The Geometry of Magnetic and Vortex Reconnection

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Abstract. Reconnection is an important process of structure formation in fluid dynamics, occurring in the form of vortex reconnection in hydrodynamics as well as in the form of magnetic reconnection in plasmas. There is a close analogy between the quantities involved in both phenomena but, surprisingly, the process of magnetic reconnection, although complicated by the presence of a magnetic field, is geometrically simpler than vortex reconnection; it may thus serve as good starting point to understand the geometry of vortex reconnection.

A general covariant definition of reconnection is given and, starting from a simple analytic model of magnetic reconnection, the basic process of reconnection is analyzed. The model is then modified to meet the additional constraints of vortex reconnection. It is shown that, although the evolution of the vorticity near the reconnection site is stationary and two-dimensional the flow velocity is inevitably three-dimensional, and time dependent. Explicit expressions for the reconnected flux and the reconnection time are given.

1 Introduction

The notion of reconnection is found in many fields of physics: in hydrodynamics [1] for the reconnection of vortex tubes, in plasma physics for magnetic reconnection [2], in the theory of superfluids [3] for the reconnection of quantized vortex elements, as well as in cosmology for the interaction of cosmic strings [4][1]. In this contribution we will consider only reconnection as a process in the evolution of a divergence-free vector field, therefore involving the first two examples. The divergence-free field is the common ingredient to both vortex reconnection in hydrodynamics (HD) and magnetic reconnection in magnetohydrodynamics (MHD). In HD the evolution of the vorticity field is determined by the curl of the Navier-Stokes equation, while the evolution of the magnetic field in MHD is determined by the curl of Ohm's law. Both equations have the same structure:

$$\partial_t \boldsymbol{G} - \nabla \times (\boldsymbol{v} \times \boldsymbol{G}) = \nabla \times \boldsymbol{N} \quad \text{with:} \tag{1}$$

$\boldsymbol{v}$: the velocity of the fluid flow

$\boldsymbol{G}$: the transported field; $\boldsymbol{G} = \nabla \times \boldsymbol{v}$ in HD, $\boldsymbol{G} = \boldsymbol{B}$ in MHD;

$\boldsymbol{N}$: The non-ideal term

$\boldsymbol{N} = \partial_k \left(\nu(\partial_k v^i + \partial_i v^k)\right) \boldsymbol{e}_i$ (Viscous, incompressible HD);

$\boldsymbol{N} = \eta \nabla \times \boldsymbol{B}$ (Resistive MHD) ,

[1] The references given here are only examples that provide an entry to the literature

where ∂_i denotes the derivative with respect to the coordinate x^i, $\boldsymbol{e}_i$ is the corresponding unit vector and summation of repeated indices is assumed.

Because only the evolution of the divergence-free field is relevant for the definition of reconnection the other equations of hydrodynamics (HD) and magnetohydrodynamics (MHD), i.e. the equations of continuity, energy and the momentum balance in MHD are not considered here.

The main difference between HD and MHD is that in HD the transporting field, $\boldsymbol{v}$, and the field which is transported, $\boldsymbol{w}$, are coupled by $\boldsymbol{w} = \nabla \times \boldsymbol{v}$. A corresponding equation in MHD ($\boldsymbol{B} = \nabla \times \boldsymbol{v}$) does not exist. This extra freedom allows for more simple reconnection solutions in MHD compared to HD. It is therefore natural to start with an example of magnetic reconnection

2 Magnetic reconnection

Note first that for vanishing resistivity equation (1) is ideal, i.e. the magnetic field is frozen into the fluid flow and the magnetic flux integrated over a comoving surface is conserved. In this case the topology of the field ($\boldsymbol{B}$) is conserved as well and no reconnection is possible.

$$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = 0 \ , \qquad (2)$$
$$\Rightarrow \int \boldsymbol{B} \cdot \boldsymbol{n} \ da = const.$$

Only if the non-idealness $\boldsymbol{N}$ is non-vanishing is reconnection possible. Consider for instance a configuration with an initially anti-parallel two-dimensional magnetic field as shown in Fig. 1. A localized non-idealness $\boldsymbol{N}$ perpendicular to the plane in which the magnetic field lies can produce a curl which reduces the initial anti-parallel field components according to (1) and adds a new perpendicular component, such that the field lines are reconnected. The non-idealness

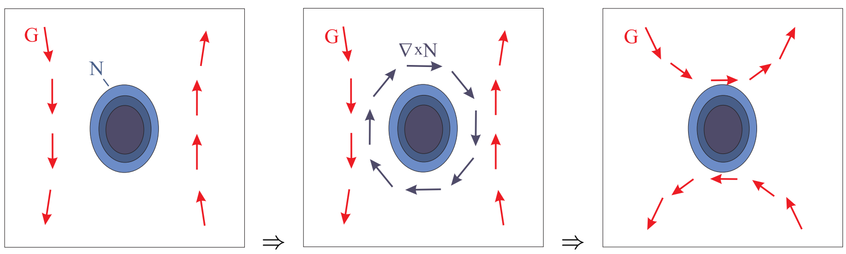

Fig. 1. Basic reconnection process with a localized non-idealness $\boldsymbol{N}$ perpendicular to the plane shown.

$\boldsymbol{N}$ or more exact $\nabla \times \boldsymbol{N}$ plays an important role in the process. $\nabla \times \boldsymbol{N}$ has to be present to allow for reconnection but it also has to be localized, that is there has to be an outside region where $\nabla \times \boldsymbol{N}$ is negligible, at least on the time scales under consideration. Otherwise we would have a situation of global diffusion of the magnetic field and the term ‘reconnection’ would be meaningless because there is no way of identifying field lines or flux tubes in time. Note that the localization of $\nabla \times \boldsymbol{N}$ is not as restrictive as it may look at first sight. It is not necessary that $\boldsymbol{N}$ is localized, only the curl of $\boldsymbol{N}$ has to be localized. The localization is usually not a serious problem in MHD because in the most of the applications of magnetic reconnection the resistivity is very low and enhanced only in thin current sheets. In contrast in HD the viscosity is usually assumed to be constant and the localization of $\nabla \times \boldsymbol{N}$ in HD is, therefore, a result of the structure of vortex sheets in most cases.

Note that ‘localization’ also means that there is no closed flux within the region where $\nabla \times \boldsymbol{N}$ is not negligble, otherwise the same argument as above applies to this closed flux. But if all flux of the non-ideal region is connected to the ‘ideal’ ($\nabla \times \boldsymbol{N} = 0$) surrounding, then the conservation of flux in the outside region implies a conservation of flux in the non-ideal region as well. Therefore, the effect of a localized $\boldsymbol{N}$ is only to dissipate energy and rearrange the flux but not to disspate the flux (at least on the time scale on which the reconnection acts). Hence it should be possible to describe the process as being ‘ideal’, i.e. satisfying (2) but with a new transport velocity $\boldsymbol{u}$ which may have a singularity. For example

$$\boldsymbol{B} = [y, kx, 0], \quad \boldsymbol{u} = [-x\ k\ E_z/(k^2x^2 + y^2), y\ E_z/(k^2x^2 + y^2), 0] \tag{3}$$

is a stationary solution (Fig. 2) of (2), which has a singularity at the origin

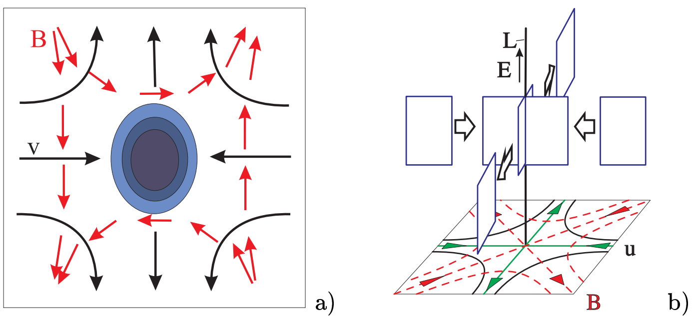

Fig. 2. (**a**) The flow structure and (**b**) the evolution of cross-sections of the magnetic flux under the singular flow $\boldsymbol{u}$ for a stationary reconnection process

of the coordinate system such that it transports the magnetic flux in a finite time along the x-z plane onto the z-axis where the flux is split and reconnected (Fig. 2b). The flux is then transported outwards along both directions of the z-y plane. The example can be derived for a magnetic field $\boldsymbol{B}$ satisfying

$$\begin{aligned} &\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \eta\ \nabla \times \boldsymbol{B} \quad \text{with} \\ &\boldsymbol{v} = [-x, y, 0], \quad \boldsymbol{E} = [0, 0, E_z], \quad \eta = (E_z - kx^2 - y^2)/(k-1) \end{aligned} \tag{4}$$

The solution holds only in the region where $\eta > 0$. Since both $\boldsymbol{E}$ and $\boldsymbol{J}$ are perpendicular to $\boldsymbol{B}$ the transport velocity of the magnetic flux is simply

$$\boldsymbol{u} = (\boldsymbol{E} - \eta \boldsymbol{J}) \times \boldsymbol{B}/B^2 \ .$$

Note that $\boldsymbol{u}$ is not the fluid velocity, but where η vanishes $\boldsymbol{u}$ and $\boldsymbol{v}$ coincide.

Now (2) is only a special case of an equation which leads to reconnective solutions. Since the transport velocity has to become infinite at the reconnection line (the line where $\boldsymbol{u}$ is singular), an adequate description is a covariant one and the most general form, which leads to an equation like (2) (see [5],[6]) is

$$\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha U^\nu F_{\nu\beta} = 0 \tag{5}$$

$$\Leftrightarrow \begin{cases} \partial_0(U^0\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B}) + \nabla(\boldsymbol{E}\cdot\boldsymbol{U}) = 0 \\ U^0\partial_0\boldsymbol{B} - \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) - \nabla U^0 \times \boldsymbol{E} = 0 \end{cases}$$

$$\Leftrightarrow L_U\omega_F^2 = 0 \tag{6}$$

where $F_{\alpha\beta}$ denotes the electromagnetic field tensor. The four-velocity $U^{(4)}=(U^0, U^1, U^2, U^3)=(U^0, \boldsymbol{U})$ is related to the usual velocity in space by

$$\boldsymbol{u} := \frac{\boldsymbol{U}}{U^0} = \frac{\mathrm{d}\boldsymbol{X}/\mathrm{d}s}{\mathrm{d}X^0/\mathrm{d}s}, \tag{7}$$

and $L_U\omega_F^2 = 0$ is the Lie-derivative of the 2-form of the electromagnetic field with respect to the flow $U^{(4)}$. The Lie-derivative theorem then implies that the electromagnetic flux integrated over a two-dimensional comoving surface (C) is constant with respect to the parameter s which describes the $U^{(4)}$-flow in space-time.

$$(6) \Rightarrow \int_C F dA = \int_C F_{\mu\nu} dx^\mu dx^\nu = const.$$

The system (5) can be derived from

$$U^\nu F_{\nu\beta} = \partial_\beta \Phi \Leftrightarrow \begin{cases} \boldsymbol{E}\cdot\boldsymbol{U} = -\partial_0\Phi \\ U^0\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B} = \nabla\Phi \ . \end{cases} \tag{8}$$

These equations can be considered as the most general form of an Ohm's law which lead to a conservation of electromagnetic flux in the form of (5). The potential Φ is not relevant for the two-dimensional reconnection example considered above because, for this case, $\boldsymbol{E}\cdot\boldsymbol{B}$ vanishes and $U^{(4)}$ can be chosen such that $\Phi = 0$ holds everywhere. It is, however, important for the case of reconnection in a non-vanishing magnetic field [7], which requires $\boldsymbol{E}\cdot\boldsymbol{B} \neq 0$ and hence $\Phi \neq 0$.

An additional advantage of the covariant formulation is that now the singularity in $\boldsymbol{u}$ can be represented as a null of $U^{(4)}$ and is thus open to an analysis of its structure. For instance, for the example given above the corresponding 4-velocity is given by

$$U^{(4)} = [(k^2x^2 + y^2), -kE_zx, E_zy, 0] \ . \tag{9}$$

This solution is typical for reconnection because $U^0 > 0$ requires that U^0 vanishes quadratically at the null of $U^{(4)}$, while the space components are antisymmetric and thus in lowest order are linear in x and y near the null. The existence of a null of $U^{(4)}$ with an X-point structure in the space components can be used for a definition of magnetic reconnection [6]. In general the reconnection process requires a line in space along which $U^{(4)}$ vanishes, the reconnection line, denoted by L in Fig. 2 (b). For the example given this line is the z-axis, but in general the line can be curved and moving, and coincides with a field line of $\boldsymbol{B}$ as long as its velocity is small. In this case the total reconnected flux is given by the integral along the part of reconnection line where $U^{(4)}$ vanishes and over the time interval when this occurs, one has

$$\Phi_{\mathrm{rec}} = \iint \boldsymbol{E} \cdot d\boldsymbol{l}\ dt\ .$$

Correspondingly $\int \boldsymbol{E} \cdot d\boldsymbol{l}$ is the reconnection rate for a stationary process.

It is worth mentioning that a uni-directional magnetic field ($\boldsymbol{B} = b_z(x, y)\boldsymbol{e}_z$) can also show an evolution, which requires a non-continuous $\boldsymbol{u}$. An example is where two initially isolated regions of positive b_z in an environment of negative b_z merge. We do not consider this type of 'reconnection' here since it does not require a 'cut and paste' of magnetic flux and, correspondingly, the reconnected flux as defined above vanishes in these cases.

3 Vortex reconnection

As mentioned in the introduction there is no one-to-one correspondence between solutions of magnetic reconnection and vortex reconnection since the latter have to satisfy $\boldsymbol{w} = \nabla \times \boldsymbol{v}$. Moreover, apart from solutions with constant viscosity, the non-ideal term $\boldsymbol{N}$ is more complicated in HD.

We start by constructing a stationary two-dimensional solution for vortex reconnection, that is a solution where all quantities depend only on two space coordinates (x, y) and the vorticity vectors lie in the x-y-plane. Using the analogy with magnetic reconnection we assume that the vorticity has the form $\boldsymbol{w} = [y, kx, 0]$ as in example (3). However, the flow velocity $\boldsymbol{v} = [-x, y, 0]$, which we used in this example, does not satisfy $\boldsymbol{w} = \nabla \times \boldsymbol{v}$. It has to be modified with an additional z-component $v_z(x, y)$, such that

$$\boldsymbol{w} = \nabla v_z(x, y) \times \boldsymbol{e}_z,$$

which is inevitable if $\boldsymbol{v}$ is not to depend on z.

Moreover, from magnetic reconnection we know that an electric field along the reconnection line (the z-axis in our example) has to be present for a non-vanishing reconnection rate. The analogue of the electromagnetic field tensor in hydrodynamics is the vorticity tensor

$$W_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \quad \mu, \nu \in \{0, 1, 2, 3\}$$

The 'electric' field component of $W_{\mu\nu}$ is therefore $-\partial_t \boldsymbol{V} - \nabla V^0$. Because, $V^{(4)}$ or $\boldsymbol{v}$, respectively, is a real fluid velocity, which we assume to be non-relativistic, we have $V^0 = 1$ and $\boldsymbol{V} = \boldsymbol{v}$. Thus the 'electric' field is given by $-\partial_t \boldsymbol{v}$. For a stationary reconnection process this field has to be constant, hence $v_z \sim t$ in our model.

Taking these conditions into account a full solution is given by

$$\boldsymbol{v} = [-x, y, y^2/2 - kx^2/2 - tE_z] \tag{10a}$$
$$\nu = E_z/(k-1) - k/(3\ k-1)\ x^2 - 1/(k-3)\ y^2 \tag{10b}$$
$$p = p_0 + (k+1)/(6\ k-2)\ x^2 - (k+1)/(2\ k-6)\ y^2\ . \tag{10c}$$

This solution satisfies (1) with a non-ideal term for viscous, incompressible HD, as given in (1), for a region near the reconnection axis where $p > 0$, $\nu > 0$ and $k > 3$.

Before discussing the properties of this solution we remark that concerning the covariant transport (5) of the vorticity tensor, the magnetic reconnection in example (3) and the above example of a vorticity reconnection are identical. They both satisfy (8) for $\Phi = 0$ and $U^{(4)}$ given by (9).

$$-U^0 \partial_t \boldsymbol{v} + \boldsymbol{U} \times \boldsymbol{w} = 0 \tag{11}$$
$$U^0 \boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B} = 0 \tag{12}$$

The solution (10a) is only a solution for the neighborhood of the reconnection line due to the simple polynomial assumption for the fields. Therefore, it cannot reproduce all the complex features of vortex reconnection such as 'bridging', secondary reconnection, etc. This simple system, however, already contains several basic properties of vortex reconnection. First, although we started with the most elementary reconnection process, with respect to the fluid velocity $\boldsymbol{v}$ the process is not stationary any more, since the v_z component grows linearly with time. Thus there is no true stationary vortex reconnection. Moreover, the existence of a non-vanishing v_z component breaks the two-dimensionality present in the vorticity field. This complicates the flow structure of the process enormously. But despite this complexity in the flow velocity, the reconnection process of the vorticity is comparatively simple.

Because we know the transport velocity of the vorticity we can calculate the time necessary for two flux tubes to reconnect in our example. This merging time is given by the time which is required for a point (x_0) on the x-axis to reach the z-axis under the flow $\boldsymbol{u}$. Hence

$$\int_{x_0}^{0} dt = \int_{x_0}^{0} U^0/U^x\ dx = -\int_{x_0}^{0} kx/E_z\ dx = x_0^2\ k/(2E_z)\ .$$

This result is in accordance with numerical results [8], which show a quadratic dependence of the reconnection time on the distance between the flux tubes. Note that this is not an accidental feature of our model, but is due to the generic structure of reconnecting flows in space-time as explained above.

In analogy to magnetic reconnection, the reconnected flux is given by the integral along the reconnection line. The reconnected flux and the reconnection rate are therefore,

$$\Phi_{\mathrm{rec}} = -\iint \partial_t \boldsymbol{v} \cdot d\boldsymbol{l}\ dt \qquad \frac{\mathrm{d}\Phi_{\mathrm{rec}}}{\mathrm{d}t} = -\int \partial_t \boldsymbol{v} \cdot d\boldsymbol{l}\ . \tag{13}$$

4 Conclusions

It has been shown that the process of reconnection of a divergence-free field is basically a flux conserving process and thus can be represented by an ideal advection of the flux under a velocity field $\boldsymbol{u}$ which has a singularity. A transition to the covariant equations in space-time allows one to resolve this singularity, which was shown to be a null point of the corresponding 4-velocity $U^{(4)}$. Within this concept we strictly distinguish between the transport velocity of the flux, $\boldsymbol{u}$ or $U^{(4)}$ in space-time, respectively, and the fluid flow ($\boldsymbol{v}$). Starting from an elementary two-dimensional stationary model for magnetic reconnection, a similar solution for vortex reconnection can be constructed. This solution is, however, stationary and two-dimensional only with respect to the velocity $U^{(4)}$, whereas the construction of the corresponding flow field $\boldsymbol{v}$ requires a three-dimensional time-dependent solution. Thus, with respect to $\boldsymbol{v}$, vortex reconnection is inherently three-dimensional and time-dependent, even in its most elementary solution. It also shows that care has to be taken if passive tracers are used to follow vortex tubes because, within the region where the non-idealness is relevant, the vortex velocity $\boldsymbol{u}$ and the fluid velocity $\boldsymbol{v}$ differ significantly.

The covariant formulation also shows that the analogous quantity to the electric field in MHD is $-\partial_t \boldsymbol{v}$ in HD and thus the reconnection rate, which is known from magnetic reconnection to be the integral of the electric field parallel to the reconnection line, is given by $-\int \partial_t \boldsymbol{v} \cdot d\boldsymbol{l}$.

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