Link Invariants of Electromagnetic Fields

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The cross-helicity integral is known in fluid dynamics and plasma physics as a topological invariant which measures the mutual linkage of two divergence-free vector fields, e.g., magnetic fields, on a three-dimensional domain. Generalizing this concept, a new topological invariant is found which measures the mutual linkage of three closed two-forms, e.g., electromagnetic fields, on a four-dimensional domain. The integral is shown to detect a separation of the cross helicity between two of the fields with the help of the third field. It can be related to the triple linking number known in knot theory. Furthermore, it is shown that the well-known three-dimensional cross helicity and the new four-dimensional invariant are the first two examples of a series of topological invariants which are defined by $n-1$ field strengths $F = dA$ on a simply connected n-dimensional manifold M^n.

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Introduction.—In the physics of fluids one is often interested in quantities which remain unchanged under smooth deformations induced by fluid motion. These quantities are usually referred to as "topological." A well-known example is that of magnetic helicity in plasma physics; for a magnetic field, $B_{ij} = \partial_i A_j - \partial_j A_i$, an evolution according to an ideal induction equation,

$$\partial_t B_{ij} + \partial_i(v^k B_{kj}) - \partial_j(v^k B_{ki}) = 0, \tag{1}$$

can be envisaged as a flux-preserving deformation along the velocity field $\mathbf{v}$. In vector notation the above equation reads $\partial_t \mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0$.

This equation not only implies the conservation of magnetic flux through a comoving surface and preserves magnetic field lines (known as Alfvéns theorem), it also conserves the magnetic helicity [1]:

$$H(B) = \int_{M^3} \epsilon^{ijk} A_i \partial_j A_k \sqrt{|g|} d^3x = \int_{M^3} A \wedge dA. \tag{2}$$

The last expression gives the integral in terms of differential forms and shows that the metric dependence cancels between the volume element and the ϵ tensor. (In the following we give all expressions also in the more compact notation of differential forms, because it simplifies the proofs and explicitly shows the topological nature of the quantities.) In order to have a gauge-invariant quantity, the domain M^3 has to have a boundary ∂M^3 which is "magnetic"; i.e., there is no magnetic flux through the boundary: $B_{ij} v^i w^j|_{\partial M^3} = 0$ for all vectors $\mathbf{v}$ and $\mathbf{w}$ tangent to the boundary.

Magnetic helicity has attracted a great deal of attention (see [2] and references therein) since it provides a lower bound for the magnetic energy [3–6] and is a very robust invariant. Magnetic helicity can be considered as a special case of the cross helicity, which is defined for two distinct magnetic fields B and B':

$$H(B, B') = \int_{M^3} \epsilon^{ijk} A_i \partial_j A'_k \sqrt{|g|} d^3x = \int_{M^3} A \wedge dA'. \tag{3}$$

The cross helicity is a measure of the mutual Gauß linking between the field lines of B and B'. It is gauge independent provided ∂M^3 is a magnetic boundary for both fields, and it is also preserved under ideal dynamics, i.e., an evolution according to Eq. (1) for B and B'. Substituting $B = B'$ in (3) returns the expression for the self-helicity (2).

Helicity is of three-dimensional nature due to the integration over M^3, so the question arises whether there exists a generalization to higher dimensions, especially to the physically interesting case of a four-dimensional space-time. In this Letter we extend the concept of cross helicity to the four-dimensional case and give an explicit interpretation in terms of the dynamics of electromagnetic fields. The new invariant is based on the so-called Novikov invariant [7] described in [8,9], which received additional geometrical interpretation in [10,11]. In the last section we demonstrate that the concept can be further generalized to arbitrary dimensions $n > 3$.

Link-invariant in four dimensions.—In the following any field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is interpreted as an electromagnetic field over a simply connected space-time domain M^4. Given three such field strengths, the new invariant is defined as

$$\begin{aligned} N_{ijk}(F^i, F^j, F^k) &:= \int_{M^4} \epsilon^{\kappa\lambda\mu\nu} A^i_\kappa A^j_\lambda \partial_\mu A^k_\nu \sqrt{|g|} d^4x \\ &= \int_{M^4} A^i \wedge A^j \wedge dA^k, \end{aligned} \tag{4}$$

if the following conditions hold:

$$\epsilon^{\kappa\lambda\mu\nu} \partial_\kappa A^i_\lambda \partial_\mu A^k_\nu \sim F^i \wedge F^k = 0, \tag{5}$$

$$\epsilon^{\kappa\lambda\mu\nu} \partial_\kappa A^j_\lambda \partial_\mu A^k_\nu \sim F^j \wedge F^k = 0, \tag{6}$$

$$F^k_{\mu\nu} V^\mu W^\nu(p) = 0 \forall V, W \in T_p \partial M^4. \tag{7}$$

The first two conditions imply conservation of the cross helicities, which can be seen from rewriting, e.g., (5) as

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$\nabla_\kappa(\epsilon^{\kappa\lambda\mu\nu}A^i_\lambda\partial_\mu A^k_\nu) = 0$. The third condition implies that the three-dimensional boundary ∂M^4 is "electromagnetic," in the sense that there is no electromagnetic flux through any 2D surface $S \subset \partial M^4$. On a noncompact domain without boundary, instead of (7) we should demand a sufficiently rapid decay of the fields to ensure finiteness of the integrals.

The invariant remains unchanged under gauge transformations of the electromagnetic potentials, since a gauge transformation, $A^i \to A^i + d\psi$, yields

$$\begin{aligned} N_{ijk} &\to N_{ijk} + \int_{M^4} d\psi \wedge A^j \wedge dA^k \\ &= N_{ijk} + \int_{\partial M^4} \psi A^j \wedge dA^k + \int_{M^4} \psi F^j \wedge F^k = N_{ijk}, \end{aligned} \tag{8}$$

where in the last step we used conditions (7) and (6) to show that both integrals vanish. The gauge invariance with respect to (w.r.t.) A^j is proved the same way, while invariance w.r.t. A^k is obvious.

In order to see that the invariant is a topological quantity, consider a family of diffeomorphisms generated by a vector field X and parametrized by s. We can formulate the flux-conserving deformation of the field strength F under these diffeomorphisms with the help of the Lie derivative of differential forms $L_X = d \circ i_X + i_X \circ d$:

$$\partial_s F(x, s) + L_X F(x, s) = 0, \tag{9}$$

which implies

$$\partial_s A(x, s) + L_X A(x, s) = d\phi. \tag{10}$$

The change of N_{ijk} due to the deformation results from

$$\begin{aligned} \frac{d}{ds}N_{ijk} &= \int_{M^4}(\partial_s + L_X)(A^i \wedge A^j \wedge dA^k) \\ &= \int_{M^4}(\partial_s + L_X)A^i \wedge A^j \wedge dA^k + \int_{M^4} A^i \wedge (\partial_s + L_X)A^j \wedge dA^k + \int_{M^4} A^i \wedge A^j \wedge d(\partial_s + L_X)A^k. \end{aligned} \tag{11}$$

Inserting (10) we see that any deformation just amounts to gauge transformations, which leave N_{ijk} unchanged.

Note that the symmetry of N_{ijk} results in

$$N_{ijk} + N_{jki} + N_{kij} = \int_{\partial M^4} A^i \wedge A^j \wedge A^k. \tag{12}$$

The integral on the right-hand side is a third order link invariant in three dimensions as described in [12]. Its definition requires only the three conditions $d(A^i \wedge A^j) = 0$ (with cyclic permutations of the indices i, j, k) on ∂M^4. Thus, these conditions are also sufficient for the sum on the left-hand side to be well-defined and replace here the conditions (5)–(7) of the individual N_{ijk}.

Another property of the N_{ijk} is the antisymmetry with respect to the first two arguments, i.e. $N_{ijk} = -N_{jik}$. This implies that the invariant vanishes whenever the first two entries are identical. If, in contrast, the last two entries coincide, $F_j = F_k$, one recovers the Novikov invariant $N_{ijj}(F^i, F^j, F^j)$ (see [8] or [9]). In this case the condition (6) turns into the continuity equation for the self-helicity of F^j.

Interpretation.—We note first that the invariant is represented by an integral over a four-dimensional domain, e.g., a subdomain of space-time. Thus it does not measure an instantaneous property of electromagnetic fields as the helicity integral does, but a property of a (time-dependent) process. In particular, the invariant vanishes for any stationary triplet of fields.

Despite these differences we can exploit the similarities to helicity to obtain a qualitative picture for the invariant: The integral (4) is the average over the domain M^4 of the contraction of the potential A^i with the vector field $H^\kappa = \epsilon^{\kappa\lambda\mu\nu}A^j_\lambda\partial_\mu A^k_\nu$. This vector field—by condition (6)—preserves the volume form. Therefore, we can, following a method demonstrated in [9] for magnetic helicity, apply a result from ergodic theory: We replace $A^i_\mu H^\mu$ by its average w.r.t. the flow ψ_t of H without changing the value of the integral,

$$A^i_\mu H^\mu \to \lim_{t\to\infty}\frac{1}{t}\int_0^t A^i_\mu H^\mu d\tau.$$

This additional averaging procedure involves the integration of the potential along the field lines of H, and for a closed field line Γ, this, by Stokes's theorem, yields the electromagnetic flux through a surface bounded by the field line

$$\int_0^t A^i_\mu H^\mu d\tau = \int_I A^i(\psi_t(x)) = \int_\Gamma A^i = \int_{\partial^{-1}\Gamma} F^i.$$

To treat the ergodic case where field lines are not closed, one has to introduce a so-called "system of short paths," which provide a closure of the curves in the asymptotic sense [9,13]. We can therefore conclude: *The new invariant is an average measure of the electromagnetic flux of F^i through (surfaces asymptotically bounded by) field lines of the cross helicity $A^j \wedge F^k$.* However, in contrast to the three-dimensional case, where an analogous statement holds true for helicity, this result does *not* admit an intuitive interpretation as linkage between field lines, since in dimension four any two one-dimensional objects can be untied. In order to explain why the N_{ijk} can nevertheless be considered a generalized linking invariant, and to make more precise its geometrical and physical meaning, we give an elementary example.

Elementary example.—First we note that the conditions (5) and (6) are satisfied for the following important class of applications. Consider an evolution of a magnetic field in magnetohydrodynamics (MHD) which possesses a flux-transporting velocity V, i.e., which satisfies

$$V^{\mu} F_{\mu\nu} \sim i_V F = 0. \tag{13}$$

This includes cases of ideal dynamics in MHD where V is the plasma velocity. In this case we have $V^0 = 1$ and (13) leads directly to (1). However, a velocity satisfying (13) may exist also in many other nonideal cases as demonstrated in [14]. In particular, certain forms of magnetic reconnection satisfy (13) with $V^0 = V^0(x)$. If the magnetic field has a natural decomposition into three subfields, $\mathbf{B} = \mathbf{B}^1 + \mathbf{B}^2 + \mathbf{B}^3$ (e.g., a system of three flux tubes), where each component is divergence-free and satisfies (13), conditions (5) and (6) are automatically satisfied. This is due to

$$\left.\begin{matrix} i_V F^l = 0 \\ i_V F^m = 0 \end{matrix}\right\} \Rightarrow i_V(F^l \wedge F^m) = 0 \Rightarrow F^l \wedge F^m = 0, \tag{14}$$

for any V vanishing at most on sets of measure zero. Since the last result holds as well for $F^l = F^m = F$ Eq. (13) implies $F \wedge F = 0$.

To construct an explicit example, we consider the dynamics of two magnetic flux tubes of the above mentioned type; i.e., each of the corresponding electromagnetic fields F^1, F^2 is respecting (13). For simplicity, we assume the flux tubes to be infinitesimally thin. Then, besides from moving within space, these idealized rings can annihilate by shrinking to a point or be created by the reversed process. They can also split or join by essentially two-dimensional reconnection [14].

Assume we let two rings create from an electric field (lowest frame of Fig. 1) and, by two successive reconnections, let them form a pair of spatially separated Hopf links (upper frame on the left side). After reversing the process the whole evolution is consistent with the boundary condition (7) and shows a process in which two regions of positive and negative magnetic helicity ($A^2 \wedge F^1$) are created for a certain period in time. This separation of magnetic helicity can be detected with the help of a third field (F^3), a flux tube that in its movement encompasses one of the separated spatial links by sweeping out a sphere S as shown in Fig. 2. This will result in a nonvanishing value for N_{321}. The crucial property here is that the regions of opposite polarity are not connected by magnetic fields for a certain interval in time and hence are not only the result of a gauge transformation.

The example shows that N_{ijk} can measure processes involving helicity separation. Such processes are known to occur, for instance, in the solar atmosphere (see, e.g., [15]) and are important for the understanding of the solar dynamo. We therefore adopt the following interpretation: *The invariant N_{ijk} detects the topological separation of cross helicity.*

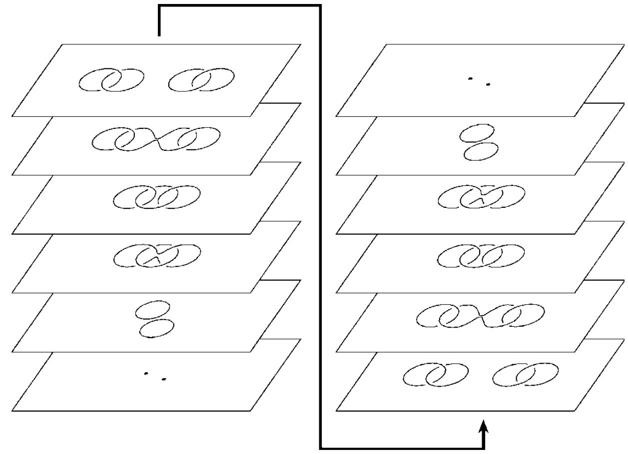

FIG. 1. Space-time evolution of two flux rings which form separated regions of opposite magnetic helicity.

The relation to the triple linking number.—In the above example the property $F^l \wedge F^l = 0$ for all three fields allows us to relate the invariant to the triple linking number known in knot theory. In space-time the corresponding electromagnetic fields of our example are localized on the "world sheets" swept over by the flux rings in time. It is possible to find a gauge for the corresponding potential such that it is supported on a three-dimensional domain bounded by the world sheet. In particular, the fields F^1 and F^2 form two tori T_1, T_2 in space-time. The cross helicity $A^2 \wedge F^1$ is localized on a subset of T_1 and can be gauged such that it is represented by a longitudinal circle of the torus. The support of A^3, a three-dimensional ball bounded by the world sheet of the third ring, has necessarily nonempty intersection with this helicity loop. Counting such (signed) intersections is a well-known concept in the theory of knots and links and gives rise to the so-called triple linking numbers of closed surfaces K in $\mathbb{R}^4$ as described in [16]:

$$tlk(K^i, K^j | K^k) := \sum \mathrm{sgn}(\partial^{-1} K^i \cap \partial^{-1} K^j \cap K^k).$$

Denoting the electromagnetic flux transversal to a surface

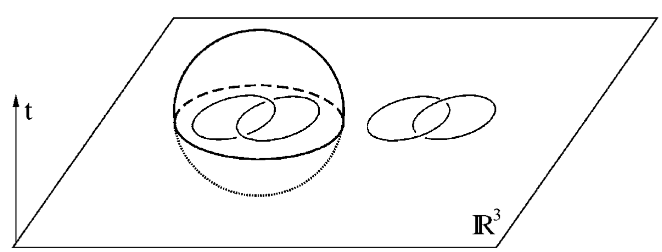

FIG. 2. A flux ring enclosing in its history two linked flux rings.

K^i as Φ^i, this yields

$$N_{321} = \Phi^3 \Phi^2 \Phi^1 tlk(S, T_2|T_1),$$

and for the example given $tlk(S, T_2|T_1) = tlk(S, T_1|T_2) = 1$, while $tlk(T_1, T_2|S) = 0$.

It should be noted that it is only due to $F \wedge F = 0$ that we can associate surfaces to electromagnetic field strengths and give an interpretation in terms of the linking of surfaces. The latter is also found in [10]. We nevertheless consider the invariant N_{ijk} a linking invariant of two-form field strengths in general, since it reproduces the triple linking number whenever such an assignment exists.

Higher-dimensional generalization.—It is possible to give a generalization of the new invariant to arbitrary dimensions. Using $n-1$ two-form field strengths $F = dA$ on a simply connected n-dimensional manifold M^n, we define the gauge-invariant, topological expressions

$$N_{i_1 \ldots i_{n-1}} := \int_{M^n} A^{i_1} \wedge A^{i_2} \wedge \ldots A^{i_{n-2}} \wedge F^{i_{n-1}}, \tag{15}$$

provided that

$$F^j \wedge F^{i_{n-1}} = 0 \forall j \in (i_1 \ldots i_{n-2})$$

and

$$F^{i_{n-1}}_{\mu\nu} V^\mu W^\nu (p) = 0 \forall V, W \in T_p \partial M^n.$$

In order to establish a relation to higher-dimensional linking, we consider field strengths supported on closed $n-2$-dimensional manifolds K^j in $\mathbb{R}^n$. Assuming unit fluxes Φ^j, an algebraic evaluation of the invariant can once again be carried out by counting signed intersections:

$$N_{i_1 \ldots i_{n-1}} = \sum \mathrm{sgn}(\partial^{-1} K^{i_1} \cap \ldots \cap \partial^{-1} K^{i_{n-2}} \cap K^{i_{n-1}}).$$

This expression reproduces the algebraic formulas for the Gauß-linking and the triple linking number in dimensions $n = 3$ and $n = 4$, respectively, and can thus be seen as a natural generalization to arbitrary dimensions.

Conclusion.—In this Letter we have shown how the concept of cross helicity can naturally be extended to higher dimensions. Similar to the three-dimensional case, these cross helicities are topological, gauge-invariant quantities that are based on linking numbers of boundaries of codimension two. In dimension four, we provided an interpretation of the new invariant in terms of electromagnetic fields. However, applications to other closed two-forms, as, for instance, the vorticity in hydrodynamics, are conceivable. Examining a simple example configuration, it was demonstrated that the invariant is a topological measure for (electromagnetic) processes within a space-time domain involving local separation of cross-helicity density.

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