ON CERTAIN NONLINEAR PSEUDOPARABOLIC VARIATIONAL INEQUALITIES WITHOUT INITIAL CONDITIONS

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We consider a nonlinear pseudoparabolic variational inequality in a tube domain semibounded in variable $t$. Under certain conditions imposed on coefficients of the inequality, we prove the theorems of existence and uniqueness of a solution without any restriction on its behavior as $t \to -\infty$.

It is known that fluid filtration in media with double porosity [1], heat transfer in a heterogeneous medium [2], and moisture transfer in soil [3] are modelled by boundary-value problems for pseudoparabolic equations. The general theory of such equations and boundary conditions for them were a subject of investigations of many authors [3–15]. For example, problems without initial conditions for some pseudoparabolic systems were investigated in [14, 15].

Pseudoparabolic variational inequalities make it possible to obtain the conditions for correct solvability of some other boundary-value problems for pseudoparabolic equations.

In the present paper, we prove the correctness of a nonlinear pseudoparabolic variational inequality without initial conditions in the class of functions with arbitrary behavior as $t \to -\infty$.

Note that certain parabolic variational inequalities without initial conditions were investigated in [16–18]. Moreover, the conditions for unique solvability of a pseudoparabolic inequality generated by a linear pseudoparabolic operator were obtained in [19]. In this case, the behavior of a solution was restricted by the condition that it increase not faster than $e^{-\lambda t}$ as $t \to -\infty$, where $\lambda$ is determined by coefficients of the inequality. The results presented in [19] cannot be obtained from the present paper.

Let $\Omega$ be a bounded region of the space $\mathbb{R}^n$ with the boundary $\partial \Omega \subset C^1$, $Q_T = \Omega \times (-\infty, T)$, $T < \infty$, $\Omega_{t_1} = \Omega \times (t_1, t_2)$, $t_1 < t_2 < T$, $\Omega_T = Q_T \cap \{ t = \tau \}$, let $V$ be a closed subspace continuously and compactly imbedded in $L^2(\Omega)$, $H^1(\Omega) \cap W^{1,p}(\Omega) \subset V \subset H^1(\Omega) \cap W^{1,p}(\Omega)$, $p > 2$, and let $K$ be a closed convex subset in $V$ which contains the zero element.

We define a norm in the space $V$ as the norm of the space $H^1(\Omega) \cap W^{1,p}(\Omega)$.

Consider the following variational inequality:

$$
\int_{Q_{t_1,t_2}} v(t) (v - u) + \sum_{i,j=1}^n b_{ij}(x,t) (v_{x_i} - u_{x_j}) v_{x_j} - u_{x_j} \\
+ \sum_{i,j=1}^n a_{ij}(x,t) u_{x_i} (v_{x_j} - u_{x_j}) + \frac{1}{2} \sum_{i,j=1}^n b_{ij,t}(x,t) (v_{x_i} - u_{x_i}) (v_{x_j} - u_{x_j}) \\
+ \sum_{i=1}^n c_i(x,t) u_{x_i} (v - u) + c_0(x,t) u (v - u) \\
+ \sum_{i=1}^n \alpha_i(x) |u_{x_i}|^{p-2} u_{x_i} (v_{x_i} - u_{x_i}) + g(x) |u|^{p-2} u (v - u) - f_0(x,t) (v - u)
$$
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\[ - \frac{1}{2} \sum_{i=1}^{n} f_i(x, t) (v_{x_i} - u_{x_i}) \] \[ \geq \frac{1}{2} \int_{Q_T} \left[ \sum_{i,j=1}^{n} b_{ij}(x, t_2) (v_{x_i} - u_{x_i})(v_{x_j} - u_{x_j}) + (v-u)^2 \right] dx \]

\[ - \frac{1}{2} \int_{Q_T} \left[ \sum_{i,j=1}^{n} b_{ij}(x, t_1) (v_{x_i} - u_{x_i})(v_{x_j} - u_{x_j}) + (v-u)^2 \right] dx. \]  

(1)

**Definition 1.** A solution of inequality (1) is a function \( u(x, t) \) such that: \( u \in L^2_{\text{loc}}((-\infty, T]; H^1(\Omega)), u \in L^2_{\text{loc}}((-\infty, T]; w_{-\infty}^p(x)) \) and \( u \in K \) for almost all \( t \in (-\infty, T] \), \( u(x, t) \) satisfies (1) for all \( t_1, t_2 \in (-\infty, T], t_1 < t_2 \), and for all functions \( v(x, t) \) such that \( v \in H^2_{\text{loc}}((-\infty, T]; H^1(\Omega)) \cap L^p_{\text{loc}}((-\infty, T]; W^{1,p}(\Omega)) \) and \( v \in K \) for almost all \( t \in (-\infty, T] \).

Let the coefficients of inequality (1) satisfy, respectively, the following conditions:

**condition A_1:**

\[ a_{ij} \in L^\infty(Q_T), \quad i, j = 1, \ldots, n, \quad \sum_{i,j=1}^{n} a_{ij}(x, t) \xi_i \xi_j \geq a_0 \sum_{i=1}^{n} \xi_i^2, \quad a_0 > 0, \]

for almost all \( \xi \in \mathbb{R}^n \) and almost all \( (x, t) \in Q_T \),

**condition A_2:**

\[ \alpha_i \in L^\infty(\Omega), \quad \alpha_i(x) \geq \alpha_0 > 0, \quad i = 1, \ldots, n, \]

**condition B:**

\[ b_{ij}(x, t) = b_{ji}(x, t), \quad b_{ij}, b_{ij} \in L^\infty(Q_T), \quad i, j = 1, \ldots, n, \]

\[ b_k \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} \frac{\partial^k b_{ij}(x, t)}{\partial t^k} \xi_i \xi_j \leq b_k \sum_{i=1}^{n} \xi_i^2, \]

\[ b_0 > 0, \quad k = 0, 1; \quad b = \min \{ b_1, -b^1 \} \]

for all \( \xi \in \mathbb{R}^n \) and almost all \( (x, t) \in Q_T \),

**condition C:**

\[ c_i \in L^\infty(Q_T), \quad i = 1, \ldots, n, \quad \sup_{Q_T} \sum_{i=1}^{n} c_i^2(x, t) = c^1, \]

\[ c_0 \in L^2_{\text{loc}}((-\infty, T]; L^\infty(\Omega)), \quad c_0(x, t) \geq c^0 > 0, \quad (x, t) \in Q_T, \]
condition G:

\[ g \in L^\infty(\Omega), \quad g(x) \geq g_0 > 0, \quad x \in \Omega. \]

**Theorem 1.** Let the coefficients of inequality (1) satisfy conditions A, A_2, B, C, and G, and, moreover, let \((4a_0 - 2b^1) c^0 > c^1\). Then inequality (1) cannot have more than one solution.

**Proof.** Define operators \( A \) and \( B_1 \) according to the following formulas:

\[
\langle Aw_1, w_2 \rangle(t) = \int_\Omega \left( \sum_{i,j=1}^n a_{ij}(x,t) w_{1x_i} w_{2x_j} + \sum_{i=1}^n \alpha_i(x) |w_{1x_i}|^{p-2} w_{1x_i} w_{2x_j} \right) dx,
\]

\[
+ \sum_{i=1}^n c_i(x,t) w_{1x_i}^2 + c_0(x,t) w_1^2 + g(x) |w_1|^{p-2} w_1 w_2 \right) dx, \]

\[
\langle B_1 w_1, w_2 \rangle = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n b_{ij}(x,t) w_{1x_i} w_{2x_j} dx,
\]

where \( w_1, w_2 \) are arbitrary functions from \( V \).

It is easy to verify that, under the conditions of the theorem, the operator \( A - B_1 \) is uniformly monotone. Indeed,

\[
\langle (A - B_1) w^1 - (A - B_1) w^2, w^1 - w^2 \rangle \geq \int_\Omega \left[ \left( a_0 - \frac{b^1 + c^1 \delta_0}{2} \right) \sum_{i=1}^n |w_{1x_i}^1 - w_{1x_i}^2|^2 + \left( c^0 - \frac{1}{2\delta_0} \right) |w_1^1 - w_2^1|^2 \right. \]

\[
+ 2^{2-p} |a_0| \sum_{i} |w_{1x_i}^1 - w_{1x_i}^2|^p + 2^{2-p} g_0 |w_1^1 - w_2^1|^p \left. \right] dx \]

\[
\geq \beta_0 \int_\Omega \left( \sum_{i,j} b_{ij}(x,t) |w_{1x_i}^1 - w_{1x_i}^2| |(w_{1x_j}^1 - w_{2x_j}^2) + (w_1^1 - w_2^1)^2 \right) dx, \quad (2)
\]

where \( \beta_0 = (n + 1)(2-p)/2 \) \( 2^{2-p} \min \{a_0, g_0\} \left( \min \left\{ 1, \frac{1}{b_0} \right\} \right)^{p/2} \).

Let \( u^1(x, t) \) and \( u^2(x, t) \) be two solutions of inequality (1). Then, for the function \( v(x, t) = \frac{1}{2} (u^1(x, t) + u^2(x, t)) \), the following inequalities are valid:

\[
\int_{Q_{t_1,t_2}} \left[ (v - f^k)(v - u^k) + \sum_{i,j=1}^n b_{ij} v_{x_i,t} (v_{x_j} - u_{x_j}^k) + \frac{1}{2} \sum_{i,j=1}^n b_{ij} (v_{x_j} - u_{x_j}^k)(v_{x_j} - u_{x_j}^k) \right] dx dt
\]
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\[
\geq \frac{1}{2} \int_{Q_T} \left[ \sum_{i,j=1}^{n} b_{ij} (v_{x_i} - u^k_{x_i})(v_{x_j} - u^k_{x_j}) + (v - u^k)^2 \right] dx
\]

\[
- \frac{1}{2} \int_{Q_T} \left[ \sum_{i,j=1}^{n} b_{ij} (v_{x_i} - u^k_{x_i})(v_{x_j} - u^k_{x_j}) + (v - u^k)^2 \right] dx,
\]

where

\[
f^k = f_0 - \sum_{i=1}^{n} f_{i,x_i} - Au^k, \quad k = 1, 2.
\]

By summing these two inequalities, we obtain

\[
\int_{Q_{t_1,t_2}} \left[ (f^1 - f^2)(u^1 - u^2) + \frac{1}{2} \sum_{i,j=1}^{n} b_{ij} (u^1_{x_i} - u^2_{x_i})(u^1_{x_j} - u^2_{x_j}) + (u^1 - u^2)^2 \right] dx \, dt
\]

\[
\geq \frac{1}{2} \int_{Q_T} \left[ \sum_{i,j=1}^{n} b_{ij} (u^1_{x_i} - u^2_{x_i})(u^1_{x_j} - u^2_{x_j}) + (u^1 - u^2)^2 \right] dx
\]

\[
- \frac{1}{2} \int_{Q_{t_1}} \left[ \sum_{i,j=1}^{n} b_{ij} (u^1_{x_i} - u^2_{x_i})(u^1_{x_j} - u^2_{x_j}) + (u^1 - u^2)^2 \right] dx.
\]

In view of the expressions of the functions \( f^k, \ k = 1, 2, \) we can rewrite estimate (3) in the form

\[
\frac{1}{2} \int_{t_1}^{t_2} \frac{d}{dt} \left( \int_{\Omega} \left[ \sum_{i,j=1}^{n} b_{ij}(x,t)(u^1_{x_i} - u^2_{x_i})(u^1_{x_j} - u^2_{x_j}) + (u^1 - u^2)^2 \right] dx \right) dt
\]

\[
+ \int_{t_1}^{t_2} \langle (A - B_1)u^1 - (A - B_2)u^2, u^1 - u^2 \rangle \, dt \leq 0.
\]

Hence, by using estimate (2), we obtain the inequality

\[
\int_{t_1}^{t_2} (y'(t) + \beta_1(y(t))^{\nu/2}) \, dt \leq 0,
\]

where

\[
y(t) = \int_{\Omega} \left[ \sum_{i,j=1}^{n} b_{ij}(u^1_{x_i} - u^2_{x_i})(u^1_{x_j} - u^2_{x_j}) + (u^1 - u^2)^2 \right] dx,
\]
Since the numbers \( t_1 \) and \( t_2 \) are arbitrary, we obtain from estimate (4) the inequality
\[
y'(t) + \beta_1 (y(t))^{p/2} \leq 0
\]
for almost all \( t \in (-\infty, T] \).

Then, by virtue of Lemma 2 in [20], we have \( y(t) = 0 \) for almost all \( t \in (-\infty, T] \), i.e., \( u^1(x, t) = u^2(x, t) \) almost everywhere in \( Q_T \). Theorem 1 is proved.

**Theorem 2.** Let the coefficients of inequality (1) satisfy conditions \( A_f, A_B, B, C, \) and \( G \), and, moreover, let
\[
a_{ij,t} \in L^\infty(\Omega_T), \quad i, j = 1, \ldots, n, \quad c_i, t \in L^1(\Omega_T), \quad i = 0, 1, \ldots, n.
\]
Let there exist a positive number \( \gamma \) such that
\[
p_0 = 2a_0 - 2b^0 \gamma - b > 0, \quad 2(g_0 - \gamma)p_0 > h_1,
\]
\[
\int_{\Omega_T} \sum_{i=0}^{n} \left( f^2_i(x, t) + f^3_i(x, t)\right)e^{2\gamma t'} \, dt < \infty.
\]
Then there exists a solution \( u(x, t) \) of inequality (1).

**Proof.** Consider a sequence of functions \( \{\psi^i\} \) which possess the following properties: \( \psi^i \in W^{1,p}(\Omega), \quad i = 1, 2, \ldots \), the functions \( \psi^1, \ldots, \psi^k \) are linearly independent for arbitrary \( k \), and linear combinations of \( \psi^i \) are dense in \( W^{1,p}(\Omega) \).

Let
\[
u^N(x, t) = \sum_{k=1}^{N} c_k^N(t) \psi^k(x), \quad N = 1, 2, \ldots,
\]
where \( c_1^N, \ldots, c_N^N \) is a solution of the Cauchy problem
\[
\int_{\Omega_T} \left[ u^N \psi^k + \sum_{i,j=1}^{n} b_{ij} u^N_{x_i} \psi^k_{x_j} + \sum_{i,j=1}^{n} a_{ij} u^N_{x_i} \psi^k_{x_j} + \sum_{i=1}^{n} c_i u^N \psi^k + c_0 u^N \psi^k
\]
\[
+ \sum_{i=1}^{n} \alpha_i |u^N|^{p-2} u^N \psi^k_{x_i} + g |u^N|^{p-2} u^N \psi^k - f_{i0} \psi^k - \sum_{i=1}^{n} f^0_i \psi^k_{x_i} \right] \, dx
\]
\[
+ \frac{1}{\varepsilon} \langle B(u^N \psi^k), \psi^k \rangle = 0, \quad t \in [t_0, T],
\]
\[
c_k^N(t_0) = 0, \quad k = 1, 2, \ldots, N.
\]
Here, \( \varepsilon > 0 \), \( B \) is the penalty operator [16, p. 384], \( B(u) = J(u - P_ku) \), \( J \) is the operator of duality between the spaces \( H^1(\Omega) \) and \( (H^1(\Omega))^* \), \( P \) is the operator of projection on the set \( K \), and
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\[ f_i^{(0)}(x, t) = \begin{cases} f_i(x, t), & \text{if } (x, t) \in Q_{0,T}; \\ 0, & \text{if } (x, t) \in Q_{0_0}. \end{cases} \]

The existence of a solution of problem (5), (6) stems from the following \textit{a priori} estimates. Continue the functions \(c_k^N(t)\) by zero to the interval \((-\infty, t_0]\) and perform the substitution \(u^N(x, t) = v^N(x, t)e^{-\gamma t}\) in system (5). Then \(u_i^N(x, t) = v_i^N(x, t)e^{-\gamma t} - \gamma v_i^N(x, t)e^{-\gamma t}\), and problem (5), (6) acquires the form

\[
\int_{Q_T} [v_i^N \phi_k^N + \sum_{i,j=1}^n b_{ij} v_{x_i,t}^N \phi_{x_j}^k + \sum_{i,j=1}^n (a_{ij} - \gamma b_{ij}) v_{x_i}^N v_{x_j}^N] \\
+ \sum_{i=1}^n c_i v_i^N \phi_k^N + (c_0 - \gamma) v_i^N \phi_k^N + e^{-\gamma(p-2)t} \sum_{i=1}^n \alpha_i |v_{x_i}^N|^p v_i^N \phi_{x_i}^k \\
+ e^{-\gamma(p-2)t} g |v_i^N|^{p-2} v_i^N \phi_k^N \left( f_i^{(0)} \phi_k^N + \sum_{i=1}^n f_i^{(0)} \phi_{x_i}^k \right) e^{\gamma t} \] dx + \frac{1}{\varepsilon} \langle B(v^N), \phi_k \rangle = 0, \quad (7)
\]

\(v_i^N(t_0) = 0\). \quad (8)

Multiplying each equation of system (7), respectively, by the function \(c_k^N(t)e^{\gamma t}\), summing them over the index \(k\) from 1 to \(N\), and integrating over the segment \([t_1, \tau] \subset [t_0, T]\), we obtain

\[
\int_{Q_{t_1, \tau}} [v_i^N v_i^N + \sum_{i,j=1}^n b_{ij} v_{x_i,t}^N v_{x_j}^N + \sum_{i,j=1}^n (a_{ij} - \gamma b_{ij}) v_{x_i}^N v_{x_j}^N] \\
+ \sum_{i=1}^n c_i v_i^N v_i^N + (c_0 - \gamma)(v_i^N)^2 + e^{-\gamma(p-2)t} \left( \sum_{i=1}^n \alpha_i |v_{x_i}^N|^p + g |v_i^N|^p \right) \\
- \left( f_i^{(0)} + \sum_{i=1}^n f_i^{(0)} v_{x_i}^N \right) e^{\gamma t} \right] dx dt + \frac{1}{\varepsilon} \langle B(v^N), v_i^N \rangle dt = 0. \quad (9)
\]

On the basis of the conditions of the theorem, one easily obtains the following estimates from equality (9):

\[
\int_{Q_{t_1}} \left( |v_i^N|^2 + \sum_{i=1}^n |v_{x_i}^N|^2 \right) dx \leq \mu_1 F_{0, \gamma}, \quad (10)
\]

\[
\int_{Q_{t_1, \tau}} \left( |v_i^N|^2 + \sum_{i=1}^n |v_{x_i}^N|^2 \right) dx dt \leq \mu_1 F_{0, \gamma}, \quad (11)
\]

\[
\int_{Q_{t_1, \tau}} e^{-\gamma(p-2)t} \left( |v_i^N|^p + \sum_{i=1}^n |v_{x_i}^N|^{p-2} \right) dx dt \leq \mu_1 F_{0, \gamma}. \quad (12)
\]
\[
\int_{t_1}^{T} \langle B(v^N), v^N \rangle dt \leq \mu_1 \varepsilon F_{0, \gamma},
\]

where \( \tau \in [t_1, T] \), the constant \( \mu_1 \) is independent of \( \varepsilon, n, \) and \( t_1 \), and

\[
F_{0, \gamma} = \int \sum_{Q_\tau} |f_i(x, t)|^2 e^{2 \gamma t} dx dt.
\]

We differentiate equation (7) with respect to the variable \( t \), multiply each equation, respectively, by the function \( (c^N_k(t) + \gamma c^N_k(t)) e^{\gamma t} \), then sum them over \( k \) from 1 to \( N \) and integrate over the segment \([t_1, \tau]\). As a result, we obtain the equality

\[
\int_{Q_{t_1, \tau}} \left[ v^N_{w,t} v^N_t + \sum_{i,j=1}^{n} b_{ij} v^N_{x_i,t} v^N_{x_j,t} + \sum_{i,j=1}^{n} (a_{ij} - \gamma b_{ij} + b_{ij,t}) v^N_{x_i,t} v^N_{x_j,t} \right]
\]

\[
+ \sum_{i=1}^{n} c_i v^N_{x_i,t} v^N_t + (c_0 - \gamma)(v^N_t)^2 \right] dx dt
\]

\[
+ (p - 1) \int_{Q_{t_1, \tau}} e^{-\gamma(p-2)t} \left[ \sum_{i=1}^{n} \alpha_i |v^N_{x_i}|^{p-2} (v^N_{x_i,t})^2 + g |v^N|^{p-2} (v^N_t)^2 \right] dx dt
\]

\[
- \gamma (p-2) \int_{Q_{t_1, \tau}} e^{-\gamma(p-2)t} \left[ \sum_{i=1}^{n} \alpha_i |v^N_{x_i}|^{p-2} v^N_{x_i,t} v^N_t + g |v^N|^{p-2} v^N_t v^N_t \right] dx dt
\]

\[
+ \int_{Q_{t_1, \tau}} \left[ \sum_{i,j=1}^{n} (a_{ij,t} - \gamma b_{ij,t}) v^N_{x_i,t} v^N_{x_j,t} + \sum_{i=1}^{n} c_i v^N_{x_i,t} v^N_t + c_0 v^N_t v^N_t \right]
\]

\[
- \left( f^0_{0,t} + \gamma f^0_{0,t} \right) v^N_t + \sum_{i=1}^{n} \left( f^0_{i,t} + \gamma f^0_{i,t} \right) v^N_{x_i,t} \right) e^{\gamma t} \right] dx dt + \frac{1}{\varepsilon} \int_{t_1}^{T} \langle B_t(v^N), v^N_t \rangle dt = 0.
\]

(14)

Taking into account the inequality \( \langle B_t(v^N), v^N_t \rangle \geq 0 \) [16, p. 413], conditions of the theorem, and estimates (10)–(12), we easily obtain the inequality

\[
\int_{\Omega_t} \left[ |v^N_t|^2 + \sum_{i=1}^{n} |v^N_{x_i,t}|^2 \right] dx + \int_{Q_{t_1, \tau}} \left[ \left( a_0 - \gamma b_0 + \frac{1}{2} b_1 - \frac{1}{2} \delta_0 c^1 - \delta_1 \right) \sum_{i=1}^{n} |v^N_{x_i,t}|^2 + \right.
\]

\[
+ \left. \left( c_0 - \gamma - \frac{1}{2\delta_0} - \delta_2 \right) |v^N_t|^2 \right] dx dt
\]

\[
\leq \mu_2 e^{-\gamma(p-2)t} \int_{\Omega_t} \left( |v^N|^p + \sum_{i=1}^{n} |v^N_{x_i}|^p \right) dx + \mu_1 F_{0, \gamma} + \mu_2 F_{1, \gamma}
\]

(15)
from equality (14). In (15), the constant $\mu_2$ is independent of $\varepsilon$, $N$, and $t_1$, $\delta_0 > 0$, $\delta_1 > 0$, $\delta_2 > 0$, and

$$F_{\gamma,t} = \int_\Omega \sum_{i=1}^n |f_{i,t}(x,t)|^2 e^{2\gamma t} \, dx \, dt.$$ 

On the basis of estimate (12) and the Fatou lemma, we have

$$\int_{t_1}^T \liminf_{t \to t_1} \|v^N\|_{W^{1,p}(\Omega)}^p e^{-\gamma(t-2)t} \, dt \leq \mu_1 F_{0,\gamma}.$$ 

Hence, 

$$e^{-\gamma(t-2)t} \liminf_{t \to t_1} \|v^N\|_{W^{1,p}(\Omega)}^p < \infty$$

for almost all $t \in [t_1, T]$. Then there exists $\bar{t} \in [T-1, T]$ such that the specified lower boundary is finite for $\tau = \bar{t}$. By replacing $\tau$ by $\bar{t}$ and passing, if necessary, to a subsequence, we can consider that

$$e^{-\gamma(t-2)\bar{t}} \|v^N(\bar{t})\|_{W^{1,p}(\Omega)}^p < \mu_3,$$  

where the constant $\mu_3$ is independent of $\varepsilon$, $N$, and $t_1$.

On the basis of the conditions of the theorem, one can choose the numbers $\delta_0$, $\delta_1$, and $\delta_2$ such that inequality (15) will imply [in view of estimate (16)] the estimate

$$\int_{Q_{0,t}} \left( |v^N_t|^2 + \sum_{i=1}^n |v_{x_i,t}^N|^2 \right) dx \, dt \leq \mu_4 (F_{0,\gamma} + F_{1,\gamma}),$$

where the constant $\mu_4$ is independent of $\varepsilon$, $N$, and $t_1$.

Taking into account estimates (10)-(12), (17) and the monotonicity of the operators $A_0$, $B$, where the operator $A_0$ is defined by the formula

$$\langle A_0 w^1, w^2 \rangle = \int_\Omega \left( \sum_{i=1}^n \alpha_i |w_{x_i}^1|^p - p^{-2} w_{x_i}^1 w_{x_i}^2 + g |w^1|^p - p^{-2} w^1 w^2 \right) dx,$$

$$w^1, w^2 \in W^{1,p}(\Omega),$$

we can assert the existence of a boundary point $v^{\nu_0}(x,t)$ of the sequence $\{v^N(x,t)\}$ which satisfies the equality

$$\begin{align*}
\int_{Q_{0,t}} &\left[ v^{\nu_0} w + \sum_{i,j=1}^n b_{ij} v^{\nu_0}_{x_i,w} w_{x_j} + \sum_{i,j=1}^n (a_{ij} - \gamma b_{ij}) v^{\nu_0}_{x_i,w} w_{x_j} + \sum_{i=1}^n c_i v^{\nu_0}_{x_i,w} + (c_0 - \gamma) v^{\nu_0} w \right] dx \, dt \\
&+ \int_{t_1}^T \left( e^{-\gamma(t-2)t} \langle A_0 v^{\nu_0}, w \rangle + \frac{1}{\varepsilon} \langle B(v^{\nu_0}), w \rangle \right) dt \\
&= \int_{Q_{0,t}} \left( f_0 w + \sum_{i=1}^n f_i w_{x_i} \right) e^{2\gamma t} dx \, dt \end{align*}$$

(18)
for an arbitrary function \( w \in L^p_{\text{loc}}((-\infty, \tau]; W^{1,p}(\Omega)) \), where \( t_1 \) is an arbitrary number from \([t_0, \tau]\).

Moreover, estimates (10)–(12) and (17) are valid for the function \( v^\tau_0(x, t) \). If one successively sets \( t_0 = \tau - 1 \), \( t_0 = \tau - 2 \), \( t_0 = \tau - k \), \( \ldots \), then one obtains a new sequence of functions \( \{v^k(x, t)\} \), each of which is a solution of Eq. (18) and satisfies estimates (10)–(12), (17). Therefore, the given sequence also has a boundary point \( v^k(x, t) \) that satisfies Eq. (18) and estimates (10)–(12), (17). Thus, one can select a subsequence \( \{v^k(x, t)\} \subset \{v^e(x, t)\} \) such that

\[
v^e_k \to v \quad \text{weakly in} \quad L^p((t_1, t_2), W^{1,p}(\Omega)),
\]

\[
v^e_k \to v \quad \text{weakly in} \quad L^2((t_1, t_2), H^1(\Omega)),
\]

\[
v^e_k \to v_t \quad \text{weakly in} \quad L^2((t_1, t_2), H^1(\Omega)),
\]

\[
v^e_k \to v \quad \text{uniformly in} \quad C([t_1, t_2], H^1(\Omega))
\]
as \( e \to 0 \) for arbitrary \( t_1, t_2 \in (-\infty, \tau], t_1 < t_2 \).

Taking into account the monotonicity of the operators \( A_0, \mathcal{B} \), we have

\[
A_0 v^k \to A_0 v \quad \text{weakly in} \quad L^p((t_1, t_2), (W^{1,p}(\Omega))^*),
\]

\[
\mathcal{B}(v^e_k) \to \mathcal{B}(v) \quad \text{weakly in} \quad L^2((t_1, t_2), (H^1(\Omega))^*)
\]
as \( e \to 0 \). By using equality (18), which is satisfied by the functions \( v^k(x, t) \) for \( \tau = t_2 \), we obtain

\[
\mathcal{B}(v^e_k) \to 0 \quad \text{weakly in} \quad L^2((t_1, t_2), (H^1(\Omega))^*)
\]

Hence, \( \mathcal{B}(v) = 0 \), i.e., \( v \in K \) for almost all \( t \in (-\infty, \tau] \). Now consider equality (18) for the functions \( v^k(x, t) \) and \( \tau = t_2 \) by setting \( w = (z - u^k)e^{-\gamma t}, u^k = v^k e^{-\gamma t} \), where \( t \in K \) for almost all \( t \in (-\infty, \tau] \), \( z \in H^1_{\text{loc}}((-\infty, \tau]; H^1(\Omega)) \cup L^p_{\text{loc}}((-\infty, \tau]; W^{1,p}(\Omega)) \):

\[
\int_{Q_{t_1, t_2}} \left[ u^k(z - u^k) + \sum_{i,j=1}^n b_{ij} u^k_{x_i_j}(z_{x_j} - u^k_{x_j}) + \sum_{i,j=1}^n a_{ij} u^k_{x_i}(z_{x_j} - u^k_{x_j}) \right. \\
+ \sum_{i=1}^n c_i u^k_{x_i}(z - u) + c_0 u^k(z - u^k) \\
+ g |u^k|^{p-2} u^k(z - u^k) - f_0(z - u^k) - \sum_{i=1}^n f_i(z_{x_i} - u^k_{x_i}) \left. \right] \, dx \, dt \\
= \frac{1}{2} \int_{t_1}^{t_2} \langle B(z e^{-\gamma t}) - B(u^k e^{-\gamma t}), z - u^k \rangle \, dt \geq 0.
\]
After elementary transformations of the integral

\[
\int_{\Omega_{t_1,t_2}} \left[ u_k^i(z-u_k) + \sum_{i,j=1}^{n} b_{ij,t} u_{x_i,t}^k (z_{x_j} - u_{x_j}^k) \right] dx dt,
\]

inequality (19) takes the form

\[
\int_{\Omega_{t_1,t_2}} \left[ z^i(z-u_k^i) + \frac{1}{2} \sum_{i,j=1}^{n} b_{ij} z_{x_i,t}^k (z_{x_j}^k - u_{x_j}^k) (z_{x_j}^k - u_{x_j}^k) \right.
\]

\[
+ \sum_{i,j=1}^{n} b_{ij} z_{x_i,t}^k (z_{x_j}^k - u_{x_j}^k) + \sum_{i,j=1}^{n} a_{ij} u_{x_i}^k (z_{x_j}^k - u_{x_j}^k)
\]

\[
+ \sum_{i=1}^{n} \alpha_i |u_{x_i}^k|^p - 2 u_{x_i}^k (z_{x_i}^k - u_{x_i}^k) + \sum_{i=1}^{n} c_i u_{x_i}^k (z - u_{x_i}^k) + c_0 u^k (z - u^k)
\]

\[
+ g |u^k|^p - 2 u^k (z - u^k) - f_0 (z - u^k) - \sum_{i=1}^{n} f_i (z_{x_i} - u_{x_i}^k) \right] dx dt
\]

\[
\geq \frac{1}{2} \int_{\Omega_{t_1,t_2}} \left[ \sum_{i,j=1}^{n} b_{ij} (z_{x_i} - u_{x_i}^k) (z_{x_j} - u_{x_j}^k) + |z - u^k|^2 \right] dx
\]

\[- \frac{1}{2} \int_{\Omega_{t_1,t_2}} \left[ \sum_{i,j=1}^{n} b_{ij} (z_{x_i} - u_{x_i}^k) (z_{x_j} - u_{x_j}^k) + |z - u^k|^2 \right] dx. \quad (20)
\]

By setting \( z = u \) in (20), we obtain strong convergence of the sequence \( \{ u^k(x,t) \} \) to the function \( u(x,t) \) in the space \( W^{1,p}(\Omega) \). Therefore, one can pass to the limit in inequality (20) as \( k \to \infty \). In this case, we obtain inequality (1), i.e., the function \( u(x,t) \) is the required one. Theorem 2 is proved.

REFERENCES