

On the Nature of Three-Dimensional Magnetic Reconnection

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Abstract. Three-dimensional reconnection in a finite diffusion region is completely different in many respects from two-dimensional reconnection at an X-point. In two dimensions a magnetic flux velocity can always be defined: two flux tubes can break at a single point and rejoin to form two new flux tubes. In three dimensions, we demonstrate that a flux tube velocity does not generally exist. The magnetic field lines continually change their connections throughout the diffusion region rather than just at one point. The effect of reconnection on two flux tubes is generally to split them into four flux tubes rather than to rejoin them perfectly. During the process of reconnection each of the four parts flips rapidly in a virtual flow that differs from the plasma velocity in the ideal region beyond the diffusion region.

1. Introduction

Two-dimensional reconnection has been studied for many years and is now fairly well understood, but the way reconnection operates in three dimensions has only been studied relatively recently (see, e.g., Priest and Forbes, 2000 for a review). General considerations have been given by Schindler, Hesse and Birn (1988), who categorised different kinds of reconnection and put forward the notion of *general magnetic reconnection*. In a companion paper Hesse and Schindler (1988) analysed the process in terms of Euler potentials. More recently, Hornig (2001) and Hornig and Rastätter (1998) have proposed an elegant covariant formulation of three-dimensional reconnection.

Several numerical experiments have added to our understanding of 3D reconnection. For example, Lau and Finn (1996) followed the evolution of a pair of twisted flux tubes. Dahlburg, Antiochos and Norton (1997) and Linton, Dahlburg and Antiochos (2000) found that two isolated flux tubes could either bounce, merge, reconnect normally or tunnel, depending on their twist and inclination. Also, Galsgaard and Nordlund (1997) found that an initial configuration including eight null points reconnected by separator reconnection.

Since very little is known about the fundamental nature of 3D reconnection and we are still in an exploratory phase, we shall focus on the implications of the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}) \quad (1)$$

alone and so, by analogy with the early stages of dynamo theory, we adopt a kinematic approach and ignore the implications of the equation of motion. Equation (1) arises by taking the curl of Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma}, \quad (2)$$

where $\mathbf{j} = \nabla \times \mathbf{B}/\mu$ and $\eta = (\mu\sigma)^{-1}$ may depend on position. It describes how the magnetic field changes in time due to two effects on the right-hand side of (1), namely, the advection of the magnetic field with the plasma and the diffusion of the magnetic field through the plasma. In most of the universe, the magnetic Reynolds number ($R_m = \frac{v_0 L_0}{\eta}$) is much larger than unity and so the first term on the right-hand side of (1) dominates, the magnetic field is frozen to the plasma and Ohm's law reduces to

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0}.$$

The exception is in current concentrations where the electric current ($\mathbf{j} = \nabla \times \mathbf{B}/\mu$) and magnetic gradient become very large, namely at locations where reconnection is occurring.

For a given magnetic field $\mathbf{B}(x, y, z, t)$, the plasma velocity (\mathbf{v}) satisfies the induction equation (1) or equivalently Ohm's law (2) together with Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (3)$$

By comparison, in general we can describe the motion of magnetic flux provided a *flux-conserving velocity* (\mathbf{w}) exists; i.e., provided there exists a \mathbf{w} such that

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{w} \times \mathbf{B}). \quad (4)$$

Here we ask: what is the basic nature of 3D reconnection in the absence of null points and how does it differ from 2D reconnection? Section 2 summarises the main features of 2D reconnection, while Section 3 shows that none of these features are present in 3D reconnection. For example, as proved in Section 3, for 3D reconnection with a finite diffusion region, in general a flux tube velocity does not exist. Finally, an example is described of the way reconnection occurs in a simple magnetic configuration.

2. The Basic Properties of 2D Reconnection

In two dimensions, magnetic reconnection possesses several fundamental properties, which we shall enumerate below. Many of these have previously been implicit rather than being explicitly stated and have been assumed to be an intrinsic part of the nature of reconnection, but wrestling with the fundamentals of 3D reconnection has forced us to examine whether indeed they are essential properties of reconnection as a whole or just of 2D reconnection.

(i) A differentiable flux tube velocity (\mathbf{w}) satisfying Equation (4) always exists everywhere except at null points. This velocity has a hyperbolic singularity at an X-type null point of \mathbf{B} , where the reconnection takes place. The magnetic flux moves at the velocity \mathbf{w} and slips through the plasma, which itself moves at \mathbf{v} .

(ii) The mapping of field lines in 2D is discontinuous. Consider the field of an X-type neutral point inside a rectangular boundary, say. A footpoint (A_1) on the left-hand boundary below the separatrix in Figure 1 maps along the magnetic field to a point B_1 on the right-hand boundary. As one moves, however, across the separatrix from A_1 to A_2 , there is a sudden jump in the mapping point from the right-hand boundary to a point D_2 on the right-hand boundary. At the same time there is a discontinuity in the mapping of points C_1 to D_2 and C_2 to B_2 . This mapping discontinuity is associated with the fact that field lines break at one point.

(iii) While they are in the diffusion region, field lines preserve their connections. The exception is the X-point, where the field lines break and their connections are changed.

(iv) Reconnecting flux tubes rejoin perfectly (Figure 2). That is, to any tube that is going to reconnect, there exists a counterpart with which it will rejoin perfectly. The net

effect is that two tubes move towards the diffusion region (at velocity $\mathbf{w} = \mathbf{v}$), they break, and they rejoin perfectly to form two new flux tubes that move out (at $\mathbf{w} = \mathbf{v}$).

(v) When a flux tube is partly in a diffusion region (D), $\mathbf{w} = \mathbf{v}$ on both parts of the tube that are outside, while $\mathbf{w} \neq \mathbf{v}$ on the segment that lies in the diffusion region. Thus the two wings of the flux tube (outside D) in Figure 3 are moving with the plasma, while the central segment (in D) is slipping through the plasma.

As an example, consider a steady-state magnetic field of an X-point with uniform current having components

$$B_x = y, \quad B_y = a x$$

where a is a constant. The flux tube velocity \mathbf{w} is determined by

$$\mathbf{E} + \mathbf{w} \times \mathbf{B} = \mathbf{0}$$

(with $\mathbf{E} = E_0 \hat{\mathbf{z}}$ constant) to be

$$w_x = \frac{-E_0 a x}{a^2 x^2 + y^2}, \quad w_y = \frac{E_0 y}{a^2 x^2 + y^2}.$$

It can be seen that \mathbf{w} has a point singularity at the X-point $(x, y) = (0, 0)$. Indeed, a signature of two-dimensional reconnection is that the inflow speed (w_y) of magnetic flux behaves like y^{-1} along the inflow (the y -axis), while the outflow speed (w_x) behaves like $-x^{-1}$ along the outflow (the x -axis).

Once these basic properties have been assumed and established, even implicitly, it is then possible to build models for different regimes of reconnection and, for instance, to calculate the reconnection rates (e.g., Priest and Forbes, 2000).

The question then arises: does the same description in terms of flux tube velocities work in three dimensions?

3. The Properties of 3D Reconnection

In three dimensions, surprisingly, none of the above properties carry over and so the nature of reconnection is profoundly different from our accustomed way of thinking in two dimensions.

(i) A flux tube velocity (\mathbf{w}) does not generally exist, as proved in the following theorem.

THEOREM: For an isolated 3D diffusion region, a flux-conserving flow (\mathbf{w}) does not exist in general.

Proof: Consider a finite diffusion region (D). Suppose \mathbf{w} does exist. Then we have

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{w} \times \mathbf{B}). \quad (5)$$

Using Faraday's law (3) we can uncurl this equation to obtain:

$$\mathbf{E} + \mathbf{w} \times \mathbf{B} = \nabla F, \quad (6)$$

where F is a scalar potential.

Outside the diffusion region (D), $\mathbf{w} = \mathbf{v}$ and $\nabla F = 0$, and so without loss of generality we may assume $F = 0$. However, taking the dot product of (6) with \mathbf{B} makes the second term on the left of (6) vanish, so that

$$\mathbf{B} \cdot \nabla F = \mathbf{B} \cdot \mathbf{E},$$

which may be integrated to give

$$F = \int E_{\parallel} ds, \quad (7)$$

where the integration is along a field line and E_{\parallel} is the component of \mathbf{E} parallel to \mathbf{B} .

Split the surface (S) of the diffusion region into two parts, on one (S_a) of which the magnetic field lines are entering the diffusion region and on the other (S_b) of which the field lines are leaving D . Suppose $F = F_a = 0$ on field lines before they enter D through S_a and $F = F_b$ at the points where they leave through S_b (Figure 4). Then, in general, equation (7) implies that F_b is non-zero and so F does not vanish on field lines beyond S_b , i.e., outside D . But this is a contradiction and so we conclude that \mathbf{w} does not in general exist.

(ii) The mapping of field lines is continuous. In general for regions in which there is no 3D null point the mapping of field lines from one part of the boundary of a region to another is continuous. Consider for example the field $(B_x, B_y, B_z) = (y, x, 1)$ inside a cube, formed by adding a uniform field in the z -direction to a 2D X-point (Figure 5). In this case a point A_1 on the bottom of the cube will map to a point B_1 on the top. This time, as the location of the initial point moves slowly from A_1 to A_2 , the final point moves rapidly but continuously along the top of the box from B_1 to D_2 . At the same time, the footpoint C_1 maps to D_1 while C_2 maps to B_2 . In other words, the mapping is now continuous but, as the initial point moves slowly across the remnant of the separatrix that existed in 2D (called a quasi-separatrix in 3D, Priest and Démoulin, 1995), the end-point of the field line performs a rapid flipping motion (Priest and Forbes, 1992).

(iii) While they are in a 3D diffusion region, magnetic field lines continually change their connections.

(iv) Two flux tubes don't generally break and reform perfectly in 3D to give two flux tubes. Instead, each half of the original tubes joins to a different half, as indicated in Figure 6.

(v) When the two flux tubes are partly in the diffusion region and so are in the process of reconnecting, they split into four parts, each of which flips in a different manner. In particular, if we follow a flux tube through the diffusion region D we find that beyond D it moves with a velocity different from the plasma velocity. This is a manifestation of the non-existence of a flux conserving velocity \mathbf{w} .

4. Example of Reconnection in a Simple 3D Configuration

The properties discussed above are illustrated in the following example. As discussed previously, we solve here only the induction equation (1), so this example is a kinematic solution of the MHD equations. We start by specifying the magnetic field

$$\mathbf{B} = (y, k^2x, b_0) \quad (8)$$

where k and b_0 are constants. This is simply a generalisation of the field shown in Figure 5, which possesses a uniform current $\mathbf{j} = ((k - 1)/\mu)\hat{\mathbf{z}}$. The parallel electric field (E_{\parallel}) is now chosen to be localised so that, in turn, η is localised within a region D , approximately defined by $-2 \leq x, y, z \leq 2$. The full electric field (\mathbf{E}) and the plasma velocity (\mathbf{v}) are then finally calculated.

We illustrate the process as follows: two symmetric (elliptical) flux tube cross-sections are initially chosen, centred on $(\pm x_1, 0, 0)$, where $x_1 \gg 2$. For each of these cross-sections we now integrate an equal distance in either direction along the magnetic field to find two new cross-sections, either of which also defines the corresponding initial flux tube. Integrating along field lines passing through the boundaries of the resulting four cross-sections gives two initial symmetric flux tubes in the ideal region, on either side of D .

The four cross-sections are chosen such that they never pass through D , but are moved always with the ideal plasma velocity \mathbf{v} , which carries them across the quasi-separatrices of \mathbf{B} . We follow the motion of the flux tubes by integrating the field lines through the four cross-sections (A, B, C, D, in Figure 8) at each time.

As long as the flux tubes remain outside the diffusion region the tubes integrated from cross sections A and B (or C and D) coincide (see Figure 8(a)).

Once the tubes reach the diffusion region, the description becomes less straightforward. The tubes integrated from cross sections A and B (or C and D) do not coincide anymore. This is due to the non-existence of a common flux-conserving velocity \mathbf{w} . Note that we stop the integration of the flux tubes at a certain length, for clarity of the image. As soon as the tubes enter D , they begin to slip through the plasma, but at different rates, so that the identity of the initial two tubes is lost and they each split into two separate tubes, as shown in Figure 8(b). The four tube-halves then flip past each other (Figure 8(c)). Since the parts of the tubes beyond the diffusion region do not move at the ideal velocity (\mathbf{v}) they continuously change their connections with the ambient field as sketched in Figure 7. As discussed before, we can think of the tubes as flipping in a “virtual” flow. For example, the tube defined by integrating from cross-section A moves with a flow \mathbf{w}_A (Figure 7), which differs from the plasma velocity both in and beyond the diffusion region. When the tubes re-emerge from D (Figure 8(d)), we can see that the configuration of two unique tubes is not recovered, as none of the final four tubes are coincident. This non-uniqueness of the final four tubes is even more apparent when we just look at the evolution of the cross-sections of the four tubes in the $z = 0$ plane (see Figure 9). Although each pair of tube sections intersects at four points, they have certainly not fully re-connected.

5. Conclusions

We have proved that in a region of non-vanishing magnetic field a localized non-ideal term in the induction equation will in general lead to the non-existence of a flux conserving flow for the evolution of the magnetic field. The result is that two flux tubes split and flip when they enter the non-ideal region, as demonstrated in an explicit example. In addition, the example shows that, even after the reconnection process is completed, the flux tubes do not join up again. This implies that the process of three-dimensional reconnection in a region of non-vanishing magnetic field is much more involved than we would expect from two-dimensional models. We shall aim to study the process in more detail, both analytically and numerically, in future.

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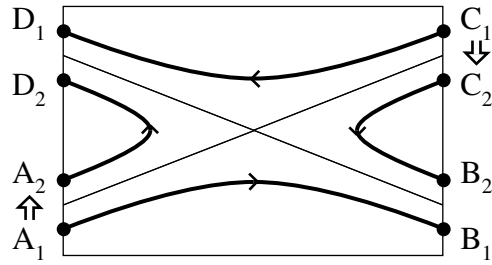


Figure 1. The mapping of field lines in 2D.

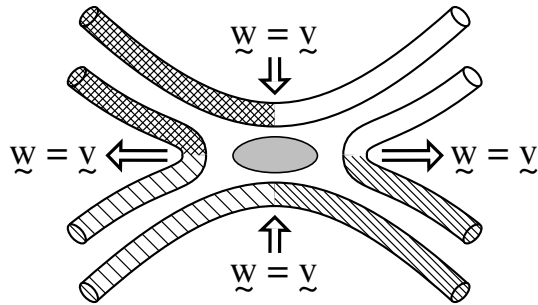


Figure 2. Breaking and rejoining of two flux tubes in 2D to form two new flux tubes.

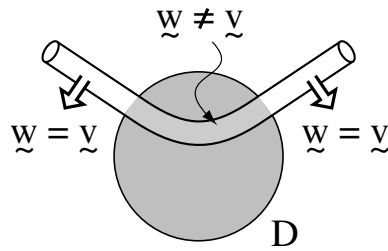


Figure 3. The behaviour of a flux tube in 2D when partly in a diffusion region.

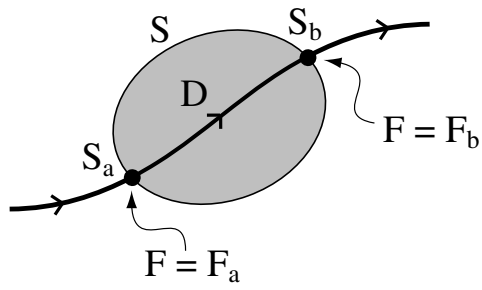


Figure 4. The notation for the potential (F) on a field line that passes through the surface (S) of a diffusion region (D) in three dimensions.

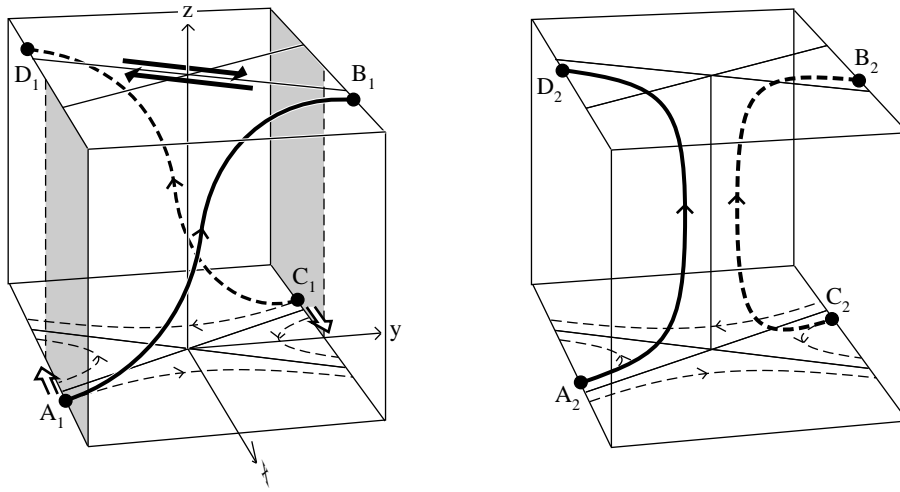


Figure 5. The mapping of field lines in 3D.

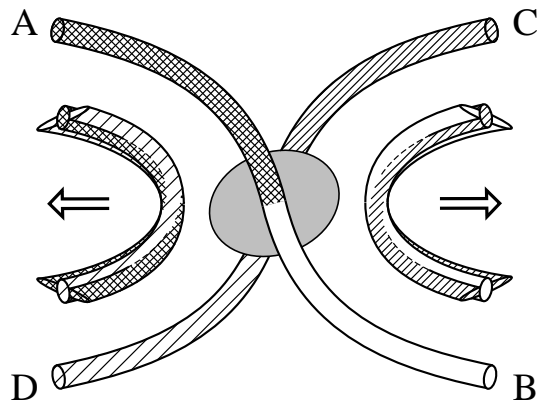


Figure 6. Breaking and partial rejoining of two flux tubes in 3D to form four new flux tubes.

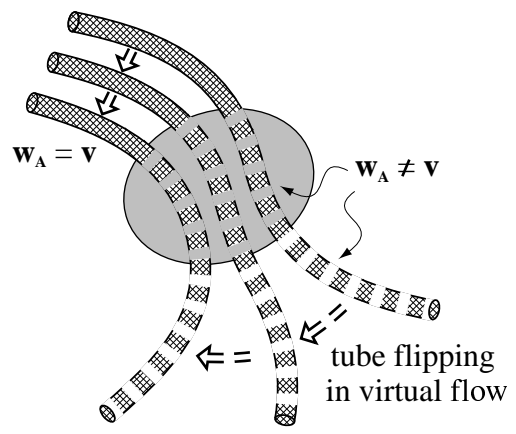


Figure 7. The flipping during reconnection of one of the four parts (A, say) of the initial two tubes in Figure 6.

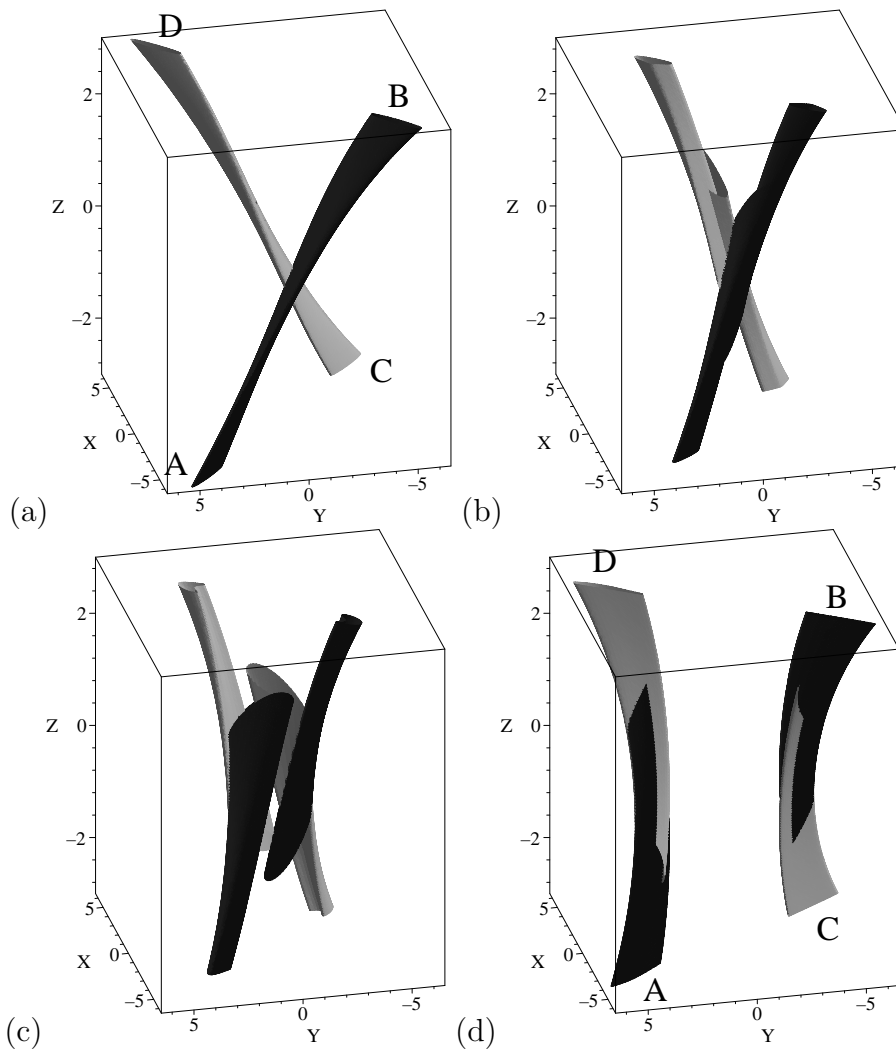


Figure 8. Kinematic reconnection of two flux tubes (integrated from the cross-sections A, B, C and D) for the magnetic field $\mathbf{B} = (y, k^2x, b_0)$. (a) shows the initial two tubes. In (b) the tubes start to split and in (c) they flip past each other. (d) shows the final four non-matching flux tubes once they have left the diffusion region.

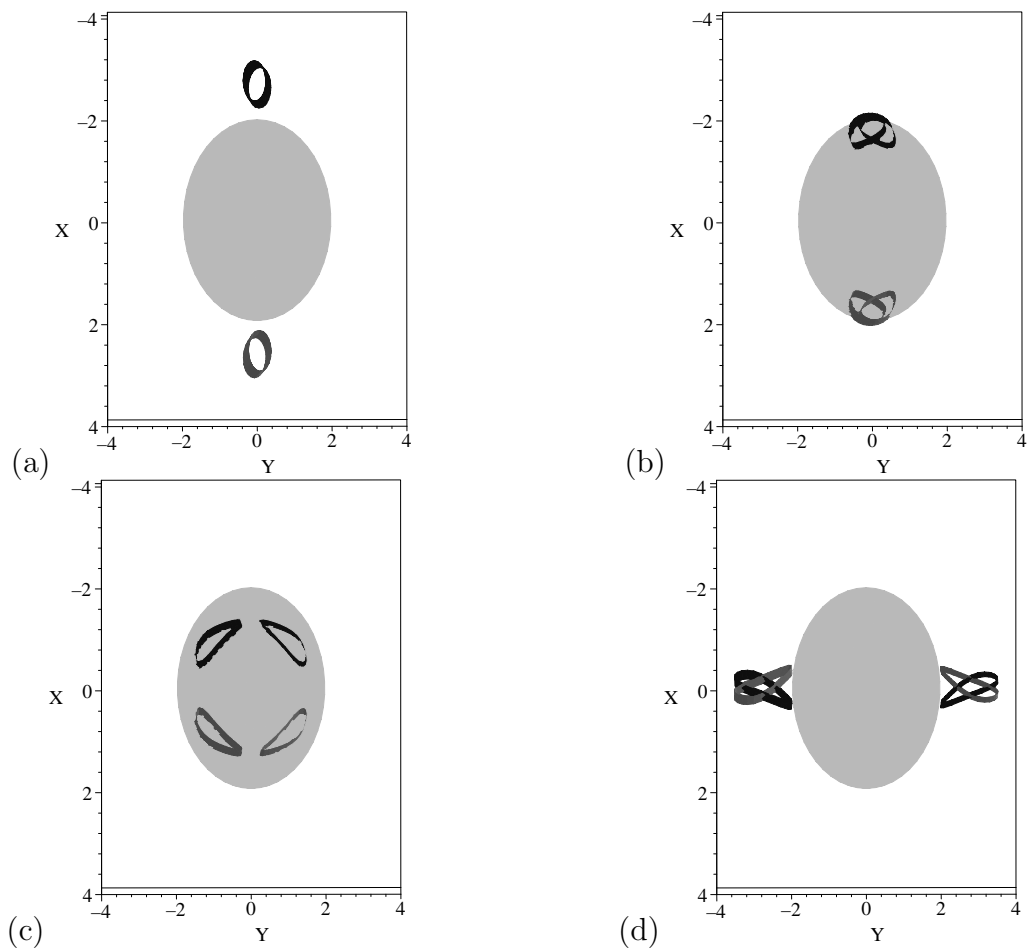


Figure 9. Cross-sections in the $z = 0$ plane of the four tubes shown in Figure 8. The shaded region shows the diffusion region. In (a) we see the two initial tubes, and in (b) and (c) we see them split and flip past each other. In (d) we clearly see the non-matching of the final tubes.