THE GEOMETRY OF RECONNECTION

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Abstract. Reconnection is an important process of structure formation in fluid dynamics. It occurs in the form of vortex reconnection in hydrodynamics as well as in the form of magnetic reconnection in plasmas. In both cases a divergence-free field, the magnetic or the vorticity field, respectively, alter the topology of their field lines. A first definition of the process in two dimensions is given, which is wide enough to cover both forms of reconnection, and which is independent of the underlying dissipation process. It is also demonstrated that the definition can distinguish reconnection from other non-ideal processes. With the help of simple two-dimensional examples, one for magnetic reconnection and one for vortex reconnection, similarities and differences of both types of reconnection are discussed. The transition to three-dimensional configurations shows to require a more general framework, which is found in the covariant generalization of flux conservation.

1. Introduction

Reconnection is a fundamental process in fluid systems by which a divergence-free field changes its topology. The divergence-free field may for example be the vorticity field of a hydrodynamical system or the magnetic field in a plasma. A first impression of the nature of the process is provided by Figure 1.

Aside of hydrodynamics (HD) and magnetohydrodynamics (MHD) the notion of reconnection exists also in other fields of physics such as in superfluids [1], cosmic strings [2] and liquid crystals [3]. Processes similar to reconnection are also known in knot-theory as a *surgery of framed knots* [5],

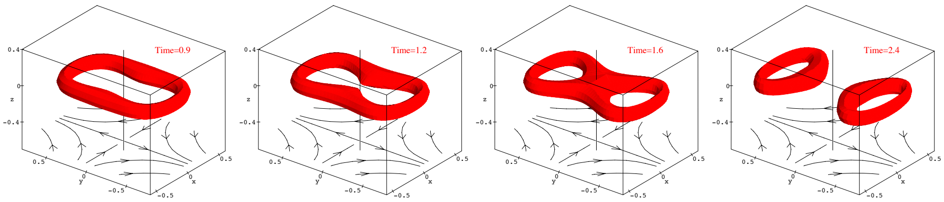

Figure 1. An example of a simple two-dimensional reconnection process. Exemplary for the evolution of the field a flux tube is followed in time. The fluid velocity is shown on the bottom of the boxes.

and in enzymology, where topoisomerase acts like reconnection on strands of the DNA [4]. However, except of vortex and magnetic reconnection all other forms of reconnection refer to singular string-like objects (cosmic strings, knots, quantized vortices, DNA) and are not based on continuous fields such as vorticity fields or magnetic fields in fluids. Therefore they are mentioned here only for completeness and are disregarded in the following. The interested reader is referred to the selected references given above.

With respect to the development of theoretical concepts vorticity or magnetic reconnection are well ahead of reconnection in other fields of physics. This is also reflected by the enormous number of more than thousand publication referring to magnetic reconnection or synonymous notions such as *merging*. Curiously vorticity reconnection (sometimes also called *cross-linking* or *cut-and-connect*) did not get that much attention although the phenomenon must have been known already for a long time in form of reconnection of smoke rings.

Among the numerous aspects under which reconnection is investigated in the literature the following exposition will focus on the topological and geometrical aspects. There are two primary reasons for this restriction. First, these are the main aspects of this book, and second, these aspects are amenable to a rigorous and very general mathematical analysis, which treats magnetic and vortex reconnection on the same basis. Especially we will prescribe macroscopic transport coefficients such as viscosity or resistivity without discussing their physical origin or quantifying them. This allows us to answer questions of how reconnection changes the topology of a magnetic or vorticity field, but not why and when it occurs. With respect to these and many other physically very important and in parts still unsolved questions, we refer the reader to the numerous publications, an entry to which is given for instance by [7] and [6] for the case of magnetic reconnection and by [8] for vorticity reconnection.

The process of reconnection is remarkable in that it occurs even for very low dissipation, that is high Reynolds numbers or magnetic Reynolds numbers, respectively. For example in the plasma of the corona, the upper atmosphere of the Sun, reconnection occurs in an environment of magnetic Reynolds numbers as high as $10^8 - 10^{12}$. Usually in these regimes the idealization of infinite Reynolds numbers is a good approximation in all other respects. Therefore it is tempting to neglect dissipation completely and use the ideal equations, either the Euler-equation for HD or the ideal Ohm's law for MHD applications. Both equations, however, preserve the topology and do not allow for reconnection. The solution to this seemingly paradox situation is the fact that plasmas and other fluids can develop self-amplifying processes such as the formation of thin current sheets, which can locally reduce the characteristic length scales and thereby the Reynolds number by several orders of magnitude. Hence, in such current sheets, dissipation is not neglectable and reconnection is enabled. Since these sheets are very thin the dissipation of energy in these sheets compared to the magnetic energy stored in the configuration is rather small and localized. Thus the restructuring of magnetic flux occurs as if there would be no dissipation. This is the reason why reconnection can be considered as a flux conserving process under non-continuous flows. But before we explain this in more detail a short survey of conservation laws in fluids is given in Section 2. It should help the reader to classify the equation of the following sections. In Section 3 we will give a short survey of the basic equations of MHD and HD and their relation to the afore mentioned conservation laws. A main problem in the analysis of reconnection is to distinguish it from other dissipative processes. Therefore Section 4 contains a discussion of so called slippage solutions and a subsection on weak forms of flux conservation, before we come to the most simple two-dimensional reconnection processes in Section 5. The last section, Section 6 is devoted to the more complicated three-dimensional reconnection.

2. Topological Conservation laws of Fields in Fluid Dynamics

The general setting for this and the following sections is a fluid confined in a simply connected compact domain $D \in \mathbb{R}^3$. The motion of the fluid is described by a velocity field $\mathbf{v}(\mathbf{x}, t)$, satisfying $\mathbf{v} \cdot \mathbf{n}|_{\partial V} = 0$, that is the normal component of the velocity on the boundary ∂D vanishes. The motion of fluid elements in the domain D is given by the *flow* $\mathbf{F}(\mathbf{x}, t)$ of the velocity field $\mathbf{v}$ which is defined by

$$\frac{\partial \mathbf{F}(\mathbf{x}, t)}{\partial t} = \mathbf{v}(\mathbf{F}(\mathbf{x}, t), t) \ , \quad \text{with} \quad \mathbf{F}(\mathbf{x}, 0) = \mathbf{x} \ . \tag{1}$$

On the one hand $\mathbf{F}(\mathbf{x}, t)$ is the *trajectory* of a fluid element initially located at $\mathbf{x}$ at $t = 0$. On the other hand, if we fix t, $\mathbf{F}(\mathbf{x}, t)$ is a *diffeomorphism*, i.e. a bijective and differentiable mapping of the domain D onto itself. The inverse of this mapping $F^{-1}(\mathbf{x}, t)$ is $F(\mathbf{x}, -t)$.

In the following we are interested in how the fluid motion affects properties of the fluid. There are four types of topological conservation laws for fields in $I\!R^3$ which lead to the conservation of corresponding integrals over k-dimensional comoving volumes $C^k(t)$.

$$\begin{aligned}
&\partial_t\ \alpha + \mathbf{v}\cdot\nabla\alpha = 0 && (2)\\
&\quad\Rightarrow \alpha(F(\mathbf{x},t),t) = const.\ ,\\
&\partial_t\mathbf{A} + \nabla(\mathbf{v}\cdot\mathbf{A}) - \mathbf{v}\times\nabla\times\mathbf{A} = 0 && (3)\\
&\quad\Rightarrow \int_{C^1(t)} \mathbf{A}\cdot\mathbf{dl} = const.\ ,\\
&\partial_t\mathbf{B} + \mathbf{v}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{v} + \mathbf{B}\ \nabla\cdot\mathbf{v} = 0 && (4)\\
&\quad\Rightarrow \int_{C^2(t)} \mathbf{B}\cdot d\mathbf{a} = const.\ ,\\
&\partial_t\rho + \nabla\cdot(\mathbf{v}\ \rho) = 0 && (5)\\
&\quad\Rightarrow \int_{C^3(t)} \rho\ d^3x = const.\ .
\end{aligned}$$

These four conservation laws correspond to four different classes of differential forms ω^k, $k = 0,1,2,3$ in $I\!R^3$. These are 0-forms (α), 1-forms ($\mathbf{A}$), 2-forms ($\mathbf{B}$) and 3-forms (ρ). The equivalent formulation in differential forms with the help of the *Lie-derivative* (cf. [9]) reads

$$\begin{aligned}
&\partial_t\omega^k + L_{\mathbf{v}}\omega^k = 0 && (6)\\
&\quad\Rightarrow \int_{C^k(t)} \omega^k = const.\ .
\end{aligned}$$

The conservation of the corresponding integral over an arbitrary k-dimensional, comoving volume, $C^k(t) = F(C^k(0),t)$, is derived from the *Lie-derivative theorem* [9]. The integrals show that we have to interprete the different forms as densities, that is scalars α, line-densities $\mathbf{A}$, surface densities $\mathbf{B}$ and volume densities ρ.

The conservation law Eq. (4) will be of particular interest. It leads to an integral of a vector field over a surface C^2 which in physics is commonly called the *flux* of $\mathbf{B}$ through C^2. Note that for a divergence-free vector field Eq. (4) is equivalent to

$$\partial_t\mathbf{B} - \nabla\times(\mathbf{v}\times\mathbf{B}) = 0\ . \tag{7}$$

Another conservation law is closely related to the above mentioned cases. Let us consider a field $\mathbf{B}$ represented by its field lines F_B,

$$\mathbf{B}(\mathbf{F}_B(\mathbf{x},s),s) := \frac{\partial\mathbf{F}_B(\mathbf{x},s)}{\partial s}\ .$$

If $\mathbf{F}_B$ is transported with the fluid flow of $\mathbf{v}$, i.e. if the field lines are conserved, we find the following conservation equation for its tangent vector field (see [10] for a derivation)

$$\begin{aligned} & \partial_t \mathbf{B} + \mathbf{v}(\mathbf{x})\cdot\nabla\mathbf{B}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\cdot\nabla\mathbf{v}(\mathbf{x}) = \lambda\mathbf{B} \\ \Leftrightarrow \quad & \partial_t \mathbf{B} + L_{\mathbf{v}}\mathbf{B} = \lambda\mathbf{B} \ . \end{aligned}$$

Here λ is a free function which corresponds to the freedom that we can multiply a vector field with an arbitrary non-vanishing function without changing its field lines. This conservation law does not result in the conservation of an integral as in the previous cases, but in the conservation of field lines of $\mathbf{B}$ as geometrical objects. This implies that a closed field line remains a closed field line, or a field line originating at a null maintains this property under the flow F. Moreover, all kinds of linkage or knottedness are preserved as well. It is worth noting that the conservation of flux (4) implies a conservation of field lines for $\lambda = -\nabla\cdot\mathbf{v}$, while the converse is not true.

In differential geometry it is easy to see that the exterior derivatives and the wedge product commute with the Lie-derivative. This implies that various derivatives and combinations of conserved quantities are in turn conserved quantities. For instance $\rho = \mathbf{A}\cdot\mathbf{B}$, $\mathbf{B} = \nabla\times\mathbf{A}$, $\hat{\mathbf{B}} = \alpha\mathbf{B}$, $\mathbf{B} = \mathbf{A}\times\hat{\mathbf{A}}$ are conserved quantities, if α, $\mathbf{A}$, $\hat{\mathbf{A}}$, $\mathbf{B}$, $\hat{\mathbf{B}}$ and ρ correspond to conserved quantities in the form of Eq. (2)-(5).

Exercise: Explicitely prove the relations given in the foregoing paragraph.

Exercise: Show that the conservation of field lines also implies a conservation of null points of the field, i.e. no isolated nulls can appear nor dissappear.

3. Hydrodynamics and Magnetohydrodynamics

We now turn to the relation between the conservation laws of the previous section and the equation governing hydrodynamics and magnetohydrodynamics, starting with MHD.

3.1. MAGNETOHYDRODYNAMICS

The set of equations of MHD consists of a continuity equation for the plasma density ρ Eq. (8), a momentum balance for the plasma velocity $\mathbf{v}$ Eq. (9), and the Ohm's law for the magnetic and electric field Eq. (10). They are supplemented by Faraday's law, $\nabla\cdot\mathbf{B} = 0$ and Maxwell's equation without displacement current, Eqs. (11) -(13).

$$\partial_t \ \rho + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{8}$$

$$\rho\,\partial_t\mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \mathbf{J}\times\mathbf{B} \tag{9}$$
$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\mathbf{J} + \mathbf{T} \tag{10}$$
$$\nabla\times\mathbf{E} + \partial_t\mathbf{B} = 0 \tag{11}$$
$$\nabla\cdot\mathbf{B} = 0 \tag{12}$$
$$\nabla\times\mathbf{B} = \mu\mathbf{J} \tag{13}$$

The Ohm's law is the result of various approximations. Depending on the plasma under consideration, aside from the resistive term $\eta\mathbf{J}$, other terms such as the Hall term or the pressure term, could be significant. We collectively denote these terms by $\mathbf{T}$. Equations Eqs. (10) - (13) can be simplified by taking the curl of Eq. (10), using Eq. (11) to substitute for the curl of $\mathbf{E}$ and finally using Eq. (13) to substitute for $\mathbf{J}$. Thus we end up with only one equation, the *induction equation* of MHD,

$$\partial_t\mathbf{B} - \nabla\times(\mathbf{v}\times\mathbf{B}) = -\nabla\times(\hat{\eta}\ \nabla\times\mathbf{B} + \mathbf{T}) \qquad (\hat{\eta} = \eta/\mu)\ , \tag{14}$$

to which Eq. (12) is only an initial condition for the magnetic field. Now the whole system, Eq. (8), Eq. (9) and Eq. (14), does not contain the electric field anymore, which justifies the name magnetohydrodynamics. The relative magnitude of the second term on the lhs. of Eq. (14) in comparison with the first term on the rhs. is given by the *magnetic Reynolds number*

$$R_M = \frac{\|\nabla\times(\mathbf{v}\times\mathbf{B})\|_{char.}}{\|\nabla\times(\hat{\eta}\nabla\times\mathbf{B}\|_{char.}} = \frac{vL}{\hat{\eta}},$$

where we used characteristic values for the velocity v, magnetic field b and derivatives $\partial_x \sim 1/L$. For large values of R_M which are especially obtained in the case of large length scales L, e.g. for astrophysical plasmas, we can neglect the resistive term in Eq. (14). With a similar consideration we can disregard the second term on the rhs. so that we end up with

$$\partial_t\mathbf{B} - \nabla\times(\mathbf{v}\times\mathbf{B}) = 0\ , \tag{15}$$

which for a divergence-free field is equivalent to Eq. (4). Thus the magnetic flux through any comoving surface is conserved, a consequence that is commonly called *Alfvén's theorem*. Equation (15) is sometimes called *ideal induction equation*, since it can be derived from the *ideal Ohm's law* $\mathbf{E}+\mathbf{v}\times\mathbf{B} = 0$, which we would have obtained in the limit of large Reynolds numbers directly from Ohm's law Eq. (10).

Exercise: Show by substituting $\mathbf{E} = -\partial_t\mathbf{A} - \nabla\phi$ in the ideal Ohm's law that we can obtain a conservation law for the vector potential $\mathbf{A}$ of the same type as Eq. (3) if we choose a suitable gauge $A \to A + \nabla\chi$, $\phi \to \phi - \partial_t\chi$. What kind of conservation law is valid for the helicity density $\mathbf{A}\cdot\mathbf{B}$ in this case?

An approximation similar to the one which leads to the ideal induction equation can be applied in hydrodynamics. Here the set of equations consists of the continuity equation (16) and the momentum balance of a viscous fluid, the *Navier-Stokes equation* (17).

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{16}$$

$$\rho \partial_t \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mathbf{F} \tag{17}$$

$$\mathbf{F} = \partial_k \left(\nu (\partial_k v^i + \partial_i v^k - 2/3 \delta_{ik} \partial_l v^l) + \partial_i \zeta \partial_l v^l \right) \vec{e}_i$$

The comparatively complicated viscous term on the right hand side of Eq. (17) is simplified considerably if we consider only incompressible fluids with constant viscosity, Eq. (18), which is a very common approximation.

$$\Rightarrow \quad \partial_t \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \triangle \mathbf{v} \quad \text{for } \nu = const., \quad \nabla \cdot \mathbf{v} = 0 \tag{18}$$

Taking the curl of the Navier-Stokes equation leads to an equation for the vorticity $\mathbf{w} = \nabla \times \mathbf{v}$:

$$\partial_t \mathbf{w} - \nabla \times (\mathbf{v} \times \mathbf{w}) = \nabla \times \left(1/\rho \nabla p + \nabla v^2/2 + 1/\rho \mathbf{F} \right) \ . \tag{19}$$

We observe that in HD an approximation which results in a conservation law in form of Eq. (4) requires not only large Reynolds numbers $R = vL/\nu$ to disregard the viscous term, but also an *isentropic* flow $p = p(\rho)$ for $\nabla \times (1/\rho \nabla p)$ to vanish. This finally leads to

$$\partial_t \mathbf{w} - \nabla \times (\mathbf{v} \times \mathbf{w}) = 0 \ , \tag{20}$$

and the conservation of vorticity which is the so called *Kelvin's Theorem.*

Question 3.1 Can the Navier-Stokes equation, analogous to Ohm's law, be cast in form of Eq. (3)?

4. Dynamics of Divergence-free Fields

In the previous section we saw that MHD as well as HD yield under suitable approximation in a conservation equation for the flux of a divergence-free field. Since the conservation of flux also implies a conservation of field lines, the magnetic field lines or vorticity field lines are conserved as well. This feature is sometimes referred to as *frozen-in* field lines. These properties help a lot to envisage possible evolutions of magnetic or vorticity fields. We can think of an evolution of these fields as if all field lines are robes (tied together at null points of the field) which are drifting with the flow of the fluid. One can hardly underestimate the importance of this picture in MHD or HD. Many models are conceived on the basis of these comparatively simple and descriptive conception. However, the concept of frozen-in field lines and flux would not have been so successful if it would be valid *only* in the limit of infinite Reynolds numbers. Its success results from the fact that it is often tolerant with respect to small or even finite deviations of the limit of vanishing dissipation. The exact meaning of 'tolerance' in this context is specified in the following section.

4.1. FLUX CONSERVING FLOWS

Consider the evolution of a divergence-free field $\mathbf{G}$ which instead of being governed by an ideal flux conserving equation, is now disturbed by a non-ideal term $\nabla \times \mathbf{N}$,

$$\partial_t \mathbf{G} - \nabla \times (\mathbf{v} \times \mathbf{G}) = -\nabla \times \mathbf{N} \ . \tag{21}$$

Note that an arbitrary evolution of $\mathbf{G}$ can be cast in this form as long as $\mathbf{N}$ is not specified. We now pose the question under which conditions the evolution of $\mathbf{G}$ can be considered as flux conserving *despite* of the non-ideal term $\nabla \times \mathbf{N}$? That is, under which conditions does a velocity field $\mathbf{u}$ exist which instead of $\mathbf{v}$ satisfies

$$\partial_t \mathbf{G} - \nabla \times (\mathbf{u} \times \mathbf{G}) = 0, \qquad \mathbf{u} = \mathbf{v} + \Delta\mathbf{v} \ . \tag{22}$$

Thus it remains to find a solution $(\Delta\mathbf{v}, \Phi)$ of

$$\Delta\mathbf{v} \times \mathbf{G} = -\mathbf{N} + \nabla\Phi \ .$$

A necessary condition for such a solution is

$$\mathbf{G} \cdot \mathbf{N} = \mathbf{G} \cdot \nabla\Phi \ . \tag{23}$$

For $\mathbf{G} \neq 0$ this condition is also a sufficient and the solution is given by

$$\Delta\mathbf{v} = (\mathbf{N} - \nabla\Phi)_\perp \times \mathbf{G}/G^2 \ , \tag{24}$$

where $\perp$ denotes the component perpendicular to $\mathbf{G}$.

A wide class of configurations ($\mathbf{G}$,$\mathbf{v}$,$\mathbf{N}$) which satisfies condition (23) are strictly two-dimensional (2D) configurations, i.e. configurations where $\mathbf{G}$ and $\mathbf{v}$ depend only on two spacial coordinates and only have components in these two coordinate directions, e.g. $\mathbf{G} = G_x(x,y,t)\mathbf{e}_x + G_y(x,y,t)\mathbf{e}_y$ in euclidian coordinates. In this case the two-dimensionality in Eq. (21) requires $\nabla \times \mathbf{N} \cdot \mathbf{e}_z = 0$ and thus $\mathbf{N} = N_z(x,y)\mathbf{e}_z + \nabla\Phi_n(x,y)$. Setting $\Phi = -\Phi_N$ satisfies Eq. (23) and Eq. (24) yields

$$\Delta\mathbf{v} = N_z(x,y)\mathbf{e}_z \times \mathbf{G}/G^2 \ . \tag{25}$$

If we require that $\Delta\mathbf{v}$ vanishes on a boundary where $\mathbf{N}$ vanishes the velocity $\Delta\mathbf{v}$ is *uniquely* determined up to a component parallel to $\mathbf{G}$.

Whenever a solution for $\Delta\mathbf{v}$ and therefore for $\mathbf{u}$ in Eq. (22) exists, the field lines of $\mathbf{G}$ are moving with a velocity $\mathbf{u}$ different from the fluid velocity $\mathbf{v}$. An observer will therefore see a slippage of the field lines with respect to the fluid elements. The slippage velocity $\Delta\mathbf{v}$ is proportional to the non-ideal term $\mathbf{N}$ and can be chosen to zero when $\mathbf{N}$ tends to zero. This is the reason why the concept of frozen-in flux works in so many cases *although* it is derived from an highly idealized situation. A small deviation in terms of a small $\mathbf{N}$ will lead in most cases only to a small and therefore for observers hardly noticeable difference between the velocity of the field lines and the fluid velocity. This observation is the basis for the following definition.

Definition 4.1 *Let* $\mathbf{G}$ *be divergence-free field in a domain* D *of a fluid with velocity* $\mathbf{v}$*. If the flux-conserving equation (7) does not hold for* $\mathbf{v}$*, but for a different velocity field* $\mathbf{u}$ *with* $\mathbf{u}_{\partial D} = \mathbf{v}_{\partial D}$ *we call this a flux conserving flow or a slippage solution. The velocity* $\mathbf{u}_\perp - \mathbf{v}_\perp$ *is the slippage velocity.*

Note that a flux conserving flow is in general not unique and moreover there need not even be a unique minimal slippage velocity (in the sense of a maximum norm), since the problem of finding a $\Delta\mathbf{v}$-solution involves the existence of a global potential Φ and is therefore a global one. The condition that $\mathbf{u}$ has to satisfy the same boundary condition as $\mathbf{v}$ was included to ensure that the existence of a solutions for two adjacent domains D_1 and D_2 implies the existence of a solution for the union $D_1 \cup D_2$.

Remark 4.2 The velocity $\mathbf{u}$ does not need to be any particle velocity. In particular, similar to a phase velocity it can be greater than the speed of light without violating any physical laws, if G^2 is sufficiently small in Eq. (24).

Exercise: Given an Ohm's law in MHD with Hall term and a pressure term included but with vanishing resistivity,

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = (en)^{-1}\mathbf{J} \times \mathbf{B} - (en)^{-1}\nabla P \ .$$

state the conditions under which flux conserving solutions exist. What is the meaning of $\mathbf{u}$ in this case?

Exercise: Show that the total helicity,

$$\int_V \mathbf{A} \cdot \mathbf{G} \ d^3x, \qquad \mathbf{n} \cdot \mathbf{G}|_{\partial V} = 0 \ ,$$

is independent of the choice of the vector potential $\mathbf{A}$ of $\mathbf{G}$, if integrated over a volume where $\mathbf{G}$ is tangent to the boundary, and that it is constant in time for every flux conserving flow (see [12] for more about helicity).

4.2. A WEAK FORM OF FLUX CONSERVATION

Although we often find flux conserving flows, there are still many cases where the effect of the non-ideal term can not be hidden in a slippage velocity. This is especially the case for reconnection, but reconnection is also in a certain sense almost flux conserving. To understand this effect one has to know the subtle difference between dissipation of energy and 'dissipation' of flux of a divergence-free field.

To demonstrate this difference consider an idealized field which consist of only one flux tube (FT) of the form of a torus, that is $\mathbf{G} = 0$ everywhere outside the flux tube. If the whole field is dissipated according to

$$\partial_t \mathbf{G} = \eta \Delta \mathbf{G}, \tag{26}$$

then the total energy of the field $\int_{FT} G^2/2 \; d^3x$ decreases with t and the same is true for the flux $\int_C \mathbf{G} \cdot d\mathbf{A}$ of the torus. Obviously there is no flux conserving flow for $\mathbf{G}(\mathbf{x}, t)$ is this case.

The situation changes significantly if for example η and therefore the dissipation in Eq. (26) is non-vanishing only in one half of the torus. Again the total energy of the field decreases, but this time the flux of the tube has to stay constant. There is no dissipation of flux whenever there is at least one cross-section of the tube on which the dissipation is zero or more general, for which we have a flux conserving solution. In other words the dissipation of flux requires a dissipative process in the whole volume of the flux tube, while dissipation of energy is effective also locally. Note, however, that this *weak* form of flux conservation does not imply the existence of the *strong* form of flux conservation i.e. a flux conserving flow. For example, if the $\mathbf{G}$-field is twisted in the dissipative half torus of the foregoing example, the twist will vanish for $t \to \infty$ since $\nabla \times \mathbf{G}$ tends to zero. This will change the connectivity of field lines within the flux tube and is in contradiction to the existence of a flux conserving flow.

The weak form of flux conservation is encountered especially in fluids which have very low Reynolds numbers and can develop essential dissipation only in very localized regions, as for instance in current sheets or vorticity sheets. Usually these regions are too small to contain closed flux tubes and hence all the flux within the dissipation region is connected to the dissipation-free environment. Reconnection, for example, is a form of weak flux conservation which does not allow for a slippage solution.

5. 2D Reconnection

The so called two-dimensional reconnection is a model for a real reconnection process in three-dimensional (3D) space which is approximately invariant in one spacial direction (here the z-direction). Thus all quantities depend only on two spatial coordinates (x and y) and the time. Moreover we require that $\mathbf{G}$ and $\mathbf{v}$ have no z-component. The domain D of the (x, y) coordinates is a compact subset of $\mathbb{R}^2$ with a boundary ∂D where the non-ideal term $\mathbf{N}$ vanishes. Thus we have the same situation which lead us to the solution Eq. (25). Obviously the existence of such solutions is only questionable at points where $\mathbf{G} = 0$. In the generic situation where $\mathbf{N}$ is non-zero at the null of $\mathbf{G}$ we will have a singularity of $\Delta\mathbf{v}$. The type of this singularity depends on the local structure of $\mathbf{G}$ at the null. Since there are only two structurally stable types of nulls for two-dimensional fields, a hyperbolic null (X-point) and an elliptic null (O-point), we get two typical singularities of $\Delta\mathbf{v}$ at null points of $\mathbf{G}$.

An elliptic null of $\mathbf{G}$ generates a $\Delta\mathbf{v}$ which points radially inwards or outwards depending on whether $\mathbf{N}$ and $\nabla \times \mathbf{G}$ are parallel or antiparallel (see Figure 2). For example for $\mathbf{N} = [0, 0, 1]$ and $\mathbf{G} = [-y, x, 0]$, $\mathbf{N}$ and $\nabla \times \mathbf{G}$ are aligned and thus

$$\Delta\mathbf{v} = \mathbf{N} \times \mathbf{G}/G^2 = (x^2 + y^2)^{(-1)}[-x, -y, 0] \ .$$

It has a singularity of type $1/\sqrt{(x^2 + y^2)}$ due to the linearity of a generic, i.e. structurally stable null of $\mathbf{G}$. These solutions have to be interpreted as either a destruction of flux, if $\Delta\mathbf{v}$ points towards the null, or a creation of flux if $\Delta\mathbf{v}$ is directed away from the null. The rate at which flux is created or annihilated at the null is equal to the change of flux per time through a certain surface. This surface (see Fig. 2b) is chosen to be tangent to the field lines of $\mathbf{u}$ and comoving with the flow of $\mathbf{u}$ except for one boundary which is comoving with the null. Since this surface is comoving except for one boundary its flux can change only by a transport of $\mathbf{G}$-flux across this boundary, i.e. across the null line. The transport velocity of the flux across this boundary is the limit of $\mathbf{u}(\mathbf{x}) - \tilde{\mathbf{v}}$ (where $\tilde{\mathbf{v}}$ is the velocity of the null) for $\mathbf{x}$ tending to $\mathbf{x}_0$, the site of the null. Thus we have for the rate of flux creation

$$\begin{aligned} \frac{d}{dt}\int_C \mathbf{G}\cdot d\mathbf{A} &= \lim_{\mathbf{x}\to\mathbf{x}_0}\int_z (\mathbf{u}(\mathbf{x}) - \tilde{\mathbf{v}}) \times \mathbf{G}\cdot d\mathbf{l} \\ &= \lim_{\mathbf{x}\to\mathbf{x}_0}\int_z \Delta\mathbf{v}(\mathbf{x}) \times \mathbf{G}\cdot d\mathbf{l} = -N_z(\mathbf{x}_0, t)\Delta z \ . \end{aligned} \quad (27)$$

or simply $N_z(\mathbf{x}_0, t)$ at the null if we normalized per unit length in z direction.

For the other type of null, the hyperbolic null point or X-point, the same consideration leads to a pattern as shown in Fig. 3a). The singularity again generically scales as $1/\sqrt{(x^2 + y^2)}$, but is now of X-type (stagnation flow). The analysis of the loss of flux from the previous case holds here for the inflow direction, while for the outflow direction we have a creation of

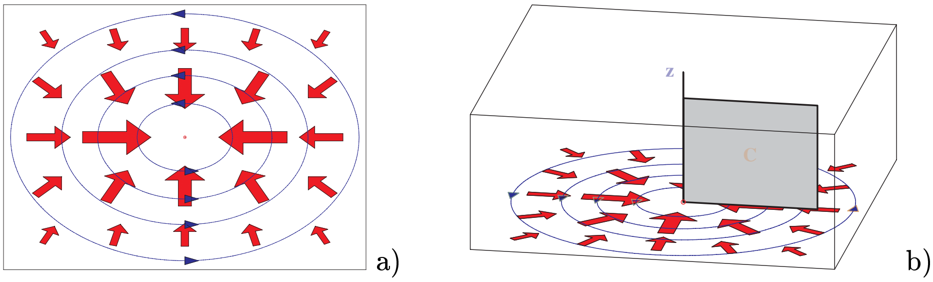

Figure 2. a) Local solution for $\Delta \mathbf{v}$ at an elliptic null of $\mathbf{G}$ for a non-vanishing non-ideal term $\mathbf{N}$. The direction of the arrows depends on the orientation of $\mathbf{N}$ with respect to the circulation of $\mathbf{G}$. b) The surface of integration used to determine the rate of flux annihilation.

flux at the same rate, as can be read of from (27). Thus the flux appears to be transported in a finite time from both sides onto the null line (along the inflow direction), and then reappears along the outflow direction but now connecting different parts of the incoming strands of $\mathbf{G}$-field lines. This leads to the impression that incoming flux is cut along the null line and thereupon newly glued together when leaving in the outflow direction, as shown in Fig. 1 for the example of a flux tube. In Fig. 3b) the evolution of the cross sections of this tube in the vicinity of the null point is shown. This feature justifies the name *reconnection*.

The rate at which flux is cut and reconnected along the null line, is called the *reconnection rate*, and is essentially given by Eq. (27). Since in the process of reconnection flux is simultaneously lost in one direction and created along the other direction there is no use in giving the reconnection rate a sign, hence,

$$\frac{d}{dt}\Phi_{rec} = \|N_z(\mathbf{x}_0, t)\|\Delta z \ . \qquad (28)$$

One should be aware that in early models of magnetic reconnection the term reconnection rate was often associated with the Alfvénic Mach number of the plasma velocity in the inflow region. Also note that the reconnection rate as given by Eq. (28) still holds if the null point has the X-point topology but is not linear in x and y, for example $\mathbf{G} = [y, x^3, 0]$. Examples of MHD solutions with null points of this kind can be found in [6]. These considerations lead us to the following definition of reconnection.

Definition 5.1 *A two-dimensional solenoidal field* $\mathbf{G}$ *in a domain* D *of a fluid with velocity* $\mathbf{v}$ *shows reconnection if a solution for* $\mathbf{u}$ *in Eq. (22) exists with* $\mathbf{u} = \mathbf{v}$ *on the boundary of* D *which has a singularity at an x-type null of* $\mathbf{G}$ *such that the reconnection rate (Eq. 28) is finite.*

Although most models of reconnection are stationary there is no doubt that in reality reconnection will be a time dependent process. It will start

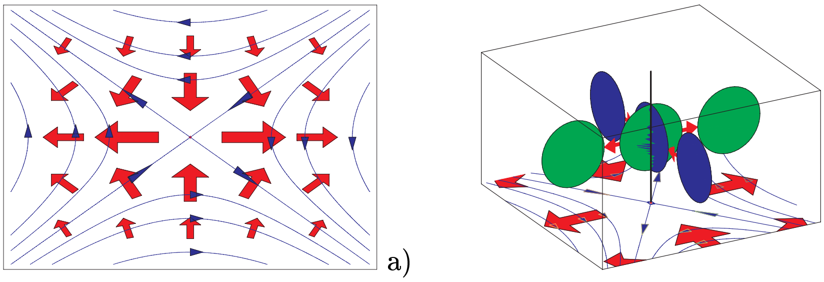

Figure 3. a) Local solution for $\Delta\mathbf{v}$ at a hyperbolic null of $\mathbf{G}$ for a non-vanishing non-ideal term $\mathbf{N}$. The direction of the arrows depends on the orientation of $\mathbf{N}$ with respect to the orientation of $\mathbf{G}$. b) The evolution of two circular cross sections of $\mathbf{G}$-flux tubes, which are transported in the singular $\Delta\mathbf{v}$-flow.

with the occurrence of $\mathbf{N} \neq 0$ at the null of $\mathbf{G}$ at time t_i and it will end as soon as $\mathbf{N}$ is again zero at $t = t_e$. Integrating over the whole process yields the total reconnected flux:

$$\Phi_{rec} = \int_{t_i}^{t_e} \int_z \Delta\mathbf{v} \times \mathbf{G} \cdot d\mathbf{l}\ dt = -\Delta z \int_{t_i}^{t_e} N_z(\mathbf{x}_0, t) dt \ .$$

Exercise: Show that under 2D reconnection the total helicity (see previous exercise) is still conserved.

5.1. MHD-EXAMPLE

Consider the following local stationary solution of the set of MHD-equations (8) - (13) with $\mathbf{T} = 0$ in the vicinity of a hyperbolic null point of a 2D magnetic field.

$$\begin{aligned}
\rho &= const. && (29)\\
\mathbf{B} &= [y, kx, 0], \quad k > 1 && (30)\\
\mathbf{v} &= [-x, y, 0], && (31)\\
\mathbf{E} &= [0, 0, E_z], && (32)\\
\hat{\eta} &= (E_z - kx^2 - y^2)/(k-1) && (33)\\
p &= p0 - 1/2(\rho + k^2 - k)x^2 - 1/2(\rho + 1 - k)y^2 && (34)
\end{aligned}$$

All quantities are normalized to their characteristic values and the solution holds only in the region near the null where η, the resistivity, and the pressure are positive. There are much more elaborated analytical and numerical solutions, but for the purpose of demonstrating the fundamental process this one is already sufficient and has the advantage of being comparatively simple. Note that the non-ideal term is represented here by a resistive term $\mathbf{N} = \eta\mathbf{J}$, as in most models of magnetic reconnection. However, there are plasmas where the resistive term is not important and other terms such as the electron inertia take over the role of the non-ideal term. This shows the advantage of not relating $\mathbf{N}$ to any particular dissipation mechanism in our definition.

If we want to answer the question whether reconnection occurs in our example we first have to solve Eq. (22) for $\mathbf{u}$ which yields

$$\mathbf{u} = (\mathbf{E} - \eta\mathbf{J}) \times \mathbf{B}/B^2 = (k^2x^2 + y^2)^{-1}[-kE_zx, E_zy, 0] \; . \tag{35}$$

Therefore $\mathbf{u}$ shows a singularity at the x-point of $\mathbf{B}$ which also leads to a finite reconnection rate of $\|E_z\| = \|N_z\|$ per unit length in z-direction, and thus satisfies the definition of reconnection. Note that due to the stationarity of the system $\mathbf{u} \equiv 0$ would have been a solution for Eq. (22) as well. However, this solution does not satisfy the condition $\mathbf{u} = \mathbf{v}$ on the boundary and is excluded by definition 5.1. (The boundary in our example is given by the implicit equation $\eta(x, y) = 0$.) This shows the necessity of the boundary condition in the definition in order to avoid trivial solutions.

5.2. HD-EXAMPLE

Although HD and MHD share the same type of equation for the evolution of the vorticity field and the magnetic field, there is no one-to-one correspondence between models of magnetic reconnection and vorticity reconnection. The primary reason is that in HD a solution of the vorticity equation also has to satisfy $\mathbf{G} = \mathbf{w} = \nabla \times \mathbf{v}$. Such a restriction is absent in MHD. Moreover, apart from solutions with constant viscosity, the standard expressions for the non-ideal term $\mathbf{N}$ is more complicated in HD. These two reasons complicate the analysis of reconnection in HD enormously which in turn explains why there are much fewer models or investigations of reconnection in HD compared to reconnection in plasmas. Another more physical reason is that usually the Reynolds numbers of hydrodynamical applications are much lower than those of e.g. astrophysical plasmas. Thus the reconnection processes are much less violent and often masked by an omnipresent dissipation.

We start by constructing a stationary two-dimensional solution for vortex reconnection, that is a solution where all quantities depend only on two spacial coordinates (x, y) and the vorticity vectors lie in the x-y-plane. Using the analogy with magnetic reconnection we assume that the vorticity has the form $\mathbf{w} = [y, kx, 0]$ as in the previous example of magnetic reconnection. However, the old flow velocity $\mathbf{v} = [-x, y, 0]$, does not satisfy $\mathbf{w} = \nabla \times \mathbf{v}$. If $\mathbf{v}$ is not to depend on z it has to be modified with an additional z-component $v_z(x, y, t)$, such that

$$\mathbf{w} = \nabla v_z(x, y, t) \times \mathbf{e}_z \ .$$

From our previous consideration of reconnection in two-dimensional systems we know that reconnection can only occur if the non-ideal term (in the HD case the parenthesis on the right hand side of Eq. (19)) has a non-vanishing z-component at the X-point. Note that due to the two-dimensionality of the system the only term which can provide this component is the viscous term. Therefore the time-derivative of $\mathbf{v}$ in Eq. (17) has to balance this z-component of the viscous term. For a (with respect to $\mathbf{w}$) stationary reconnection process $\partial_t \mathbf{v}$ has to be constant in time, hence $v_z \sim t$ in our model.

Taking these conditions into account a full solution is given by

$$\begin{aligned} \mathbf{v} &= [-x, y, y^2/2 - kx^2/2 - tE_z] \ , && (36) \\ \nu &= E_z/(l-1) - k/(3\ k-1)\ x^2 - 1/(k-3)\ y^2 \ , && (37) \\ p &= p_0 + (k+1)/(6\ k-2)\ x^2 - (k+1)/(2\ k-6)\ y^2 \ . && (38) \end{aligned}$$

This solution satisfies Eq. (17) with a non-ideal term for viscous, incompressible flows in a region near the reconnection axis where $p > 0$, $\nu > 0$ and $k > 3$. Eq. (36)-(38) are to be considered as a solution only for the neighborhood of the reconnection line due to the simple polynomial ansatz for the vorticity. Therefore, it cannot reproduce all the complex features of vortex reconnection such as ‘bridging’, secondary reconnection, etc. This simple system, however, already contains several basic properties of vortex reconnection. First, although we started with the most elementary reconnection process, the process is not stationary anymore with respect to the fluid velocity $\mathbf{v}$, since the v_z component grows linearly in time. Thus there is no true stationary vortex reconnection. Moreover, the existence of a non-vanishing v_z component breaks the two-dimensionality which was present in the example of magnetic reconnection. This complicates the flow structure of the process enormously. But despite of this complexity in the flow velocity, the reconnection process of the vorticity is comparatively simple. So both examples satisfy Eq. (22) with the purely two-dimensional and flux conserving but singular flow $\mathbf{u} = (k^2x^2 + y^2)^{-1}[-kE_z x, E_z y, 0]$.

Because we know the transport velocity of the vorticity we can calculate the time necessary for two flux tubes to reconnect in our example. This merging time is given by the time which is required for an arbitrary point x_0 on the x-axis to reach the z-axis under the flow $\mathbf{u}$. Hence

$$\int_{x_0}^{0} dt = \int_{x_0}^{0} 1/u^x \ dx = -\int_{x_0}^{0} kx/E_z \ dx = x_0^2 \ k/(2E_z) \ .$$

This result is in accordance with numerical results [13], which show a quadratic dependence of the reconnection time on the distance between the flux tubes. Note that this is not an accidental feature of our model, but is due to the generic structure of reconnecting flows in space-time.

In analogy to magnetic reconnection, the reconnected flux is given by the integral along the reconnection line. The reconnected flux and the reconnection rate are therefore,

$$\Phi_{rec} = -\iint \partial_t \mathbf{v} \cdot d\mathbf{l}\; dt\ , \qquad \frac{d\Phi_{rec}}{dt} = -\int \partial_t \mathbf{v} \cdot d\mathbf{l}\ . \tag{39}$$

6. 3D Reconnection

Many physical phenomena show strong distinctions in the transition from two to three dimensions and so does reconnection. However, the crucial difference occurs already in the transition from a field $\mathbf{G}$ with two components to a field with all three components, even if they depend only on two spatial coordinates. For instance, in the example of magnetic reconnection Eq. (34) we can add a constant component b_z in z-direction to the magnetic field and modify the electric field into $\mathbf{E} = [-yb_z, -xb_z, E_z]$. This does not affect the stationarity of the solution nor the structure of the plasma flow $\mathbf{v}$ or the non-ideal term $\eta\mathbf{J}$. However, no flux conserving flows, neither singular nor smooth, exist since the necessary condition Eq. (23) can not be satisfied anymore. Correspondingly the evolution of flux tubes in this case is completely different from the 2D reconnection. This is shown in Figure 4 where we followed two flux tubes entering from a dissipation-free outside region into a non-ideal region with a magnetic field structure as in the given example with a small constant b_z component. The flux tubes are integrated at every instant starting from their cross sections in the dissipation-free outside region which are comoving with $\mathbf{v}$. The sequence of images show that the flux tubes loose their identity, that is they split, as soon as they enter the non-ideal region. Thereupon, the strands of the different ends anchored in the ideal outside region, flip around each other and eventually merge as they leave the non-ideal region, but now differently connected. Thus with respect to the initial and final state of the flux tubes the process is similar as in the two-dimensional case. But contrary to the 2D case the whole evolution of the flux tubes is smooth and there is no point in time where a cutting process or something similar occurs.

Since this pattern of reconnection occurs for arbitrarily small b_z it should not be considered as a new type of reconnection aside of the 2D case, but as the generic one, whereas the 2D case with $b_z = 0$ is an exceptional and structurally unstable case. Thus we need to incorporate this process into

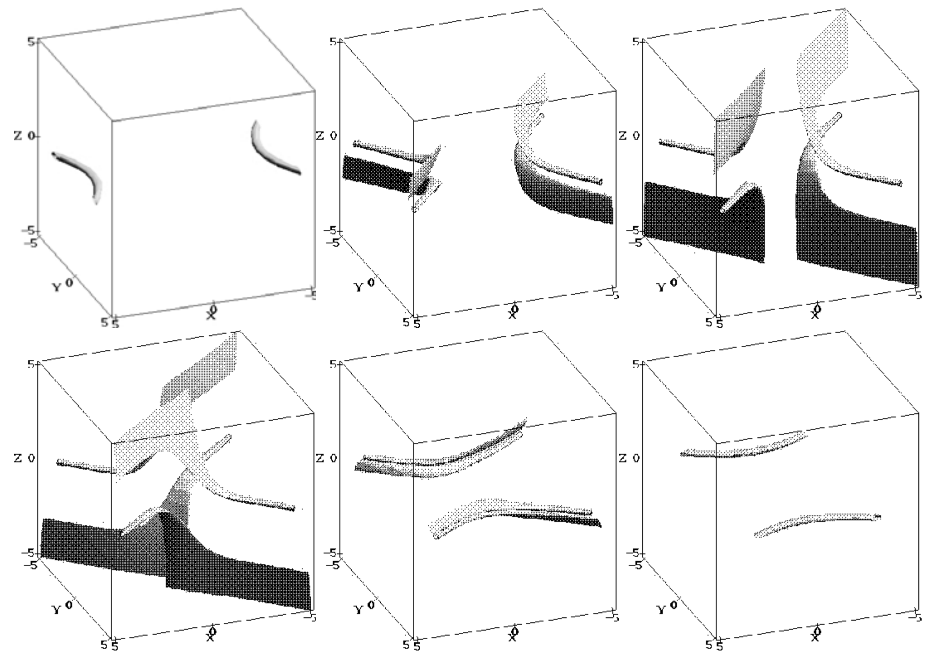

Figure 4. Reconnection in the case of a non-vanishing **G**-field. The non-ideal term **N** is localized near the z-axis.

our notion of reconnection if this notion should be applicable to the real, three-dimensional space.

The key to describe both kinds of reconnection on the same basis is to generalize the notion of flux conservation. First note that the flux conservation Eq. (7) is the non-relativistic limit of the covariant form of flux conservation in 4-space with coordinates $x^0 = ct, x^1, x^2, x^3$:

$$\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha U^\nu G_{\nu\beta} = 0 \tag{40}$$

$$\Leftrightarrow \begin{cases} \partial_0(U^0\mathbf{G}_E + \mathbf{U}\times\mathbf{G}_B) + \nabla(\mathbf{G}_E\cdot\mathbf{U}) = 0 \\ U^0\partial_0\mathbf{G}_B - \nabla\times(\mathbf{U}\times\mathbf{G}_B) - \nabla U^0\times\mathbf{G}_E = 0 \end{cases}$$

$$\Leftrightarrow L_U\omega_G^2 = 0 \tag{41}$$

where $G_{\alpha\beta}$ denotes the field tensor, with the electric components $\mathbf{G}_E$ and the magnetic components $\mathbf{G}_B$. The four-velocity $U^{(4)}= (U^0, U^1, U^2, U^3)= (U^0, \mathbf{U})$ is related to the usual velocity in space by

$$\mathbf{u} := \frac{\mathbf{U}}{U^0} = \frac{d\mathbf{F}_G/ds}{dF_G^0/ds}, \quad \text{where } U^{(4)} = \frac{dF_G(x,s)}{ds} \tag{42}$$

and F_G is the flow of $U^{(4)}$ in 4-space. Note that the parameter is now s instead of t, but they are related by $U^0 = d(ct)/ds$. Here it is important to note that these equations (40) are covariant with respect to *any* coordinate transformation in 4-space, in particular they are not restricted to Minkowski transformation; also $U^{(4)}$ is not normalized to 1 or c.

For the MHD-case the field tensor is the electromagnetic field and $\mathbf{G}_E$ and $\mathbf{G}_B$ are the usual electric and magnetic field. For the hydrodynamic case $G_{\alpha\beta}$ is the vorticity tensor, i.e. $\mathbf{G}_E = -\partial_0 \mathbf{v} - \nabla v^0$ and $\mathbf{G}_B = \nabla \times \mathbf{v}$.

$L_U \omega_G^2 = 0$ in Eq. (41) is the Lie-derivative of the 2-form of the G field with respect to the flow $U^{(4)}$. The Lie-derivative theorem [9] then implies that the flux integrated over a two-dimensional comoving surface C is constant with respect to the parameter s which describes the $U^{(4)}$-flow in space-time.

$$\text{Eq. (41)} \Rightarrow \int_C G dA = \int_C G_{\mu\nu} dx^\mu dx^\nu = const. \tag{43}$$

If the time component U^0 of the 4-flow $U^{(4)}$ is constant, e.g. $U^0 \equiv 1$, the general parameter of the flow s coincides with the time t, and the covariant form of the flux conservation yields the ordinary flux conservation for the magnetic part.

$$\text{Eq. (7)} \stackrel{U^0 = const.}{\Longrightarrow} \int_C \mathbf{G}_B \cdot d\mathbf{A} = const.$$

The advantage of the covariant formulation is that now the singularity in $\mathbf{u}$ which we encountered in reconnection can be represented as a smooth $U^{(4)}$-flow with a null at the former singularity. For instance, for the singular flow Eq. (35) a smooth 4-velocity is given by

$$U^{(4)} = [U^0, U^x, U^y, U^z] = [(k^2x^2 + y^2), -kE_z x, E_z y, 0] \ . \tag{44}$$

More about the relation between singular flows in 3-space and their smooth representation in 4-space can be learned if we integrate the system Eqs. (40) to

$$U^\nu G_{\nu\beta} = \partial_\beta \Phi \ \Leftrightarrow \ \begin{cases} \mathbf{G}_E \cdot \mathbf{U} = -\partial_0 \Phi \\ U^0 \mathbf{G}_E + \mathbf{U} \times \mathbf{G}_B = \nabla \Phi \ . \end{cases} \tag{45}$$

From this it is obvious that whenever we can set $\Phi \equiv 0$, the time component of the equation is a trivial implication of the equation for the space components. Thus fields $\mathbf{G}_E, \mathbf{G}_B$ which allow for a certain $U^{(4)}$-flow also satisfy

$$\mathbf{G}_E + \mathbf{u} \times \mathbf{G}_B = 0 \ ,$$

with the corresponding 3-vector flow $\mathbf{u} = \mathbf{U}/U^0$. This in turn also implies Eq. (22), i.e. the conservation of $\mathbf{G}_B$-flux under $\mathbf{u}$. This is a *a posteriori* justification of our interpretation that the flux is conserved under a singular velocity field in the 2D reconnection case.

The 3D reconnection case, however, requires a non-vanishing Φ because in this case $\mathbf{G}_E \cdot \mathbf{G}_\mathbf{B} = 1/U^0 \mathbf{G}_B \cdot \nabla\Phi \neq 0$ as in the example at the beginning of this section. Note that for this example ($\mathbf{E} = [-yb_z, -xb_z, E_z]$, $\mathbf{B} = [y, kx, b_z]$ $(k > 0)$) a solution for the covariant equations (40) would be

$$U^{(4)} = (y^2 + kx^2 + 1)^{(-1)}[(y^2 + kx^2), -x, y, 0]; \quad \Phi = -b_z xy \ .$$

This flow is smooth and its 3-flow $\mathbf{u} = \mathbf{U}/U^0$ shows the same singularity as in the 2D example. However, due to a non-vanishing Φ the covariant flux conservation does not imply the existence of the ordinary $\mathbf{G}_B$-flux conservation nor the conservation of $\mathbf{G}_B$ field lines. This is the reason why we see the strange splitting and flipping of flux tubes in Figure 4.

This example shows that there exists a more general kind of flux conservation which covers also the case of reconnection in a non-vanishing $\mathbf{G}_B$-field. Correspondingly it is possible to give a definition of reconnection on the basis of the covariant flux conservation instead of Eq. (22)(see [12]) analogous to the definition for 2D reconnection. This however requires additional considerations which are beyond the scope of this article.

Exercise: Show by means of the above given example that the total helicity is in general not conserved under 3D reconnection.

7. Conclusions

The process of reconnection is an important process capable of altering the topology of a divergence-free field. In most applications of fluid dynamics reconnection is driven by a dissipative term, e.g. a resistive or viscous term. It is therefore important to distinguish reconnection from other processes which involve dissipation. It turns out that reconnection is special in the sense that the flux of a field is not reduced but is only restructured. Therefore the effect of reconnection can be modeled with the help of flux conserving flows with singularities of a specific type. This helps to understand the effect of reconnection on the structure of a field even if the details of the underlying dissipation mechanism are unknown as it is often the case in magnetic reconnection, and it also reveals the similarities of magnetic and vortex reconnection.

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