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Magnetic diffusion and the motion of field lines

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Diffusion of a magnetic field through a plasma is discussed in one-, two- and three-dimensional configurations, together with the possibility of describing such diffusion in terms of a magnetic flux velocity, which, when it exists, is in general non-unique. Physically useful definitions of such a velocity include doing so in terms of the energy flow or in such a way that it vanishes in a steady state. Straight field lines (or plane flux surfaces) diffuse as if flux is disappearing at a neutral sheet, whereas circular field lines (or cylindrical flux surfaces) do so as if flux is disappearing at an 0-type neutral line. In three dimensions it is not always possible to define a flux velocity, for example when the magnetic flux through a closed field line is changing in time. However, in at least some such cases it is possible to describe the behaviour of the magnetic field in terms of a pair of quasi-flux-velocities.

Keywords: Magnetohydrodynamics; Magnetic diffusion; Plasmas

1. Introduction

In most of the universe the magnetic field behaves as if it is frozen to the plasma, so that magnetic field lines may be said to move around with the plasma velocity. However, in small regions, of very strong electric current concentration, typically filaments or sheets, the magnetic field can slip through the plasma and reconnect, with far-reaching consequences such as changes of magnetic topology and conversion of magnetic energy into bulk kinetic energy, heat and fast particle energy (e.g., Priest and Forbes, 2000). This is responsible for a wide range of dynamic phenomena, including solar flares and coronal heating on the Sun and the flux transfer events and geomagnetic substorms in the Magnetosphere.

Understanding the fundamental processes of advection and diffusion of magnetic field is a key part of describing the behaviour of magnetic fields in magnetohydrodynamics. In an ideal medium the magnetic field (**B**), electric field (**E**) and

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plasma velocity ($\mathbf{v}$) are related by

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0}, \tag{1}$$

but in a non-ideal medium the right-hand side is non-zero. In particular, in a resistive medium this becomes

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \nabla \times \mathbf{B} \tag{2}$$

and the corresponding form of the induction equation is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \nabla \times (\eta \nabla \times \mathbf{B}). \tag{3}$$

Physically, this may be interpreted as saying that the magnetic field changes in time due to two effects on the right-hand side, namely the advection of magnetic field with the plasma and its diffusion through the plasma. In the particular case where the magnetic diffusivity (η) is uniform in space, equation (3) reduces to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}. \tag{4}$$

In such a diffusive medium, a magnetic flux velocity ($\mathbf{w}$), if it exists for a given magnetic field variation, satisfies

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{w} \times \mathbf{B}).$$

Our present understanding of the existence and properties of $\mathbf{w}$ is described in section 2.

The aim if this article is partly to understand how diffusion occurs and magnetic flux disappears in a variety of situations, and partly to see whether the concept of the motion of field lines, which has proved so useful in an ideal medium, may also be employed in a diffusive medium as an aid to understanding diffusion. In this article we focus on the fundamental question of pure resistive diffusion of magnetic field in the absence of a plasma flow, which is a more primitive question than the nature of reconnection that has been discussed elsewhere (Schindler *et al.*, 1988; Priest and Forbes, 2000; Priest *et al.*, 2003). In this case equation (3) for the evolution of the magnetic field reduces to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\eta \nabla \times \mathbf{B}). \tag{5}$$

Section 2 summarises our current understanding (e.g., Hornig, 2001) of the nature of a magnetic flux velocity ($\mathbf{w}$), while section 3 describes the behaviour of a one-dimensional magnetic field, and sections 4 and 5 extend the discussion to fields in two and three dimensions, respectively.

2. The concept of a magnetic flux velocity (w)

Under ideal evolution, equation (1) holds and the magnetic field is frozen to the plasma, so that the curl of (1) gives

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = \mathbf{0}. \tag{6}$$

One of the consequences of this is the conservation of magnetic flux (Alfvén's frozen-flux theorem),

$$\int_C \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \text{constant},$$

i.e. the flux through a comoving surface C (a surface moving with $\mathbf{v}$) is conserved. This in turn implies the conservation of magnetic field lines, together with conservation of magnetic nulls and of knots and linkages of field lines. The far reaching consequences of (6) for the evolution of the magnetic field are derived from the algebraic form of the equation; they make no use of the fact that $\mathbf{v}$ is the plasma velocity. Thus we can ask whether also for non-ideal evolution such as (3) a velocity exists which yields an equation of the form (6). This velocity will in general differ from the plasma velocity and hence we write

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{w} \times \mathbf{B}) = \mathbf{0}, \tag{7}$$

and call $\mathbf{w}$ a flux transporting velocity.

For the case when the ideal Ohm's law (1) holds the velocity ($\mathbf{w}$) with which the magnetic field lines may be said to move can be identified with the plasma velocity ($\mathbf{v}$). For more general cases we have to answer the question about the existence and uniqueness of such a flux transport velocity. In order to gain some insight into these questions we integrate (7) to be able to compare it with other forms of Ohm's law. This yields

$$\mathbf{E} + \mathbf{w} \times \mathbf{B} = \nabla F, \tag{8}$$

where F is an arbitrary function (a function of integration). We can compare this with an arbitrary Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{N}, \tag{9}$$

where $\mathbf{N}$ denotes an arbitrary non-ideal term, that is, a term which is not just the gradient of a scalar and therefore can break the ideal flux transport equation (6). For the existence of a flux transporting velocity we have to rewrite equation (9) in form of (8) so that $\mathbf{N}$ must have the form

$$\mathbf{N} = \underbrace{(\mathbf{v} - \mathbf{w})}_{:=\mathbf{u}} \times \mathbf{B} + \nabla F. \tag{10}$$

A sufficient condition (provided $\mathbf{B} \neq \mathbf{0}$) to represent $\mathbf{N}$ in form of (10) and hence for the existence of $\mathbf{u}$ (or $\mathbf{w}$, respectively) is

$$\mathbf{B} \cdot \nabla F = \mathbf{B} \cdot \mathbf{N} \equiv \mathbf{B} \cdot \mathbf{E}. \tag{11}$$

This equation can always be solved if for instance $\mathbf{E} \cdot \mathbf{B} = 0$, that is, if $\mathbf{N}$ is perpendicular to $\mathbf{B}$. Then $F \equiv 0$ is a trivial solution. Important examples are the resistive two-dimensional case ($\mathbf{N} = \eta \mathbf{j}$), which we shall consider in the following sections, and the case when $\mathbf{N}$ represents a Hall term: $\mathbf{N} = (ne)^{-1} \mathbf{j} \times \mathbf{B}$.

If $\mathbf{B} \cdot \mathbf{N} = \mathbf{B} \cdot \mathbf{E} \neq \mathbf{0}$ we can still solve (11) if there exists a certain surface (called here a "transversal" surface) such that all magnetic field lines cross this surface exactly once. In this case we can obtain F from an integration of (11) along magnetic field lines. We parametrize the magnetic field line by $\mathbf{x}(s)$ and start from a point $\mathbf{x}(0)$ on the transversal surface C and integrate $N_\parallel$ along a magnetic field line to $\mathbf{x} = \mathbf{x}(s)$:

$$F(\mathbf{x}) = \int_0^s N_\parallel \, \mathrm{d}s + F(\mathbf{x}(0)); \quad \frac{\mathrm{d}\mathbf{x}(s)}{\mathrm{d}s} = \frac{\mathbf{B}}{\|\mathbf{B}\|}; \quad \mathbf{x}(0) \in C; \quad N_\parallel = \frac{\mathbf{N} \cdot \mathbf{B}}{\|\mathbf{B}\|}. \tag{12}$$

The condition that the field lines cross the surface only once ensures that the integration does not lead to ambiguities depending on whether we integrate forward or backward along the field lines, as could be the case if we had closed loops.

However, there are also cases (see section 5) where (11) has no solutions. This is for instance the case if there are closed magnetic field lines with

$$\oint N_\parallel \, \mathrm{d}s \neq 0,$$

which implies that no flux transporting velocity exists. In addition, boundary conditions on F or $\mathbf{w}$ can prevent the existence of a solution. This is for instance the case for three-dimensional reconnection. Recently it has been discovered (Hornig and Priest, 2003; Priest *et al.*, 2003) that, during three-dimensional magnetic reconnection at an isolated diffusion region, a flux velocity ($\mathbf{w}$) in general does not exist and instead can be replaced by a pair of flux velocities ($\mathbf{w}_{\text{in}}$ and $\mathbf{w}_{\text{out}}$). $\mathbf{w}_{\text{in}}$ describes the behaviour of field lines that are anchored on one side of the diffusion region, while $\mathbf{w}_{\text{out}}$ describes those anchored on the other side.

Note that F, if it exists, is determined only up to an arbitrary function (an initial condition for F on the transversal surface). Once F is determined we can solve (8) for the perpendicular component of $\mathbf{w}$:

$$\mathbf{w}_\perp = \frac{(\mathbf{E} - \nabla F) \times \mathbf{B}}{B^2}. \tag{13}$$

The $\mathbf{w}_\perp$ is a flux transporting velocity; in other words, there is conservation of flux and field line topology with respect to $\mathbf{w}$ with $\mathbf{u}$ being a slippage velocity of the plasma relative to the magnetic field lines. Particular care has to be taken at points where $\mathbf{B}$ vanishes. At null points of $\mathbf{B}$ the transport velocity might become singular, indicating either reconnection or a loss or generation of magnetic flux.

Thus $\mathbf{w}_\perp$ is well-defined, apart from null points of the magnetic field. However, even the perpendicular component of $\mathbf{w}$ is in general not unique due to the non-uniqueness of F (Hornig and Schindler, 1996; Hornig, 2001). This non-uniqueness results from the freedom to add to any solution $(F, \mathbf{u})$ a solution $(\tilde{F}, \tilde{\mathbf{u}})$ of the "homogenous" equation

$$\tilde{\mathbf{u}} \times \mathbf{B} = \nabla \tilde{\mathbf{F}}.$$

However, uniqueness can be achieved by specifying corresponding boundary or initial conditions for $\mathbf{w}$. Examples for this will be given in section 3.1.

In the case of pure resistive diffusion considered in this article $\mathbf{E} = \eta \nabla \times \mathbf{B}$, and so, provided $\mathbf{w}$ exists, (i.e., provided (11) is satisfied) (8) becomes

$$\eta \nabla \times \mathbf{B} + \mathbf{w} \times \mathbf{B} = \nabla F, \tag{14}$$

while (13) reduces to

$$\mathbf{w}_\perp = \frac{(\eta \nabla \times \mathbf{B} - \nabla F) \times \mathbf{B}}{B^2}.$$

3. Diffusion of magnetic field with straight field lines

Consider first for simplicity the way in which a one-dimensional magnetic field ($\mathbf{B} = B(x, t)\hat{\mathbf{y}}$) diffuses, for which (5) reduces to the equation

$$\frac{\partial B}{\partial t} = \frac{\partial}{\partial x}\left(\eta \frac{\partial B}{\partial x}\right). \tag{15}$$

3.1. *Uniform diffusivity*

In particular, suppose the diffusivity is uniform ($\eta = \eta_0$) and the magnetic field initially has a step profile

$$B(x, 0) = \begin{cases} +B_0, & x > 0, \\ -B_0, & x < 0, \end{cases}$$

representing an infinitesimally thin current sheet. If the magnetic field is held fixed at two points ($\pm\ell$) so that

$$B(\ell, t) = -B(-\ell, t) = B_0,$$

the solution to the diffusion equation

$$\frac{\partial B}{\partial t} = \eta_0 \frac{\partial^2 B}{\partial x^2} \tag{16}$$

is

$$B(x,t) = B_0 \frac{x}{\ell} + \frac{2B_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \exp\left(-n^2 \pi^2 \eta_0 t/\ell^2\right) \sin\left(\frac{n\pi x}{\ell}\right). \tag{17}$$

(The corresponding solution for an infinite range and an arbitrary initial profile is given in Appendix A.1.)

As can be seen in figure 1, the magnetic field diffuses away very rapidly towards the steady-state solution to (16), namely, $B(x) = B_0(x)/l$. Indeed, after a time of only $t = l^2/\eta_0$, the field is within a factor 10^{-4} of its final profile.

Energetically, a Poynting flux $\mathbf{E} \times \mathbf{B}/\mu = -(\eta/\mu)\partial B/\partial x\ \hat{\mathbf{x}}$ into the boundaries $x = \pm\ell$ and a decrease in the magnetic energy is balanced by Ohmic heating (j^2/σ). In the final steady state there is ohmic heating $j^2/\sigma = (\eta_0/\mu)(B_0/\ell)^2$ per unit length, which is provided by a continual inflow of energy through the boundaries at rates $\eta B_0^2/(\mu\ell)$ from both sides.

In this one-dimensional case a flux velocity ($\mathbf{w} = w\hat{\mathbf{x}}$) does exist and (14) becomes

$$\eta \frac{\partial B}{\partial x} + wB = E_0(t),$$

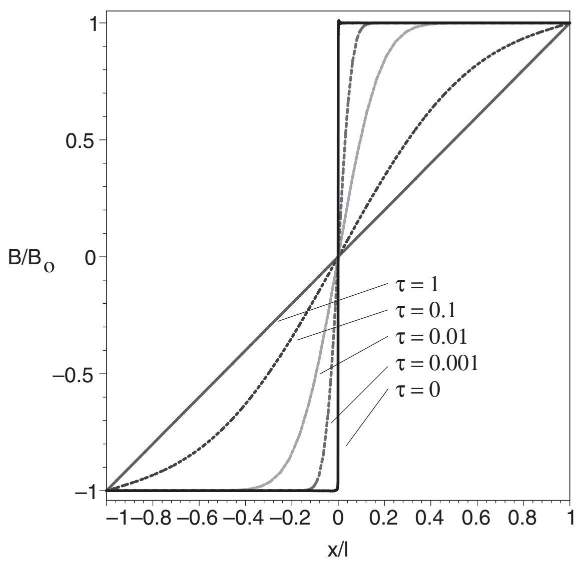

Figure 1. Diffusion of a one-dimensional field, having an initial step-function profile, with dimensionless time $\tau = \eta_0 t/\ell^2$.

with solution

$$w = -\frac{\eta}{B}\frac{\partial B}{\partial x} + \frac{E_0}{B}, \tag{18}$$

where E_0 is an arbitrary function of t, which represents a nonuniqueness in the form of the flux velocity.

So what are physically reasonable ways of choosing E_0? One is to choose $E_0 = 0$, so that $\mathbf{w} = \mathbf{E} \times \mathbf{B}/B^2$ and $\mathbf{w}$ is then a flux velocity associated with the energy flow, as shown in figure 2a. The field lines are initially stationary (except at the origin) and later move towards the origin with a singular velocity at the null.

We can say that the field is evolving as if the field lines are moving towards the origin and annihilating or disappearing there at a neutral sheet. As time increases, the flux velocity increases everywhere in magnitude towards its steady-state value.

In general, the solution (18) with B given by (17) is

$$w_x = -\frac{\eta_0}{\ell}\left[\frac{1 + \sum_1^\infty \exp\left(-n^2\pi^2\eta_0 t/\ell^2\right)\cos(n\pi x/\ell) - (E_0\ell/\eta_0 B_0)}{(x/\ell) + (2/\pi)\sum_1^\infty (1/n)\exp(-n^2\pi^2\eta_0 t/\ell^2)\sin(n\pi x/\ell)}\right]$$

and so we could instead choose $E_0(t) = \eta(\partial B/\partial x)_0$, i.e.,

$$E_0(t) = \frac{\eta_0 B_0}{\ell_0}\left(1 + 2\sum_1^\infty \exp\left(-n^2\pi^2\eta_0 t/\ell^2\right)\right),$$

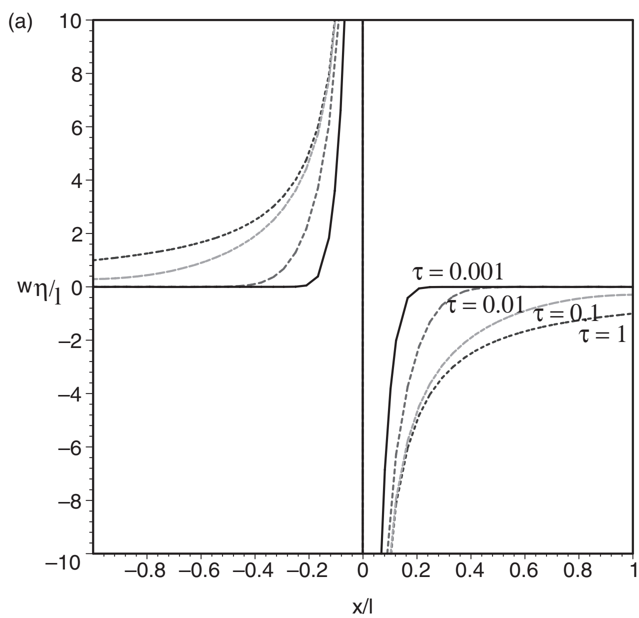

Figure 2. The flux velocity when (a) $E_0 = 0$, (b) $E_0 = \eta(\partial B/\partial x)_0$ and (c) $E_0 = \eta B_0/\ell$.

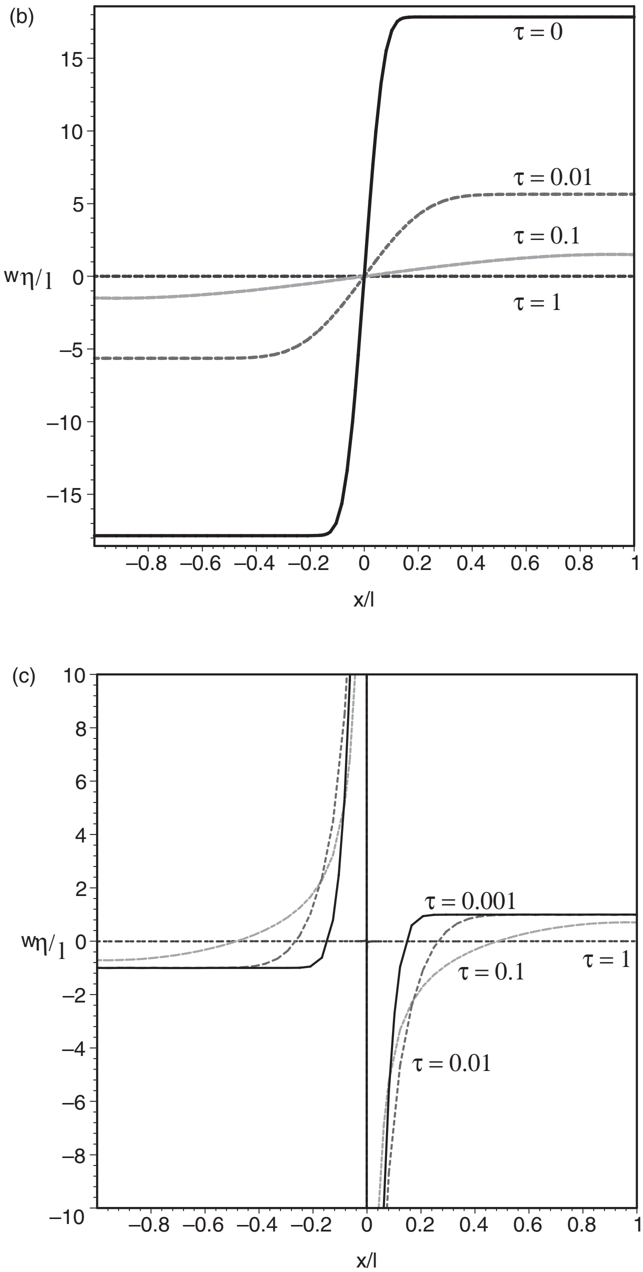

Figure 2. Continued.

which makes the flux velocity nonsingular at the origin, as shown in figure 2b. In this case the field lines move outwards towards the boundaries and again their velocity is nonzero in the final steady state.

A third possibility would be to choose E_0 constant in time in such a way that the flux velocity vanishes in the final steady state ($B(x) = B_0 x/\ell$). This can be done by putting

$E_0 = \eta B_0/\ell$ (figure 2c). In this case the field lines have a nonzero velocity initially which is singular at the origin and decreases everywhere towards zero in the final state. Whereas in case (a) the field lines all approach the origin and in case (b) they all approach $\pm\infty$, in this third case there is a combination of both types of behaviour.

3.2. *Spatially varying diffusivity* $\eta(x)$

In many applications of diffusion theory (Priest and Forbes, 2000), such as in the solar corona, the magnetic diffusivity (η) is not constant but varies spacially and/or temporally, either because of a temperature-dependence of η or because of a dependence on the electric current. We therefore suppose briefly in this section that the magnetic diffusivity varies in space, so that the magnetic field satisfies (15) in place of (16). Then to what equilibrium does the field diffuse and what is the behaviour of the field lines? Suppose first of all for simplicity that the diffusivity is piecewise uniform with

$$\eta(x) = \begin{cases} \eta_i, & 0 < x < L, \\ \eta_e, & L < x < a \end{cases} \tag{19}$$

and that the magnetic field has boundary conditions

$$B(0,t) = 0, \qquad B(a,t) = B_0.$$

In addition, the jump conditions at $x = L$ are that B and $\eta\partial B/\partial x$ be continuous. The resulting steady-state solution is piecewise linear and has the form

$$B(x) = \begin{cases} B_i(x) = B_0 \dfrac{x}{a}\left(\dfrac{\eta_e a/(\eta_i L)}{\eta_e/\eta_i + a/L - 1}\right), & 0 < x < L\ , \\[2ex] B_e(x) = B_0\left(\dfrac{x}{a} - 1\right)\left(\dfrac{a/L}{\eta_e/\eta_i + a/L - 1}\right) + B_0, & L < x < a. \end{cases} \tag{20}$$

A particular class of solutions to this problem is

$$\frac{B(x,t)}{B_0} = \begin{cases} \dfrac{x}{a}\left(\dfrac{\eta_e a/(\eta_i L)}{\eta_e/\eta_i + a/L - 1}\right) + \alpha \exp\left(-\Lambda^2 \dfrac{\eta_i t}{a^2}\right)\sin\left(\Lambda \dfrac{x}{a}\right), & 0 < x < L, \\[2ex] \left(\dfrac{x}{a} - 1\right)\left(\dfrac{a/L}{\eta_e/\eta_i + a/L - 1}\right) + 1 & \\[2ex] +\alpha \exp\left(-\Lambda^2 \dfrac{\eta_i t}{a^2}\right) \dfrac{\sin(\Lambda L/a)\sin\left(\Lambda\sqrt{\eta_i/\eta_e}(x/a - 1)\right)}{\sin\left(\Lambda\sqrt{\eta_e/\eta_i}(L/a - 1)\right)}, & L < x < a, \end{cases}$$

where α is arbitrary and Λ is any solution of

$$\tan(\Lambda L/a) = \sqrt{\eta_i/\eta_e}\tan\left[\Lambda\sqrt{\eta_i/\eta_e}(L/a - 1)\right]. \tag{21}$$

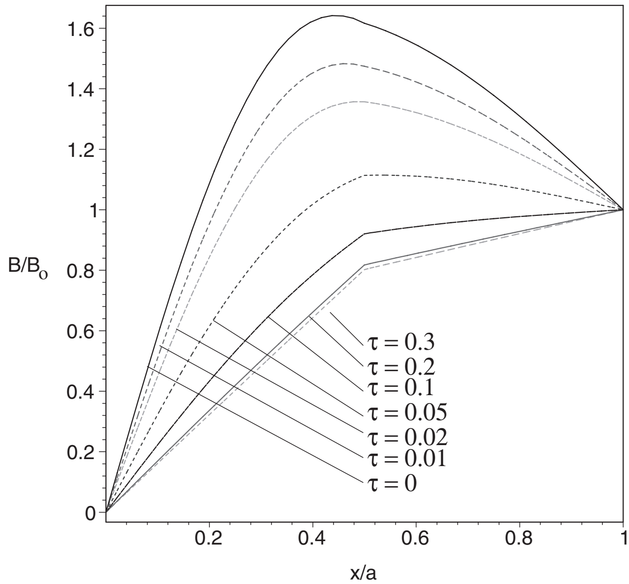

Figure 3. The diffusion of a magnetic field of the form (21) with a piecewise linear diffusivity and $\eta_i/\eta_e = 1/4$, $L/a = 1/2$, $\alpha = 1$, $\Lambda = 4.37$.

One member of this family is plotted in figure 3, showing how an initial state diffuses towards the final state (20). With $E_0 = 0$, all the field lines move towards the origin.

Analytical solutions to (15) may be constructed for a variety of different forms of $\eta(x)$. For example, a linear profile yields solutions in terms of Bessel functions, while a quadratic profile gives Legendre functions and a square-root profile gives Airy functions (Wilmot-Smith, PhD thesis, in preparation).

We may also solve (15) numerically to study the effect of an enhanced diffusion in a finite region (which may arise in, for example, the solar corona). In particular, figure 4 shows the effect of a diffusivity of the form

$$\eta = \frac{1}{2}\eta_0\left\{\tanh[40(x-0.25)] - \tanh[40(x-0.75)]\right\}$$

on an initial profile

$$B(x,0) = B_0 x^2/\ell^2$$

in the domain $0 \le x \le \ell$. The diffusivity has the value η_0 in the range $0.25 < x < 0.75$ and drops rapidly to essentially zero outside that range. Integrating (15) from a to b, say, where η vanishes at both a and b, gives

$$\frac{\partial}{\partial t}\int_a^b B\,\mathrm{d}x = \int_a^b \frac{\partial}{\partial x}\left(\eta\frac{\partial B}{\partial x}\right)\mathrm{d}x \tag{22}$$

$$= \left[\eta\frac{\partial B}{\partial x}\right]_a^b = 0, \tag{23}$$

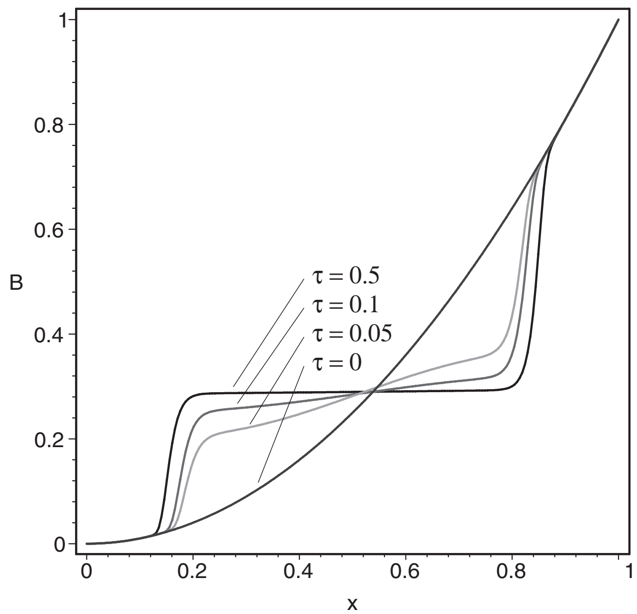

Figure 4. The effect of diffusion in an isolated region ($0.25 < x < 0.75$) showing (a) the magnetic profile for $F_0 = 1$ at different times $\tau = \eta t$.

and so the magnetic flux in a region of non-zero diffusivity lying between a and b must remain constant.

It can be seen from figure 4 that the initial field flattens out in the region of enhanced diffusivity and so forms current sheets at the ends of this region in such a way as to preserve the total flux in the domain. The flux velocities are in this case outwards from the centre of the diffusing domain towards the boundaries.

4. Diffusion of a magnetic field with circular field lines

Having determined how magnetic field lines diffuse in one dimension, and found that they can disappear either at a neutral sheet or at infinity, let us now turn to diffusion in two dimensions. The simplest approach mathematically is to consider one-dimensional solutions of 2D fields of the form $B(r,t)\hat{\boldsymbol{\theta}}$ having circular field lines, for which the diffusive limit of the induction equation becomes

$$\frac{\partial B}{\partial t} = \eta\left(\frac{\partial^2 B}{\partial r^2} + \frac{1}{r}\frac{\partial B}{\partial r} - \frac{B}{r^2}\right).$$

Writing the field as $B = -\partial A/\partial r$ in terms of a flux function ($A(r,t)$), this may be replaced by a simpler equation for A, namely,

$$\frac{\partial A}{\partial t} = \eta\left(\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r}\right), \tag{24}$$

which we require to solve under an initial condition of the form

$$A(r,0) = g(r). \tag{25}$$

A self-similar solution exists of the form

$$B(r,t) = \frac{Cr}{2\eta t^2} e^{-r^2/(4\eta t)},$$

which possesses a maximum value of $Ce^{-0.5}/\sqrt{2\eta t^3}$ at $r = \sqrt{2\eta t}$. As t increases, the total magnetic flux (C/t) decreases to zero, while the maximum field strength decreases and its location moves outwards. However, this solution does not satisfy (24) and it possesses the undesirable feature of having an initial flux that is infinite.

The solution subject to the initial condition (24) in a finite region $0 < r < a$ can, however, be shown (by separating the variables) to have the form

$$A(r,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2 \eta t} \mathrm{J}_0(\lambda_n r), \tag{26}$$

where J_0 is the Bessel function of order zero and

$$C_n = \frac{2}{\alpha^2} \int_0^a r g(r) \frac{\mathrm{J}_0(\lambda_n r)}{\mathrm{J}_0^2(\lambda_n a)} \, \mathrm{d}r$$

with a corresponding magnetic field

$$B(r,t) = \sum_{n=1}^{\infty} C_n \lambda_n e^{-\lambda_n^2 \eta t} \mathrm{J}_1(\lambda_n r). \tag{27}$$

The solution in an infinite region (Appendix A.2) is instead

$$A(r,t) = \frac{1}{2\eta t} \int_0^{\infty} \exp\left(-\frac{r^2+s^2}{4\eta t}\right) \mathrm{I}_0\left(\frac{rs}{4\eta t}\right) s g(s) \, \mathrm{d}s, \tag{28}$$

where I_0 is the hyperbolic Bessel function of zero order.

As an example, consider the diffusion of an isolated circular flux tube of flux F_0 at radius a with an initial field

$$B(r,0) = F_0 \, \delta(r-a)$$

and flux

$$A(r,0) = \begin{cases} 0, & r < a, \\ -F_0, & r > a. \end{cases}$$

The solution (26) then becomes

$$A(r,t) = -\frac{F_0}{2\eta t}\int_a^\infty s e^{-(s^2+r^2/(4\eta t))}\, \mathrm{I}_0\left(\frac{rs}{2\eta t}\right) \mathrm{d}s \tag{29}$$

with corresponding magnetic field

$$B(r,t) = \frac{F_0}{4\eta^2 t^2}\left\{-r\int_a^\infty s\exp\left(-\frac{r^2+s^2}{4\eta t}\right)\mathrm{I}_0\left(\frac{rs}{2\eta t}\right)\mathrm{d}s + \int_a^\infty s^2\exp\left(-\frac{r^2+s^2}{4\eta t}\right)\mathrm{I}_1\left(\frac{rs}{2\eta t}\right)\mathrm{d}s\right\}, \tag{30}$$

as shown in figure 5. It can be seen how the maximum field strength decreases in time, while the flux spreads outwards.

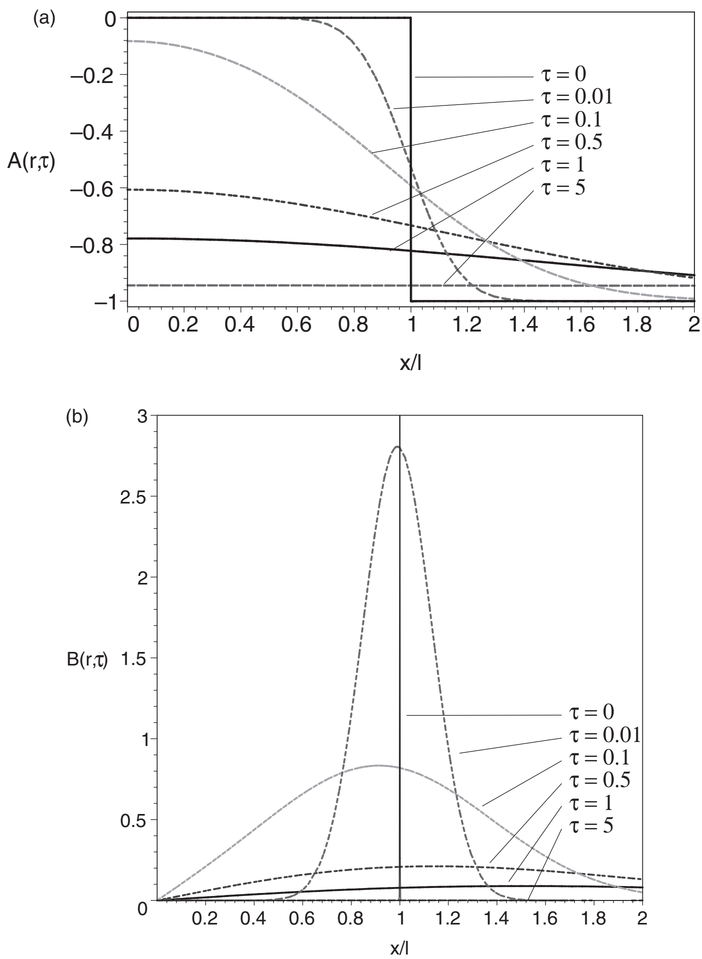

Figure 5. (a) The flux function $A(r,t)$ and (b) magnetic field $B(r,t)\hat{\boldsymbol{\theta}}$ of a circular flux tube diffusing away in time.

The resulting total flux is

$$A(0,t) - A(\infty,t) = F_0(1 - e^{-a^2/(4\eta t)}), \tag{31}$$

which decays away from an initial value of F_0 to zero over a time-scale of $a^2/(4\eta)$. The corresponding radial flux velocity is

$$w = \frac{1}{B}\left(E_0 - \eta\frac{\partial B}{\partial r}\right), \tag{32}$$

where E_0 is an arbitrary function of time, which can be chosen in a variety of ways. For example, the field line velocity associated with the Poynting flux ($\mathbf{E}\times\mathbf{B}/\mu$) has $E_0 = 0$ and so vanishes at maxima and minima of B where $\partial B/\partial r = 0$. In this case the motion of field lines is away from the maximum and towards both the 0-point and infinity, where the field lines are disappearing. Alternatively, we could choose $E_0(t)$ in such a way as to make the field line velocity vanish at infinity (or at the origin), in which case the field lines would be disappearing at the 0-point (or at infinity).

5. Magnetic field diffusion in three dimensions

As mentioned in section 2, a flux velocity ($\mathbf{w}$) exists if there is a function F such that

$$\eta\nabla\times\mathbf{B} + \mathbf{w}\times\mathbf{B} = \nabla F \tag{33}$$

holds, where $\eta\nabla\times\mathbf{B} = \mathbf{E}$. If $\mathbf{w}$ does exist, it is in general nonunique. For example, the flux velocity associated with the Poynting flux would have $\nabla F = 0$ and so would vanish where the electric current ($\nabla\times\mathbf{B}$) is zero. We could instead choose $\mathbf{w}$ to vanish when the configuration has approached a steady state, so that $\partial\mathbf{B}/\partial t = \mathbf{0}$, which implies that $\nabla\times\mathbf{E} = \mathbf{0}$ and so $\mathbf{E} = \nabla G_0$, say. For example, choosing $F = G_0$, we would have

$$\mathbf{w}\times\mathbf{B} = -\eta\nabla\times\mathbf{B} + \nabla G_0.$$

If there exists a closed magnetic field line C enclosing a surface S, then the rate of change of magnetic flux through S is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{B}\cdot\mathrm{d}\mathbf{S} = \int \nabla\times\mathbf{E}\cdot\mathrm{d}\mathbf{S} = \int_C \mathbf{E}\cdot\mathrm{d}\mathbf{l}.$$

If (33) holds, it implies that

$$\int_C \mathbf{E}\cdot\mathrm{d}\mathbf{l} = \int_C \nabla F\cdot\mathrm{d}\mathbf{l} = 0.$$

Thus, if the flux through a closed field line is indeed changing in time, it implies that (33) cannot hold and no flux velocity ($\mathbf{w}$) exists.

Consider, for example, the diffusion of a linear force-free field satisfying

$$\nabla \times \mathbf{B} = \alpha_0 \mathbf{B},$$

where α_0 is constant. The diffusive induction equation (5) then reduces to

$$\frac{\partial \mathbf{B}}{\partial t} = -\eta \alpha_0^2 \mathbf{B}$$

with solution

$$\mathbf{B}(x, y, z, t) = \mathbf{B}_0(r, \theta, \phi) e^{-\eta \alpha_0^2 t},$$

where $\mathbf{B}_0(r, \theta, \phi)$ is the initial state. As a particular case, consider the 2.5D lowest-order axisymmetric linear-force free field in a sphere $(0 \le r \le a,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi)$, namely,

$$B_{0R} = \frac{1}{r^2 \sin\theta} \frac{\partial A}{\partial \theta}, \qquad B_{0\theta} = -\frac{1}{r \sin\theta} \frac{\partial A}{\partial r}, \qquad B_{0\phi} = \frac{\alpha A}{r \sin\theta},$$

as sketched in figure 6, where $A = r^{1/2} \mathrm{J}_{3/2}(\alpha_0 r) \sin^2\theta$ and $\alpha_0 a \approx 4.49$ is the first zero of $\mathrm{J}_{3/2}(\xi)$. This possesses a closed field line (C) in the equatorial plane ($\theta = 1/2\pi$) at the value (2.46) of $\alpha_0 r$ at which $\partial A/\partial r$ has its first maximum. Within C the poloidal flux is decreasing in time, and so we know from the above general result that no flux velocity exists.

However, we may generalise the concept of a flux velocity to give a pair of flux velocities ($\mathbf{w}_p$ and $\mathbf{w}_t$) that describe the behaviour of the field as follows. The poloidal magnetic field

$$\mathbf{B}_p = B_R \hat{\mathbf{R}} + B_z \hat{\mathbf{z}}$$

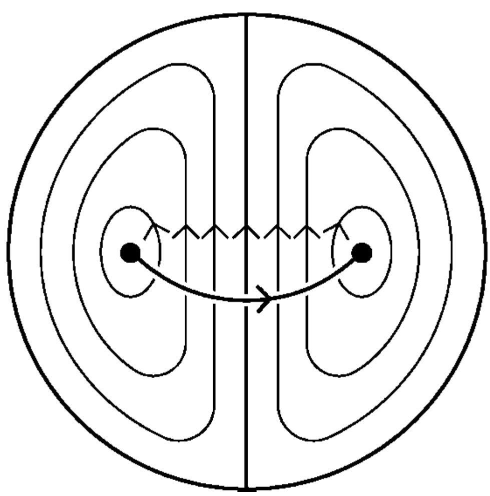

Figure 6. A diffusing magnetic field inside a sphere whose poloidal field lines are shrinking towards the toroidal line, while the toroidal field is diffusing towards the separator.

in planes $\phi =$ constant changes in time according to

$$\frac{\partial \mathbf{B}_p}{\partial t} = -\nabla \times \mathbf{E}_t,$$

while the toroidal field $\mathbf{B}_t = B_\phi \hat{\boldsymbol{\phi}}$ obeys

$$\frac{\partial \mathbf{B}_t}{\partial t} = -\nabla \times \mathbf{E}_p,$$

where $\mathbf{E}_p$ and $\mathbf{E}_t$ are the poloidal and toroidal components of the electric field. Thus we may define velocities $\mathbf{w}_p$ and $\mathbf{w}_t$ (which we refer to as "quasi-flux velocities") satisfying

$$\mathbf{E}_t + \mathbf{w}_p \times \mathbf{B}_p = \mathbf{0}$$

and

$$\mathbf{E}_p + \mathbf{w}_t \times \mathbf{B}_t = \mathbf{0},$$

which are perpendicular to $\mathbf{B}_p$ and $\mathbf{B}_t$, respectively, and describe the motions of field lines based separately on the poloidal and toroidal components, respectively.

It can be seen from figure 6 that, as the field decays in time, it behaves as if the poloidal field is shrinking at $\mathbf{w}_p$ towards the closed toroidal field line (C) and disappearing into the 0-points of the poloidal field. At the same time the toroidal field can be regarded as shrinking and disappearing at the separator joining the null points N_1 and N_2. It is since $\mathbf{w}_p$ is different from $\mathbf{w}_t$ that a single flux velocity cannot be defined.

6. Conclusions

We have discussed here a series of examples of magnetic diffusion and have shown that, in cases when a flux velocity ($\mathbf{w}$) exists satisfying

$$\eta \nabla \times \mathbf{B} + \mathbf{w} \times \mathbf{B} = \nabla F, \tag{34}$$

in general it is not unique, but that the evolution of the field may be described as if the field lines are moving in one way or another.

Straight magnetic field lines (or plane magnetic flux surfaces) diffuse in current sheets and the magnetic field behaves as if the flux is disappearing at a neutral sheet and/or infinity. Given that there is a whole family of possible flux velocities ($\mathbf{w}$), the question arises – which particular one should we choose? We suggest here that two possibilities are particularly useful physically. If one is interested in the energetics of a situation, the first is to define a flux velocity that describes the energy flow. Since the field tends to diffuse to a steady-state solution of $\nabla \times (\eta \nabla \times \mathbf{B}) = \mathbf{0}$, the second physically useful definition is to define it so that it vanishes in the final steady state.

For a finite region of nonzero diffusivity, the magnetic field diffuses away to a uniform field in the region in such a way that the flux there remains constant and current sheets of steep magnetic gradient are formed at the ends of the region. This suggests that, in a more general situation where the magnetic diffusivity is enhanced in regions where the electric current exceeds a critical threshold, enhanced diffusivity will be triggered at both ends and so will propagate both ways well beyond its original location. Thus, if enhanced resistivity is initiated in some small domain, it has the potential to spread and reduce the magnetic profile so that it remains in a state of marginal stability over the whole region subject to the appropriate boundary conditions.

Circular magnetic field lines (or cylindrical flux surfaces) diffuse in a very similar way. The field behaves as if the magnetic flux is moving either towards the 0-type neutral line or towards infinity or both and vanishing there. In particular, an isolated tube of radius a is found to have a flux that decays exponentially over a time-scale $a^2/(4\eta)$.

In three dimensions one can always define the magnetic field lines at any instant of time, but it is not always possible to describe the decay of the field in terms of the motion of field lines from one time to another. For generic magnetic fields one can identify three-dimensional null points and their fan planes, which separate magnetic flux above and below the null (e.g., Priest and Titov, 1996). In some simpler configurations (e.g., Brown and Priest, 1999; Beveridge *et al.*, 2003; Maclean *et al.*, 2005), space is divided into a series of closed laminar regions bounded by the fans.

In simple solutions for which (34) is satisfied, a flux velocity exists, which may either be associated with the Poynting flux or could be chosen to vanish in the final steady state. However, if a closed field line exists through which the magnetic flux is changing, then a flux velocity does not exist. Nevertheless, it has been suggested that the concept of a single flux velocity may, in at least some cases, be generalised to a pair of quasi-flux-velocities, each of which describes the motion of field lines associated with one component of the field.

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References

Beveridge, C., Priest, E.R. and Brown, D.S., Magnetic topologies in the solar corona due to four discrete photospheric flux regions. *Geophys. Astrophys. Fluid Dynam.*, 2003, **98**, 429–455.

Brown, D.S. and Priest, E.R., Topological bifurcations in 3D magnetic fields. *Proc. Roy. Soc. Lond.* A, 1999, **455**, 3931–3951.

Hornig, G., The geometry of reconnection. In *An Introduction to the Geometry and Topology of Fluid Flows*, edited by R.L. Ricca, pp. 295–313, 2001 (Kluwer: Dordrecht, The Netherlands).

Hornig G. and Priest, E.R., Evolution of magnetic flux in an isolated reconnection process. *Phys. Plasmas*, 2003, **10**, 2712–2721.

Hornig, G. and Schindler, K., Magnetic topology and the problem of its invariant definition. *Phys. Plasmas*, 1996, **3**, 781–791.

Maclean, R., Beveridge, C., Longcope, D.W., Brown, D.S. and Priest, E.R., A topological analysis of the magnetic breakout model for an eruptive solar flare. *Proc. Roy. Soc.* A, 2005, in press.

Priest, E.R. and Forbes, T.G., *Magnetic Reconnection: MHD Theory and Applications*, 2000 (Cambridge University Press: Cambridge, UK).

Priest, E.R., Hornig, G. and Pontin, D.I., On the nature of three-dimensional magnetic reconnection, *J. Geophys. Res.*, 2003, **108**, SSH 6.1–6.8.
Priest, E.R. and Titov, V.S., Magnetic reconnection at three-dimensional null points, *Phil. Trans. R. Soc. Lond.* A, 1996, **354**, 2951–2992.
Schindler, K., Hesse, M. and Birn, J., General magnetic reconnection, parallel electric fields, and helicity. *J. Geophys. Res.*, 1988, **93**, 5547–5557.
Spanier, J. and Oldham, B., *An Atlas of Functions*, 1987 (Hemisphere Publishing Corp.).

A. Appendix

A.1. *The General Solution $B(x,t)$ for an arbitrary initial profile in an infinite domain*

Section 3 derived the solution subject to a step-function initial profile in a finite domain for $B(x,t)$ to

$$\frac{\partial B}{\partial t} = \eta_0 \frac{\partial^2 B}{\partial x^2}.$$

The corresponding solution over an infinite range may be obtained from (17) by putting $\xi = n\pi/\ell, \mathrm{d}\xi = \pi/\ell$ and letting ℓ tend to infinity. It is

$$B(x,t) = B_0 \ \mathrm{erf}\left(x/\sqrt{4\eta_0 t}\right) \tag{A.1}$$

in terms of the error function

$$\mathrm{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-u^2}\,\mathrm{d}u. \tag{A.2}$$

Its behaviour is very similar to the finite-region problem, except that the magnetic field tends to zero everywhere as time increases.

The corresponding solution for an arbitrary initial profile $B(x,0) = f(x)$ is

$$B(x,t) = \frac{1}{\sqrt{4\pi\eta_0 t}} \int_{-\infty}^{\infty} f(x')e^{-(x-x')^2/(4\eta_0 t)}\mathrm{d}x'. \tag{A.3}$$

This may be obtained either by considering the finite-range solution over $(-\ell, \ell)$

$$B(x,t) = \sum_{n=1}^{\infty} A_n \exp\left(-\frac{n^2\pi^2}{\ell^2}\eta_0 t\right) \sin\left(\frac{n\pi x}{\ell}\right), \tag{A.4a}$$

where

$$A_n = \frac{1}{\ell}\int_{-\ell}^{\ell} F(x') \sin\left(\frac{n\pi x'}{\ell}\right) \mathrm{d}x$$

or by using Laplace transforms and Green's functions, as follows.

First of all, putting $n\pi/\ell = \xi, \pi/\ell = \mathrm{d}\xi$ and letting $\ell \to \infty$, we find

$$B(x,t) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(x')e^{-\xi^2\eta_0 t}\sin(\xi x')\sin(\xi x)\,\mathrm{d}x'\,\mathrm{d}\xi = \frac{1}{4\pi}(I_1 + I_2 + I_3), \tag{A.4b}$$

where

$$\begin{aligned} I_1 &= \int_0^\infty \int_{-\infty}^\infty f(x')e^{-\xi^2\eta_0 t}e^{\mathrm{i}\xi(x-x')}\,\mathrm{d}x'\mathrm{d}\xi, \\ &= \int_{-\infty}^\infty f(x')\exp\left(-\frac{(x-x')}{4\eta_0 t}\right)\mathrm{d}x \int_0^\infty \exp\left[-t\left(\sqrt{\eta}_0\xi - \mathrm{i}\frac{(x-x')}{2\sqrt{\eta_0 t}}\right)\right]^2 \mathrm{d}\xi, \\ I_2 &= \int_0^\infty \int_{-\infty}^\infty f(x')e^{-\xi^2\eta_0 t}e^{-\mathrm{i}\xi(x-x')}\,\mathrm{d}x'\,\mathrm{d}\xi \end{aligned}$$

and

$$I_3 = -2\int_0^\infty \int_{-\infty}^\infty f(x')e^{-\xi^2\eta_0 t}e^{\mathrm{i}\xi(x+x')}\,\mathrm{d}x'\,\mathrm{d}\xi.$$

Putting $z = \sqrt{\eta}_0\xi - \mathrm{i}(x-\xi)/[2\sqrt{\eta}_0 t]$ and $\gamma = (x-x')/[2\sqrt{\eta}_0 t]$, the first integral becomes

$$\begin{aligned} I_1 &= \int_{-\infty}^\infty f(x')\exp\left(-\frac{(x-x')^2}{4\eta_0 t}\right)\mathrm{d}x' \int_{\mathrm{i}\gamma}^{\infty+\mathrm{i}\gamma} \frac{e^{-tz^2}}{\sqrt{\eta}_0}\mathrm{d}z, \\ &= \int_{-\infty}^\infty f(x')\exp\left(-\frac{(x-x')^2}{4\eta_0 t}\right)\mathrm{d}x' \int_0^\infty \frac{e^{-y^2}}{\sqrt{\eta}_0 t}\mathrm{d}y, \end{aligned}$$

or, since $\int_0^\infty e^{-y^2}\mathrm{d}y = \sqrt{\pi}/2$, we have

$$I_1 = \frac{1}{2\sqrt{\pi\eta_0 t}}\int_{-\infty}^\infty f(x')\exp\left(-\frac{(x-x')^2}{4\eta_0 t}\right)\mathrm{d}x'.$$

Similarly, $I_2 = I_3 = I_1$ and so I reduces to (A.3), as required.

The alternative derivation is to take Laplace transforms

$$LB(x,t) = \overline{B}(x,s) = \int_0^\infty B(x,t)e^{-st}\,\mathrm{d}t$$

of (16), assuming $B(0,t) = 0$, so that

$$\eta\frac{\partial^2\overline{B}}{\partial x^2} - s\overline{B} = -f(x).$$

This is self-adjoint and so its solution may be written in terms of a Green's function $G(x, x')$ as

$$\overline{B}(x, s) = -\int_{-\infty}^{\infty} G(x, x') f(x') \, \mathrm{d}x', \tag{A.5}$$

where

$$\eta \frac{\partial^2 G}{\partial x^2} - sG = \delta(x - x') \tag{A.6}$$

and

$$G(-\infty, x') = G(x, x') = 0. \tag{A.7}$$

We require $G(x, x')$ to be continuous at $x = x'$, and so, integrating (A.6) across $x = x'$ gives

$$\left(\frac{\partial G}{\partial x}\right)_{x=x'+} - \left(\frac{\partial G}{\partial x}\right)_{x=x'-} = \frac{1}{\eta}. \tag{A.8}$$

The solution subject to boundary conditions that it vanishes at $\pm\infty$ is

$$G(x, x') = \begin{cases} ae^{\sqrt{s/\eta x}}, & x < x', \\ be^{-\sqrt{s/\eta x}}, & x > x', \end{cases}$$

where (A.7) and (A.8) imply

$$a = -\frac{1}{\sqrt{4\eta s}} e^{-\sqrt{s/\eta x'}}, \qquad b = -\frac{1}{\sqrt{4\eta x}} e^{\sqrt{x/\eta x'}}.$$

Thus, the Green's function is finally

$$G(x, x') = -\frac{1}{\sqrt{4\eta s}} e^{-\sqrt{s/\eta}|x-x'|}$$

and so the solution (A.5) becomes

$$\overline{B}(x, s) = \frac{1}{\sqrt{4\eta x}} \int_{-\infty}^{\infty} e^{-\sqrt{s/\eta}|x-x'|} f(x') \, \mathrm{d}x'.$$

Hence by taking the Laplace transform inverse

$$L^{-1}\overline{B}(x, s) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \overline{B}(x, s) e^{st} \, \mathrm{d}s,$$

we recover the required solution (A.3).

A.2. *The general solution $A(r,t)$ (28) for an arbitrary initial profile*

The general solution for $A(r, t)$ to

$$\frac{\partial A}{\partial t} = \eta_0\left(\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r}\right)$$

subject to $A(r, 0) = g(r)$ is

$$A(r, t) = \frac{1}{2\eta t}\int_0^\infty \exp\left(-\frac{r^2 + s^2}{4\eta_0 t}\right)\mathrm{I}_0\left(\frac{rs}{4\eta_0 t}\right)sg(s)\,\mathrm{d}s, \qquad (28)$$

which may be proved as follows.

Consider first the equation

$$\frac{\partial A}{\partial t} = \eta_0\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right).$$

Its solution is a natural generalisation of the solution (A.2) to the one-dimensional equation (16), namely,

$$A(x, y, t) = \frac{1}{(4\pi\eta_0 t)}\int_{-\infty}^\infty \int_{-\infty}^\infty \exp\left(-\frac{|\mathbf{r} - \mathbf{r}'|}{4\eta_0 t}\right)^2 A_0(x', y')\,\mathrm{d}x'\mathrm{d}y'.$$

In the particular case that $A_0(x', y')$ is a function of the radial coordinate (s) alone, namely, $g(s)$, this reduces to

$$\begin{aligned}
A(r, t) &= \frac{1}{4\pi\eta_0 t}\int_{s=0}^\infty \int_{\theta'=0}^{2\pi} \exp\left(-\frac{[r^2 + s^2 + 2rs\cos(\theta - \theta')]}{4\eta_0 t}\right)g(s)s\,\mathrm{d}s\,\mathrm{d}\theta', \\
&= \frac{1}{4\pi\eta_0 t}\int_{s=0}^\infty \exp\left(-\frac{r^2 + s^2}{4\eta_0 t}\right)\int_{\theta'=0}^{2\pi} \exp\left(-\frac{2rs\cos(\theta - \theta')}{4\eta_0 t}\right)g(s)s\,\mathrm{d}\theta'\mathrm{d}s, \\
&= \frac{1}{4\pi\eta_0 t}\int_{s=0}^\infty \exp\left(-\frac{r^2 + s^2}{4\eta_0 t}\right)g(s)s\mathrm{I}(s)\,\mathrm{d}s,
\end{aligned}$$

where

$$\mathrm{I}(s) = \int_{\theta'=0}^{2\pi} \exp\left(-\frac{rs}{2\eta_0 t}\cos\theta'\right)\mathrm{d}\theta'.$$

Thus, evaluating this integral using the result (Spanier and Oldham, 1987, p. 49.3)

$$\int_0^{2\pi} \exp(-y\cos z)\,\mathrm{d}z = 2\pi\mathrm{I}_0(y),$$

where I_0 is the zeroth-order hyperbolic Bessel function, we recover the required result (28).