Topological aspects in magnetic field dynamics

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Twisted magnetic fields

Twisted fields are more likely to erupt (Canfield et al. 1999).

Twist increases the stability of magnetic fields in tokamaks.
Magnetic helicity

\[ H_M = \int_V \mathbf{A} \cdot \mathbf{B} \, dV = 2n\phi_1\phi_2 \]

\[ \phi_i = \int_{S_i} \mathbf{B} \cdot d\mathbf{S} \]

Realizability condition:

\[ E_m(k) \geq k |H(k)|/2\mu_0 \]

Magnetic energy is bound from below by magnetic helicity.

Magnetic helicity conservation

\[ \text{Re}_M \to \infty \]

\[ \frac{dH_M}{dt} = 0 \]
Stability criteria

Ideal MHD: \( \mu = 0 \)

Induction equation: \( \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) \)

Woltjer (1958):
\[
\frac{\partial}{\partial t} \int_{V} \mathbf{A} \cdot \mathbf{B} \, dV = 0
\]

Taylor (1974):
\[
\frac{\partial}{\partial t} \int_{\tilde{V}} \mathbf{A} \cdot \mathbf{B} \, dV = 0
\]

Conditions:
- **Constraint**
  \[ \nabla \times \mathbf{B} = \alpha \mathbf{B} \]
- **Equilibrium**
  \[ \nabla \times \mathbf{B} = \alpha(a, b) \mathbf{B} \]
  constant along field line

\( V \) total volume \( \tilde{V} \) volume along magnetic field line
Creation of magnetic field and magnetic helicity

Mean-field decomposition: \( \mathbf{B} = \overline{\mathbf{B}} + \mathbf{b} \)

Induction equation:
\[
\partial_t \overline{\mathbf{B}} = \eta \nabla^2 \overline{\mathbf{B}} + \nabla \times (\overline{\mathbf{U}} \times \overline{\mathbf{B}} + \overline{\mathbf{E}})
\]

Electromotive force:
\[
\overline{\mathbf{E}} = \mathbf{u} \times \mathbf{b} = \alpha \overline{\mathbf{B}} - \eta_t \nabla \times \overline{\mathbf{B}}
\]

\(\alpha\) effect:
\[
\alpha = \alpha_K + \alpha_M = -\tau \mathbf{\omega} \cdot \mathbf{u} / 3 + \mathbf{j} \cdot \mathbf{b} / (3\rho)
\]

Inverse cascade:
Large- and small-scale magnetic helicity of opposite sign is created.

Leorat et al., 1975
Interlocked flux rings

\[ H_M \neq 0 \]

- Isothermal compressible gas
- Viscous medium
- Periodic boundaries

\[ H_M = 0 \]
Interlocked flux rings

$\tau = 4$

$H_M = 0$

$H_M \neq 0$
Magnetic helicity rather than actual linking determines the field decay.
**N-foil knots**

3-foil  4-foil  5-foil  6-foil  7-foil

![Cinquefoil knot](image)

\[ x(s) = \begin{pmatrix} (C + \sin s n_f) \sin[s(n_f - 1)] \\ (C + \sin s n_f) \cos[s(n_f - 1)] \\ D \cos s n_f \end{pmatrix} \]

* from Wikipedia, author: Jim.belk
Magnetic helicity is approximately conserved.

Self-linking is transformed into twisting after reconnection.
Slower decay for higher $n_f$. 

N-foil knots
N-foil knots

$H_M = 2n\phi_1\phi_2$

$n_{app} = H_M / 2\phi^2$

$H_M = (n_f - 2)n_f\phi^2 / 2$
N-foil knots

\[ \frac{2M(k)}{|H(k)|k} \]

Realizability condition more important for high \( n_f \).
IUCAA knot and Borromean rings

\[ H_M = 0 \]

- Is magnetic helicity sufficient?
- Higher order invariants?
Reconnection characteristics

3 rings

Twisted ring + interlocked rings

2 twisted rings
Reconnection characteristics

Conversion of linking into twisting

Ruzmaikin and Akhmetiev (1994)
Magnetic energy decay

\[ \frac{\langle B^2 \rangle}{\langle B_0^2 \rangle} \]

- \( t^{-1/2} \)
- \( t^{-1} \)
- \( t^{-3/2} \)

- IUCAA knot
- Borromean rings
- Helical triple rings
- Non-helical triple rings

\[ t/t_{\text{res}} \]
Fixed point index

mapping: \((x, y) \rightarrow F_z(x, y)\)

Fixed points: \(F_1(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}\)

Color coding:  

Fixed point index:  
\[ T = \sum_i t_i \quad t_i = \pm 1 \]

Yeates et al. 2011
Magnetic braid configurations

AAA (trefoil knot)       AABB (Borromean rings)
Field line tracing

Generalized flux function:

\[ A(x, y) = \int_{z=0}^{z=1} A \cdot dl \]

Reconnection rate:

\[ \sum_i \frac{dA(x_i)}{dt} \]
Conclusions

- Topology *can* constrain field decay.
- Stronger packing for high $n_f$ leads to different decay slopes.
- Higher order invariants?
- Isolated helical structures inhibit energy decay.
- Reconsider realizability condition.

- Apply fixed point method to knots (braids).
- Monitor the reconnection rate.
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Magnetic energy decay

\[ \frac{\langle B^2 \rangle}{\langle B_0^2 \rangle} \]

Graph showing the decay of magnetic energy with time, \( \tau \), for different states labeled as \( n=0 \) and \( \text{or} \) \( n=2 \). The graph includes lines labeled \( t^{-1/2} \) and \( t^{-3/2} \) to indicate different decay rates.
Simulations

- $256^3$ mesh point
- Isothermal compressible gas
- Viscous medium
- Periodic boundaries

\[
\frac{\partial A}{\partial t} = U \times B + \eta \nabla^2 A
\]

\[
\frac{D U}{D t} = -c_s^2 \nabla \ln \rho + J \times B / \rho + F_{\text{visc}}
\]

\[
\frac{D \ln \rho}{D t} = -\nabla \cdot U
\]
Linking number

Sign of the crossings for the 4-foil knot

Number of crossings increases like $n_f^2$

$$H_M \propto n_{\text{linking}}$$

$$H_M \propto n_f^2$$

$$n_{\text{linking}} = (n_+ - n_-)/2$$
Helicity vs. energy

\[ E_M \propto l_{\text{knot}} \propto n_f \]
\[ H_M \propto n_f^2 \]

Knot is more strongly packed with increasing \( n_f \).

Magnetic energy is closer to its lower limit for high \( n_f \).