints $W = Y = 0$ or $H_1 = V_1 = 0$ are imposed then the systems described in [3] are retrieved. If A_1 and ψ_1 have the form specified in (10) then the solutions obtained form the ansatz (9) can describe reconnection at one or two null points. In the former case information about the structure of the solution in the limit of negligible field curvature can be obtained considering an analogous, but simpler, system in cartesian coordinates which can be derived using the ansatz

$$\mathbf{B} = (a - \zeta)x\hat{\mathbf{x}} + [Y(x) - ay]\hat{\mathbf{y}} + [H_1(x)y + H_0(x) + \zeta z]\hat{\mathbf{z}}, \tag{11}$$

$$\mathbf{v} = \beta\left(\frac{a}{\zeta} - 1\right)x\hat{\mathbf{x}} + \left[W(x) - \beta\frac{a}{\zeta}y\right]\hat{\mathbf{y}} + [V_1(x)y + V_0(x) + \beta z]\hat{\mathbf{z}}. \tag{12}$$

The above form for the magnetic field can be interpreted as the superposition of a linear potential field with "disturbances" provided by the terms Y, H_1 and H_0 acting along y and z. The resulting solutions describe reconnection at a null point with a non-symmetric concentration of current density on the fan plane.

5 Conclusions

Properties of different classes of exact solutions of the steady MHD equations for incompressible plasmas have been presented. The differences between classes of solutions in different dimensions have been emphasized. These solutions describe magnetic reconnection in a curvilinear geometry that turns out to be particularly useful for the modeling of a large class of solar flares.

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7 References

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