Life cycles...

...result from moving and interacting cells!
Migration

Cell migration types:
- active (e.g. myxobacteria), passive (e.g. blood cells)
- collective (e.g. swarming)
- directed (e.g. chemotaxis)
- undirected (esp. diffusion → Brownian motion)

**Goal:** Mathematical model for diffusive motion of individuals (e.g. cells, molecules)
Cells and random walk

Endodermal cells disperse with a random walk movement during early gastrulation

Mean-square displacement

Endodermal cells disperse with a random walk movement during early gastrulation

$$\text{MSD} = \langle |x(t) - x(0)|^2 \rangle$$
Cellular automata – diffusion

Andreas Deutsch

January 5, 2010
Overview

LGCA model of random walk

From microscopic rules to macroscopic equations

Stability analysis
Transport in elementary CA

Rule 184:

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**simple traffic model:** all cars **travel into the same direction** (here to the right). Positions are updated synchronously. During the motion, each car can be at rest or jump to the nearest neighbor site, along the direction of motion. The rule: a car moves only if its destination cell is empty (feature: congestion).
Diffusion in CA

- one particle: stochastic synchronous CA
- many particles: problem of overcrowding
  solution: asynchronous update (dimer automata), heterogeneous rules (e.g. Toffoli rules)

**Solution with synchronous, homogeneous rules:** LGCA
We model random walk as random rotation of the particle orientations:

\[ \eta_i^O(r, k) = \sum_{l=0}^{b-1} \mu_l(r, k) [\eta_{i+1}(r, k)] . \]

\[ \mu_l(r, k) \in \{0, 1\}: \text{Boolean random variables which obey} \]

\[ \sum_{l=0}^{b-1} \mu_l = 1. \]

\[ <\mu_l> = p_l = 1/b, l = 0, \ldots b - 1. \]
Microdynamical difference equation

- **Rotation dynamics:**

\[ \eta_i^O(r, k) = \sum_{l=0}^{b-1} \mu_l(r, k) [\eta_{i+1}(r, k)] . \]

- **Rotation and propagation:**

\[ \eta_i(r + mc_i, k + 1) - \eta_i(r, k) = \left\{ \sum_{l=0}^{b-1} \mu_l(r, k) [\eta_{i+1}(r, k)] \right\} - \eta_i(r, k) = C_i(\eta(r, k)) . \]
Simulations

Random walk of single particle:
Mean-square displacement

Mean displacement:

\[ < X_k > = < \sum_{\tau=0}^{k-1} \Delta x(\tau) > = 0 \]

Mean square displacement:

\[ < X_k^2 > = \sum_{\tau_1=0}^{k-1} \sum_{\tau_2=0}^{k-1} < \Delta x(\tau_1) > < \Delta x(\tau_2) > = k < v^2(0) > , \text{ where} \]

\[ \Delta x(k) = x(k) - x(0) = \sum_{\tau=0}^{k-1} v(\tau) , \quad v(\tau) = \sum_{r,i} mc_i \eta_i(r, \tau) \]

Hypothesis: diffusive behaviour
From microscopic rules to macroscopic equations

- LGCA $\rightarrow$ finite-difference equation
- Scaling
- $\rightarrow$ Taylor approximation
- $\rightarrow$ Partial differential equation
Define $f_i(r, k) = E(\eta_i(r, k))$

(Linear finite-difference) lattice-Boltzmann equation can be derived as

$$f_i(r + mc_i, k + 1) - f_i(r, k) = \frac{1}{b} \left( \sum_{l=1}^{b} f_i(r) \right) - f_i(r)$$

for $i = 1, \ldots, b$. 
Assumptions

- **Scaling:**
  a. **time:** \( k \) (discrete) \( \rightarrow \) \( t \in \mathbb{R}^+ \) (cont.),
  timestep: \( \tau \) (discrete) \( \rightarrow \) \( g(\epsilon)\tau \in \mathbb{R}^+_0 \) (cont.), \( \tau \in \mathbb{R}^+ \)
  b. **space:** \( r \) (discrete) \( \rightarrow \) \( x \in \mathbb{R} \) (cont.),
  lattice spacing: \( \Delta x \) (discrete) \( \rightarrow \) \( h(\epsilon)\Delta x \in \mathbb{R}^+_0 \) (cont.),
  \( \Delta x \in \mathbb{R}^+ \)

- \( f_i(x, t) \) cont., differ. fct., i.e. all deriv. in \( x \) and \( t \) exist.

Then:

\[
\begin{align*}
f_i(x + mc_i\Delta x h(\epsilon), t + \tau g(\epsilon)) - f_i(x, t) &= \frac{1}{2} \left( \sum_{l=1}^{2} f_i(x, t) \right) - f_i(x, t)
\end{align*}
\]
Taylor expansion

\[ f_i(x + mc_i \Delta x h(\epsilon), t + \tau g(\epsilon)) = f_i(x, t) + (mc_i) \Delta x h(\epsilon) \delta_x f_i(x, t) + \tau g(\epsilon) \delta_t f_i(x, t) \]

\[ + \frac{1}{2} (mc_i)^2 \Delta x^2 (h(\epsilon))^2 \delta_{xx} f_i(x, t) + \frac{1}{2} \tau^2 (g(\epsilon))^2 \delta_{tt} f_i(x, t) \]

\[ + \frac{1}{2} (mc_i) \Delta x h(\epsilon) \tau g(\epsilon) \delta_{xt} f_i + O(\epsilon^3) \]
Grouping terms...

\[ m \Delta x \delta_x h(\epsilon)f_1 + \tau g(\epsilon)\delta_t f_1 + \frac{1}{2} m^2 \Delta x^2 g(\epsilon)^2 \delta_{xx} f_1 = \frac{1}{2}(f_2 - f_1) \]  
(7)

\[-m \Delta x h(\epsilon)\delta_x f_2 + \tau g(\epsilon)\delta_t f_2 + \frac{1}{2} m^2 \Delta x^2 g(\epsilon)^2 \delta_{xx} f_2 = \frac{1}{2}(f_1 - f_2) \]  
(8)

\[ J(x, t) = f_1(x, t) - f_2(x, t), \rho(x, t) = f_1(x, t) + f_2(x, t) \]

\[ m \Delta x h(\epsilon)\delta_x J + \tau g(\epsilon)\delta_t \rho + \frac{1}{2} m^2 \Delta x^2 g(\epsilon)^2 \delta_{xx} \rho = 0 \]  
(10)

\[ m \Delta x h(\epsilon)\delta_x \rho + \tau g(\epsilon)\delta_t J + \frac{1}{2} m^2 \Delta x^2 g(\epsilon)^2 \delta_{xx} J = -J \]  
(11)
Diffusion equation

\[ m \Delta x h(\epsilon) \delta_x J + \tau g(\epsilon) \delta_t \rho + \frac{1}{2} m^2 \Delta x^2 g(\epsilon)^2 \delta_{xx} \rho = 0 \]

(13)

- **diffusive scaling:** \( g(\epsilon) = \epsilon^2 \), \( h(\epsilon) = \epsilon \) (since \( \frac{h^2(\epsilon) \Delta x^2}{g(\epsilon) \tau} = \text{const} \))

\[ \rho_{tt} + \frac{1}{\tau g(\epsilon)^2} \rho_t = \frac{1}{2} \frac{(m \Delta x)^2}{\tau^2} \frac{\epsilon^2}{\epsilon^4} \rho_{xx} \]

- \( \rho_t = \frac{1}{2} \frac{(m \Delta x)^2}{\tau} \rho_{xx} - \tau \epsilon^2 \rho_{tt} \)

- \( (\epsilon \to 0) \rightarrow \rho_t = D \rho_{xx} \) (Diffusion equation)

**diffusion coefficient** \( D = \frac{1}{2} \frac{(m \Delta x)^2}{\tau} \)
Remarks

Derivations of the **diffusion equation** from

1. Fick's law \( J(x, t) = -D \frac{\delta c(x, t)}{\delta x} \) and balance equation \( \frac{\delta c(x, t)}{\delta t} = -\frac{\delta J(x, t)}{\delta x} \)

2. random walk of one particle

3. LGCA of many particles
Telegrapher’s equation

\[ m \Delta x h(\epsilon) J_{xt} + \tau g(\epsilon) \rho_{tt} = 0 \]
\[ m \Delta x h(\epsilon) \rho_{xx} + \tau g(\epsilon) J_{tx} = - J_x \]

\[ \rho_{tt} + \frac{1}{\tau g(\epsilon)} \rho_t = \frac{(m \Delta x)^2}{\tau^2} \left( \frac{h(\epsilon))}{g(\epsilon))} \right)^2 \rho_{xx} \]

**hyperbolic scaling**: \( g(\epsilon) = h(\epsilon) = \epsilon \):

**telegraph equation**: \[ \rho_{tt} + \frac{1}{\tau \epsilon} \rho_t = \frac{(m \Delta x)^2}{\tau^2} \rho_{xx} \]

with \( s := \frac{m \Delta x}{\tau} \) particle speed, \( \frac{1}{\epsilon \tau} \) reversal rate

**property**: signals can not propagate faster than \( s \), signal is slowed down by the term \( \frac{1}{\epsilon \tau} \rho_t \) (hyperbolic equation)
Telegrapher’s equation and mean-square displacement

- mean square displacement: \( < x^2 > = \int_R x^2 \rho(x, t) dx \)

\[
\begin{align*}
\frac{d^2 < x^2 >}{dt^2} + 2\lambda \frac{d < x^2 >}{dt} &= 2D\lambda \\
\rightarrow < x^2 > &= D\left\{t - \frac{1}{2\lambda}(1 - \exp(-2\lambda t))\right\} \\
\lambda t &= \frac{t}{\tau} \ll 1 : < x^2 > \approx \lambda Dt^2 \text{ (wave equation)} \\
t \rightarrow \infty : < x^2 > \approx Dt \\
\text{(diffusive process with diffusion coefficient } D = \frac{m^2}{\tau b})
\end{align*}
\]
Wave equation

- Neglect shuffling, hyperbolic scaling \( g(\epsilon) = h(\epsilon) = \epsilon \):

\[
\begin{align*}
    m\Delta x \epsilon J_{xt} + \tau \epsilon \rho_{tt} &= 0 \\
    m\Delta x \epsilon \rho_{xx} + \tau \epsilon J_{tx} &= 0
\end{align*}
\] (14) (15) (16)

- \( \rho_{tt} = \frac{(m\Delta x)^2}{\tau^2} \rho_{xx} \)

with \( s := \frac{m\Delta x}{\tau} \) particle speed

- property: signals propagate with speed \( s \) (hyperbolic equation)
Plan:

1. Discrete Fourier transformation
2. Eigenvalues of Boltzmann propagator
Lattice Boltzmann equations:

\[
f_1(r + m, k) - f_1(r, k) = \frac{1}{2} (f_2(r, k) - f_1(r, k))
\]

\[
f_2(r - m, k) - f_2(r, k) = \frac{1}{2} (f_1(r, k) - f_2(r, k))
\]

for \( i = 1, \ldots, 2 \), lattice \( \mathcal{L}, |\mathcal{L}| = L \)

Discrete Fourier mode decomposition (for \( L \)-)periodic functions \( f_i(r, k) \):

\[
f_i(r, k) = \frac{1}{L} \sum_{q=0}^{L-1} \exp\left(-\frac{2\pi}{L} qr\right) F_i(q, k)
\]

The Fourier-transformed system becomes...
LGCA model of random walk
From microscopic rules to macroscopic equations
Stability analysis

\[
\sum_q \exp\left(-\frac{2\pi i}{L} q(r + m)\right) F_1(q, k + 1) - \sum_q \exp\left(-\frac{2\pi i}{L} qr\right) F_1(q, k)
\]  

(17)

\[
= \frac{1}{2} \left( \sum_q \exp\left(-\frac{2\pi i}{L} qr\right) F_2(q, k) - \sum_q \exp\left(-\frac{2\pi i}{L} qr\right) F_2(q, k) \right)
\]  

(18)

\[
\sum_q \exp\left(-\frac{2\pi i}{L} q(r - m)\right) F_2(q, k + 1) - \sum_q \exp\left(-\frac{2\pi i}{L} qr\right) F_2(q, k)
\]  

(19)

\[
= \frac{1}{2} \left( \sum_q \exp\left(-\frac{2\pi i}{L} qr\right) F_1(q, k) - \sum_q \exp\left(-\frac{2\pi i}{L} qr\right) F_2(q, k) \right)
\]  

(20)
The vectors \( \{ \exp(-\frac{2\pi i}{L} qr), q = 0, \ldots, L - 1 \} \) form an orthogonal basis over the set of \( L \)-dimensional complex vectors since:

\[
\sum_{q=0}^{L-1} \left( \exp\left(\frac{2\pi i}{L} qr\right) \right) \left( \exp\left(-\frac{2\pi i}{L} q'r\right) \right) = L \cdot \delta_{qq'}, \tag{21}
\]

where \( \delta_{qq'} \) is the Kronecker delta.

Therefore:

\[
\exp\left(-\frac{2\pi i}{L} qm\right) F_1(q, k + 1) - F_1(q, k) = \frac{1}{2} (F_2(q, k) - F_1(q, k)) \tag{22}
\]

\[
\exp\left(\frac{2\pi i}{L} qm\right) F_2(q, k + 1) - F_2(q, k) = \frac{1}{2} (F_1(q, k) - F_2(q, k))
\]
or:

\[
F_1(q, k + 1) = \exp\left(\frac{2\pi i}{L}qm\right)\frac{1}{2}(F_1(q, k) + F_2(q, k))
\]

\[
F_2(q, k + 1) = \exp\left(-\frac{2\pi i}{L}qm\right)\frac{1}{2}(F_1(q, k) + F_2(q, k))
\]
Boltzmann propagator

Each Fourier mode $F(q, k)$ develops according to

$$F(q, k) = \Gamma(q)^k F(q, 0) \quad \forall k \in \mathbb{N},$$

where the **Boltzmann propagator** is defined as

$$\Gamma(q) = T \{ I + \Omega^0 \}$$

$$= \frac{1}{2} \begin{pmatrix} e^{i \frac{2\pi}{L} q \cdot m} & 0 \\ 0 & e^{-i \frac{2\pi}{L} q \cdot m} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

with $|\mathcal{L}| = L$. 
Spectrum of Boltzmann propagator

\[ \Gamma(q) = \frac{1}{2} \begin{pmatrix} \exp\left(\frac{2\pi i}{L} q m\right) & \exp\left(\frac{2\pi i}{L} q m\right) \\ \exp\left(-\frac{2\pi i}{L} q m\right) & \exp\left(-\frac{2\pi i}{L} q m\right) \end{pmatrix} \]

has a very simple structure such that we can determine the spectrum

\[ \Lambda_{\Gamma(q)} = \{ \lambda_1(q), \lambda_2(q) \}, \quad \det(\Gamma - \lambda I) = 0 \]

as

\[ \lambda_1(q) = \cos\left(\frac{2\pi}{L} q m\right), \]
\[ \lambda_2(q) = 0, \]

Hence, solutions corresponding to the eigenvalue \( \lambda_2(q) = 0 \) decrease to zero in one time step independently of the value of \( q \). Because \( |\lambda_1(q)| \leq 1 \) for all \( q \), no modes can grow with time, i.e. pattern formation is not possible since all inhomogeneities (with \( q \neq 0 \)) are damped out.
Extensions

Models for diffusive transport:

- include rest particles, i.e. $\tilde{b} > 0$
- 2D/3D models
Simulations

biomodeling.info
References


Websites

- mathworld.wolfram.com (CA, LGCA)
- biomodeling.info (LGCA)
CA models for...

- Transport (diffusion)
- **Reaction (growth)**
- Interaction (swarming, adhesion)
- Applications: myxobacteria, tumor growth
THANKS!