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Periodic Solution for a Three-Species Lotka-Volterra Food-Chain Model with Time Delays

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Abstract—A delayed three-species periodic Lotka-Volterra food-chain model without instantaneous negative feedback is investigated. By using Gaines and Mawhin's continuation theorem of coincidence degree theory and by constructing a suitable Lyapunov functional, a set of easily verifiable sufficient conditions are derived for the existence, uniqueness, and global stability of positive periodic solutions of the system. Computer simulations are presented to illustrate the conclusions.

Keywords—Food-chain model, Time delay, Periodic solution, Coincidence degree, Global stability.

1. INTRODUCTION

The classical Lotka-Volterra type predator-prey systems are very important in the models of multi-species population dynamics. There are considerable works on the global dynamics of Lotka-Volterra predator-prey systems (see, for example, [1–3]). It is often observed that populations in the real world tend to fluctuate. This is especially so for predator-prey interactions. There are three typical approaches for modelling such behavior

(i) introducing more species into the model, and considering the higher dimensional systems (like predator-prey interactions, see [4]);
(ii) assuming that the per capita growth function is time dependent and periodic in time;
(iii) taking into account the time delay effect in the population dynamics (see [5,6]).

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In most of the models considered so far, it has been assumed that all biological and environmental parameters are constants in time. However, any biological or environmental parameters are naturally subject to fluctuation in time. The effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumption of periodicity of the parameters is a way of incorporating the periodicity of the environment (such as seasonal effects of weather, food supplies, mating habits and so forth).

On the other hand, time delays of one type or another have been incorporated into biological models by many researchers, we refer to the monographs of [7–10], for general delayed biological systems, and to [11–16], and the references cited therein, for studies on delayed biological systems. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the population to fluctuate. Time delay due to gestation is a common example, because generally the consumption of prey by the predator throughout its past history governs the present birth rate of the predator. Therefore, more realistic models of population interactions should take into account the seasonality of the changing environment and the effects of time delays.

The purpose of this paper is to study the combined effects of periodicity of the environment and delays on the dynamics of Lotka-Volterra predator-prey systems. To do so, we discuss the following delayed three-species periodic Lotka-Volterra type food-chain system without instantaneous negative feedback

$$\begin{aligned}
\dot{x}_1(t) &= x_1(t)\left[r_1(t) - a_{11}(t)\, x_1(t-\tau_{11}) - a_{12}(t)\, x_2(t)\right],\\
\dot{x}_2(t) &= x_2(t)\left[-r_2(t) + a_{21}(t)\, x_1(t-\tau_{21}) - a_{22}(t)\, x_2(t-\tau_{22}) - a_{23}(t)\, x_3(t)\right],\\
\dot{x}_3(t) &= x_3(t)\left[-r_3(t) + a_{32}(t)\, x_2(t-\tau_{32}) - a_{33}(t)\, x_3(t-\tau_{33})\right],
\end{aligned} \tag{1.1}$$

with initial conditions,

$$\begin{gathered}
x_i(\theta) = \phi_i(\theta), \qquad \theta \in [-\tau, 0], \quad \phi_i(0) > 0,\\
\phi_i \in C([-\tau, 0), R_+), \qquad i = 1, 2, 3,
\end{gathered} \tag{1.2}$$

where $x_1(t)$, $x_2(t)$, and $x_3(t)$ denote the densities of prey, predator, and top predator population, respectively. We have assumed in (1.1) that when the predator species x_2 is absent, the prey species x_1 is governed by the well-known delay logistic equation,

$$\frac{dx_1(t)}{dt} = x_1(t)\left(r_1(t) - a_{11}(t)\, x_1(t-\tau_{11})\right),$$

where $\tau_{11} \geq 0$ denotes the delay in the negative feedback of the prey species x_1. τ_{21} and τ_{32} are the delays due to gestation, that is, mature adult predators can only contribute to the reproduction of predator biomass. In addition, we have included the terms $-a_{22}x_2(t-\tau_{22})$ and $-a_{33}x_3(t-\tau_{33})$ in the dynamics of predator x_2 and top predator x_3, respectively, to incorporate the negative feedback of predator crowding.

In this paper, for system (1.1), we always assume that the following holds.

(H1) $r_i(t), a_{ij}(t)$ are continuously positive periodic functions with period ω, $i, j = 1, 2, 3$.

It is well known that by the fundamental theory of functional differential equations [17], system (1.1) has a unique solution $x(t) = (x_1(t), x_2(t), x_3(t))$ satisfying initial conditions (1.2). It is easy to verify that solutions of system (1.1) corresponding to initial conditions (1.2) are defined on $[0, +\infty)$ and remain positive, for all $t \geq 0$. In this paper, the solution of system (1.1) satisfying initial condition (1.2) is said to be positive.

The organization of this paper is as follows. In the next section, by using the powerful and effective coincidence degree theory, a new criterion is obtained for the existence of positive ω-periodic solutions of system (1.1),(1.2). In Section 3, by means of a suitable Lyapunov functional, a set of easily verifiable sufficient conditions are derived for the uniqueness and global stability of the positive periodic solution to the system. Finally, computer simulations are presented to illustrate the conclusions.

2. EXISTENCE OF PERIODIC SOLUTION

The objective of this section is to derive sufficient condition for the existence of positive periodic solutions to system (1.1),(1.2) by using Gaines and Mawhin's continuation theorem of coincidence degree theory. To this end, we first introduce the following notations.

Let X, Y be real Banach spaces, let L: $\operatorname{Dom} L \subset X \to Y$ be a linear mapping, and $N : X \to Y$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero, if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$, and $Q : Y \to Y$, such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, then, the restriction L_P of L to $\operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. Denote the inverse of L_P by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

For convenience of use, we introduce the continuation theorem of coincidence degree theory (see [18, p. 40]) as follows.

LEMMA 2.1. *Let $\Omega \subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Assume*

(a) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;*
(b) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$;*
(c) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then, $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \operatorname{Dom} L$.

In what follows, we shall use the following notations,

$$\bar{f} = \frac{1}{\omega} \int_0^\omega f(t)\, dt, \qquad f^L = \min_{t \in [0,\omega]} f(t), \qquad f^M = \max_{[0,\omega]} f(t),$$

where f is a continuous ω-periodic function.

We are now in a position to state and prove our result on the existence of periodic solutions of system (1.1).

THEOREM 2.1. *In addition to (H1), assume further that the following holds.*

(H2) $\bar{r}_1 a_{21}^L a_{32}^L - \bar{r}_3 a_{11}^M a_{22}^M - \bar{r}_2 a_{11}^M a_{32}^M - \bar{r}_3 a_{12}^M a_{21}^M > 0$.

Then, system (1.1),(1.2) has at least one positive ω-periodic solution.

PROOF. Since solutions of (1.1),(1.2) remain positive, for all $t \geq 0$, we let

$$u_1(t) = \ln[x_1(t)], \qquad u_2(t) = \ln[x_2(t)], \qquad u_3(t) = \ln[x_3(t)], \tag{2.1}$$

and derive that

$$\begin{aligned} \frac{du_1(t)}{dt} &= r_1(t) - a_{11}(t) e^{u_1(t-\tau_{11})} - a_{12}(t) e^{u_2(t)}, \\ \frac{du_2(t)}{dt} &= -r_2(t) + a_{21}(t) e^{u_1(t-\tau_{21})} - a_{22}(t) e^{u_2(t-\tau_{22})} - a_{23}(t) e^{u_3(t)}, \\ \frac{du_3(t)}{dt} &= -r_3(t) + a_{32}(t) e^{u_2(t-\tau_{32})} - a_{33}(t) e^{u_3(t-\tau_{33})}. \end{aligned} \tag{2.2}$$

It is easy to see that, if system (2.2) has one ω-periodic solution $(u_1^*(t), u_2^*(t), u_3^*(t))^\top$, then, $x^*(t) = (x_1^*(t), x_2^*(t), x_3^*(t))^\top = (\exp[u_1^*(t)], \exp[u_2^*(t)], \exp[u_3^*(t)])^\top$ is a positive ω-periodic solution of system (1.1). Therefore, to complete the proof, it suffices to show that system (2.2) has at least one ω-periodic solution.

Take

$$X = Y = \left\{ \left(u_1(t), u_2(t), u_3(t)\right)^\top \in C\left(R, R^3\right) : u_i(t+\omega) = u_i(t),\ i = 1,2,3 \right\}$$

and

$$\left\| \left(u_1(t), u_2(t), u_3(t)\right)^\top \right\| = \sum_{i=1}^{3} \max_{t\in[0,\omega]} |u_i(t)|,$$

here, $|\cdot|$ denotes the Euclidean norm. It is easy to see that X and Y are both Banach spaces. Set

$$L : \operatorname{Dom} L \cap X \to X, \quad L\left(u_1(t), u_2(t), u_3(t)\right)^\top = \left(\frac{du_1(t)}{dt}, \frac{du_2(t)}{dt}, \frac{du_3(t)}{dt}\right)^\top,$$

where $\operatorname{Dom} L = \{(u_1(t), u_2(t), u_3(t))^\top \in C^1(R, R^3)\}$ and $N : X \to X$,

$$N \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} r_1(t) - a_{11}(t) e^{u_1(t-\tau_{11})} - a_{12}(t) e^{u_2(t)} \\ -r_2(t) + a_{21}(t) e^{u_1(t-\tau_{21})} - a_{22}(t) e^{u_2(t-\tau_{22})} - a_{23}(t) e^{u_3(t)} \\ -r_3(t) + a_{32}(t) e^{u_2(t-\tau_{32})} - a_{33}(t) e^{u_3(t-\tau_{33})} \end{bmatrix}$$

Define two projectors P and Q as

$$P \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = Q \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega} \int_0^\omega u_1(t)\, dt \\ \frac{1}{\omega} \int_0^\omega u_2(t)\, dt \\ \frac{1}{\omega} \int_0^\omega u_3(t)\, dt \end{bmatrix}, \qquad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in X = Y.$$

It is clear that

$$\operatorname{Ker} L = \left\{ x \mid x \in X,\ x = h,\ h \in R^3 \right\},$$
$$\operatorname{Im} L = \left\{ y \mid y \in Y,\ \int_0^\omega y(t)\, dt = 0 \right\} \text{ is closed in } Y,$$

and

$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 3.$$

Therefore, L is a Fredholm mapping of index zero. Obviously, P and Q are continuous projectors, such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q).$$

Furthermore, it is easy to prove that the inverse K_P of L_P has the form: $\operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$,

$$K_P(y) = \int_0^t y(s)\, ds - \frac{1}{\omega} \int_0^\omega \int_0^t y(s)\, ds\, dt.$$

Then, $QN : X \to Y$ and $K_P(I-Q)N : X \to X$ read,

$$QNx = \begin{bmatrix} \frac{1}{\omega} \int_0^\omega \left[r_1(t) - a_{11}(t) e^{u_1(t-\tau_{11})} - a_{12}(t) e^{u_2(t)} \right] dt \\ \frac{1}{\omega} \int_0^\omega \left[-r_2(t) + a_{21}(t) e^{u_1(t-\tau_{21})} - a_{22}(t) e^{u_2(t-\tau_{22})} - a_{23}(t) e^{u_3(t)} \right] dt \\ \frac{1}{\omega} \int_0^\omega \left[-r_3(t) + a_{32}(t) e^{u_2(t-\tau_{32})} - a_{33}(t) e^{u_3(t-\tau_{33})} \right] dt \end{bmatrix},$$

$$K_P(I-Q)Nx = \int_0^t Nx(s)\, ds - \frac{1}{\omega} \int_0^\omega \int_0^t Nx(s)\, ds\, dt - \left(\frac{t}{\omega} - \frac{1}{2}\right) \int_0^\omega Nx(s)\, ds.$$

It is easy to see that QN and $K_P(I-Q)N$ are continuous. By using the Arzela-Ascoli theorem, it is not difficult to prove that $\overline{K_P(I-Q)N(\bar{\Omega})}$ is compact, for any open bounded set $\Omega \subset X$. Moreover, $QN(\bar{\Omega})$ is bounded. Therefore, N is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

In order to apply Lemma 2.1, we need to search for an appropriate open, bounded subset Ω.

Corresponding to the operator equation $Lx = \lambda Nx, \lambda \in (0,1)$, we have

$$\begin{aligned} \frac{du_1(t)}{dt} &= \lambda\left[r_1(t) - a_{11}(t)\, e^{u_1(t-\tau_{11})} - a_{12}(t)\, e^{u_2(t)}\right], \\ \frac{du_2(t)}{dt} &= \lambda\left[-r_2(t) + a_{21}(t)\, e^{u_1(t-\tau_{21})} - a_{22}(t)\, e^{u_2(t-\tau_{22})} - a_{23}(t)\, e^{u_3(t)}\right], \\ \frac{du_3(t)}{dt} &= \lambda\left[-r_3(t) + a_{32}(t)\, e^{u_2(t-\tau_{32})} - a_{33}(t)\, e^{u_3(t-\tau_{33})}\right]. \end{aligned} \tag{2.3}$$

Suppose that $(u_1(t), u_2(t), u_3(t))^{\top} \in X$ is a solution of (2.3) for a certain $\lambda \in (0,1)$. Integrating (2.3) over the interval $[0,\omega]$ leads to

$$\int_0^\omega a_{11}(t)\, e^{u_1(t-\tau_{11})}\, dt + \int_0^\omega a_{12}(t)\, e^{u_2(t)}\, dt = \int_0^\omega r_1(t)\, dt, \tag{2.4}$$

$$\int_0^\omega r_2(t)\, dt + \int_0^\omega a_{22}(t)\, e^{u_2(t-\tau_{22})}\, dt + \int_0^\omega a_{23}(t)\, e^{u_3(t)}\, dt = \int_0^\omega a_{21}(t)\, e^{u_1(t-\tau_{21})}\, dt, \tag{2.5}$$

$$\int_0^\omega r_3(t)\, dt + \int_0^\omega a_{33}(t)\, e^{u_3(t-\tau_{33})}\, dt = \int_0^\omega a_{32}(t)\, e^{u_2(t-\tau_{32})} dt. \tag{2.6}$$

It is easy to see that

$$\int_0^\omega a_{ij}(t)\, e^{u_j(t-\tau_{ij})}\, dt = \int_{-\tau_{ij}}^{\omega-\tau_{ij}} a_{ij}(s+\tau_{ij})\, e^{u_j(s)}\, ds.$$

By means of the mean value theorem of integral theory, we see that there exists $\xi_{ij} \in [-\tau_{ij}, \omega-\tau_{ij}]$, such that

$$\begin{aligned} \int_0^\omega a_{ij}(t)\, e^{u_j(t-\tau_{ij})}\, dt &= a_{ij}(\xi_{ij}+\tau_{ij}) \int_{-\tau_{ij}}^{\omega-\tau_{ij}} e^{u_j(s)}\, ds \\ &= a_{ij}(\xi_{ij}+\tau_{ij}) \int_0^\omega e^{u_j(s)}\, ds, \end{aligned} \tag{2.7}$$

in which $(i,j) \in \{(1,1),(2,1),(2,2),(3,2),(3,3)\}$.

Using the mean value theorem of integral theory again directly, we derive

$$\int_0^\omega a_{ij}(t)\, e^{u_j(t)}\, dt = a_{ij}(\xi_{ij}) \int_0^\omega e^{u_j(s)}\, ds, \tag{2.8}$$

in which $\xi_{ij} \in [0,\omega]$, $(i,j) \in \{(1,2),(2,3)\}$.

On substituting (2.7),(2.8) into (2.4)–(2.6), we derive

$$a_{11}(\xi_{11}+\tau_{11}) \int_0^\omega e^{u_1(s)}\, ds + a_{12}(\xi_{12}) \int_0^\omega e^{u_2(s)}\, ds = \bar{r}_1\omega, \tag{2.9}$$

$$a_{21}(\xi_{21}+\tau_{21}) \int_0^\omega e^{u_1(s)}\, ds - a_{22}(\xi_{22}+\tau_{22}) \int_0^\omega e^{u_2(s)}\, ds - a_{23}(\xi_{23}) \int_0^\omega e^{u_3(s)}\, ds = \bar{r}_2\omega, \tag{2.10}$$

$$a_{32}(\xi_{32}+\tau_{32}) \int_0^\omega e^{u_2(s)}\, ds - a_{33}(\xi_{33}+\tau_{33}) \int_0^\omega e^{u_3(s)}\, ds = \bar{r}_3\omega. \tag{2.11}$$

It follows from (H1),(H2) and (2.9)–(2.11) that

$$\int_0^\omega e^{u_i(s)}\,ds = \frac{\omega A_i}{A}, \tag{2.12}$$

where

$$\begin{aligned}
A_1 &= \bar{r}_1 a_{23}(\xi_{23})\, a_{32}(\xi_{32}+\tau_{32}) - \bar{r}_3 a_{12}(\xi_{12})\, a_{23}(\xi_{23}) \\
&\quad + \bar{r}_1 a_{22}(\xi_{22}+\tau_{22})\, a_{33}(\xi_{33}+\tau_{33}) + \bar{r}_2 a_{12}(\xi_{12})\, a_{33}(\xi_{33}+\tau_{33}), \\
A_2 &= \bar{r}_1 a_{21}(\xi_{21}+\tau_{21})\, a_{33}(\xi_{33}+\tau_{33}) - \bar{r}_2 a_{11}(\xi_{11}+\tau_{11})\, a_{33}(\xi_{33}+\tau_{33}) \\
&\quad + \bar{r}_3 a_{11}(\xi_{11}+\tau_{11})\, a_{23}(\xi_{23}), \\
A_3 &= \bar{r}_1 a_{21}(\xi_{21}+\tau_{21})\, a_{32}(\xi_{32}+\tau_{32}) - \bar{r}_2 a_{11}(\xi_{11}+\tau_{11})\, a_{32}(\xi_{32}+\tau_{32}) \\
&\quad - \bar{r}_3 a_{11}(\xi_{11}+\tau_{11})\, a_{22}(\xi_{22}+\tau_{22}) - \bar{r}_3 a_{12}(\xi_{12})\, a_{21}(\xi_{21}+\tau_{21}), \\
A &= a_{11}(\xi_{11}+\tau_{11})\, a_{22}(\xi_{22}+\tau_{22})\, a_{33}(\xi_{33}+\tau_{33}) \\
&\quad + a_{11}(\xi_{11}+\tau_{11})\, a_{23}(\xi_{23})\, a_{32}(\xi_{32}+\tau_{32}) \\
&\quad + a_{33}(\xi_{33}+\tau_{33})\, a_{12}(\xi_{12})\, a_{21}(\xi_{21}+\tau_{21}).
\end{aligned}$$

By Assumptions (H1),(H2), it is easy to verify that $A > 0$, $A_i > 0$, $i = 1,2,3$.

Using the mean value theorem again, we derive that there exists $\eta_i \in [0,\omega]$, such that

$$u_i(\eta_i) = \ln \frac{A_i}{A}, \qquad i = 1,2,3.$$

Thus, there must exist constants $C_i > 0$, such that

$$|u_i(\eta_i)| \le C_i, \qquad i = 1,2,3. \tag{2.13}$$

It follows from (2.3) and (2.4) that

$$\int_0^\omega |u_1'(t)|\,dt < \int_0^\omega \left[r_1(t) + a_{11}(t)\, e^{u_1(t-\tau_{11})} + a_{12}(t)\, e^{u_2(t)}\right] dt = 2\bar{r}_1\omega.$$

This, together with (2.13), leads to

$$|u_1(t)| \le |u_1(\eta_1)| + \int_0^\omega |u_1'(t)|\,dt < C_1 + 2\bar{r}_1\omega := R_1. \tag{2.14}$$

It follows from (2.3) and (2.5) that

$$\begin{aligned}
\int_0^\omega |u_2'(t)|\,dt &< \int_0^\omega \left[r_2(t) + a_{21}(t)\, e^{u_1(t-\tau_{21})} + a_{22}(t)\, e^{u_2(t-\tau_{22})} + a_{23}(t)\, e^{u_3(t)}\right] dt \\
&= 2\int_0^\omega a_{21}(t)\, e^{u_1(t-\tau_{21})}\,dt \\
&= 2a_{21}(\xi_{21}+\tau_{21}) \int_0^\omega e^{u_1(s)}\,ds \\
&\le 2a_{21}^M \omega e^{C_1},
\end{aligned}$$

which, together with (2.13), yields

$$|u_2(t)| \le |u_2(\eta_2)| + \int_0^\omega |u_2'(t)|\,dt < C_2 + 2a_{21}^M \omega e^{C_1} := R_2. \tag{2.15}$$

Similarly, by (2.3) and (2.6), we obtain

$$\begin{aligned}\int_0^\omega |u_3'(t)|\,dt &< \int_0^\omega \left[r_3(t) + a_{32}(t)\,e^{u_2(t-\tau_{32})} + a_{33}(t)\,e^{u_3(t-\tau_{33})}\right] dt \\ &= 2\int_0^\omega a_{32}(t)\,e^{u_2(t-\tau_{32})}\,dt \\ &= 2a_{32}(\xi_{32}+\tau_{32})\int_0^\omega e^{u_2(s)}\,ds \\ &\le 2a_{32}^M \omega e^{C_2},\end{aligned}$$

which, together with (2.13), leads to

$$|u_3(t)| \le |u_3(\eta_3)| + \int_0^\omega |u_3'(t)|\,dt < C_3 + 2a_{32}^M\omega e^{C_2} := R_3. \tag{2.16}$$

Obviously, R_1, R_2, and R_3 in (2.14), (2.15), and (2.16) are independent of λ. Denote $M = R_1+R_2+R_3+R_0$, where R_0 is taken sufficiently large, such that the unique solution $(\alpha^*,\beta^*,\gamma^*)^\top$ of the system of algebraic equations,

$$\begin{aligned} \bar r_1 - \bar a_{11}e^\alpha - \bar a_{12}e^\beta &= 0,\\ -\bar r_2 + \bar a_{21}e^\alpha - \bar a_{22}e^\beta - \bar a_{23}e^\gamma &= 0,\\ -\bar r_3 + \bar a_{32}e^\beta - \bar a_{33}e^\gamma &= 0,\end{aligned} \tag{2.17}$$

satisfies $\|(\alpha^*,\beta^*,\gamma^*)^\top\| = |\alpha^*|+|\beta^*|+|\gamma^*| < M$.

We now take $\Omega = \{(u_1(t),u_2(t),u_3(t))^\top \in X : \|(u_1,u_2,u_3)^\top\| < M\}$. This satisfies the Condition (a) in Lemma 2.1. When $(u_1(t),u_2(t),u_3(t))^\top \in \partial\Omega\cap \operatorname{Ker} L = \partial\Omega\cap R^3$, $(u_1,u_2,u_3)^\top$ is a constant vector in R^3 with $|u_1|+|u_2|+|u_3| = M$. Obviously, we have

$$QN\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} = \begin{bmatrix}\bar r_1 - \bar a_{11}e^{u_1} - \bar a_{12}e^{u_2}\\ -\bar r_2 + \bar a_{21}e^{u_1} - \bar a_{22}e^{u_2} - \bar a_{23}e^{u_3}\\ -\bar r_3 + \bar a_{32}e^{u_2} - \bar a_{33}e^{u_3}\end{bmatrix} \ne \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

This proves that Condition (b) in Lemma 2.1 is satisfied.

Finally, we will prove that Condition (c) in Lemma 2.1 holds. Taking $J = I : \operatorname{Im} Q \to \operatorname{Ker} L, (u_1,u_2,u_3)^\top \to (u_1,u_2,u_3)^\top$ and by a standard calculation, we can derive

$$\begin{aligned}&\deg\left(JQN(u_1,u_2,u_3)^\top, \Omega\cap\operatorname{Ker} L, (0,0,0)^\top\right)\\ &= \deg\big((\bar r_1 - \bar a_{11}e^{u_1} - \bar a_{12}e^{u_2}, -\bar r_2 + \bar a_{21}e^{u_1} - \bar a_{22}e^{u_2} - \bar a_{23}e^{u_3},\\ &\qquad -\bar r_3 + \bar a_{32}e^{u_2} - \bar a_{33}e^{u_3})^\top, \Omega\cap\operatorname{Ker}L, (0,0,0)^\top\big)\\ &= -1.\end{aligned}$$

By now, we have proved that Ω satisfies all the requirements in Lemma 2.1. Hence, (2.2) has at least one ω-periodic solution. Moreover, system (1.1) has at least one positive ω-periodic solution. The proof is complete. ∎

3. UNIQUENESS AND GLOBAL STABILITY

We now proceed to the discussion on the uniqueness and global stability of the ω-periodic solution $x^*(t)$ in Theorem 2.1. It is immediate that if $x^*(t)$ is globally asymptotically stable, then, $x^*(t)$ is, in fact, unique. We first derive certain upper bound estimates for solutions of (1.1),(1.2).

LEMMA 3.1. *Let $x(t) = (x_1(t), x_2(t), x_3(t))$ denote any positive solution of system (1.1) with initial conditions (1.2). If system (1.1) satisfies (H1) and (H2), then, there exits a $T > 0$, such that*

$$0 < x_i(t) \le M_i \qquad (i = 1,2,3) \text{ for } t \ge T, \tag{3.1}$$

where

$$\begin{aligned} M_1 &= \frac{r_1^M}{a_{11}^L} e^{\tau r_1^M}, \\ M_2 &= \frac{a_{21}^M M_1 - r_2^L}{a_{22}^L} e^{(a_{21}^M M_1 - r_2^L)\tau}, \\ M_3 &= \frac{a_{32}^M M_2 - r_3^L}{a_{33}^L} e^{(a_{32}^M M_2 - r_3^L)\tau}. \end{aligned} \tag{3.2}$$

The proof of Lemma 3.1 is standard and it is similar to that of Lemma 2.1 in [19], therefore, we omit it here.

We now formulate the uniqueness and global stability of the positive ω-periodic solutions of system (1.1).

THEOREM 3.1. *In addition to (H1),(H2), assume further that the following hold:*

(H3) $\liminf_{t\to\infty} A_i(t) > 0$, $i = 1,2,3$,

where

$$\begin{aligned} A_1(t) &= a_{11}(t) - (r_1(t) + a_{11}(t) M_1 + a_{12}(t) M_2) \int_t^{t+\tau_{11}} a_{11}(s)\, ds \\ &\quad - a_{21}(t+\tau_{21}) - \sum_{k=1}^{2} M_k a_{k1}(t+\tau_{k1}) \int_{t+\tau_{k1}}^{t+\tau_{kk}+\tau_{k1}} a_{kk}(s)\, ds; \\ A_2(t) &= a_{22}(t) - \left(r_2(t) + \sum_{k=1}^{3} a_{2k}(t) M_k \right) \int_t^{t+\tau_{22}} a_{22}(s)\, ds \\ &\quad - a_{12}(t) - a_{32}(t+\tau_{32}) - M_1 a_{12}(t) \int_t^{t+\tau_{11}} a_{11}(s)\, ds \\ &\quad - \sum_{k=2}^{3} M_k a_{k2}(t+\tau_{k2}) \int_{t+\tau_{k2}}^{t+\tau_{kk}+\tau_{k2}} a_{kk}(s)\, ds; \\ A_3(t) &= a_{33}(t) - (r_3(t) + a_{32}(t) M_2 + a_{33}(t) M_3) \int_t^{t+\tau_{33}} a_{33}(s)\, ds \\ &\quad - a_{23}(t) - M_2 a_{23}(t) \int_t^{t+\tau_{22}} a_{22}(s)\, ds \\ &\quad - M_3 a_{33}(t+\tau_{33}) \int_{t+\tau_{33}}^{t+2\tau_{33}} a_{33}(s)\, ds. \end{aligned} \tag{3.3}$$

Then, system (1.1),(1.2) has a unique positive ω-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), x_3^*(t))^\top$ which is globally asymptotically stable.

PROOF. Due to the conclusion of Theorem 2.1, we only need to show the global asymptotic stability of the positive periodic solution of (1.1),(1.2). Let $x^*(t) = (x_1^*(t), x_2^*(t), x_3^*(t))^\top$ be a positive ω-periodic solution of system (1.1),(1.2), and $y(t) = (y_1(t), y_2(t), y_3(t))^\top$ be any positive solution of system (1.1).

It follows from Lemma 3.1 that there exist positive constants T and M_i $(i = 1,2,3)$, such that, for all $t \ge T$,

$$0 < x_i^*(t) \le M_i, \quad 0 < y_i(t) \le M_i, \qquad i = 1,2,3. \tag{3.4}$$

Let

$$V_{11}(t) = \left| \ln x_1^*(t) - \ln y_1(t) \right|. \tag{3.5}$$

Calculating the upper right derivative of $V_{11}(t)$ along the solution of (1.1), it follows that

$$\begin{aligned}
D^+V_{11}(t) &= \left(\frac{\dot{x}_1^*(t)}{x_1^*(t)} - \frac{\dot{y}_1(t)}{y_1(t)}\right)\operatorname{sgn}(x_1^*(t) - y_1(t)) \\
&= \operatorname{sgn}(x_1^*(t) - y_1(t))\left[-a_{11}(t)(x_1^*(t-\tau_{11}) - y_1(t-\tau_{11}))\right. \\
&\quad \left.-a_{12}(t)(x_2^*(t) - y_2(t))\right] \\
&= \operatorname{sgn}(x_1^*(t) - y_1(t))\Bigg[-a_{11}(t)(x_1^*(t) - y_1(t)) \\
&\quad - a_{12}(t)(x_2^*(t) - y_2(t)) \\
&\quad + a_{11}(t)\int_{t-\tau_{11}}^{t}(\dot{x}_1^*(u) - \dot{y}_1(u))\,du\Bigg] \\
&\le -a_{11}(t)|x_1^*(t) - y_1(t)| + a_{12}(t)|x_2^*(t) - y_2(t)| \\
&\quad + a_{11}(t)\left|\int_{t-\tau_{11}}^{t}(\dot{x}_1^*(u) - \dot{y}_1(u))\,du\right|.
\end{aligned} \tag{3.6}$$

On substituting (1.1) into (3.6), we derive

$$\begin{aligned}
D^+V_{11}(t) &\le -a_{11}(t)|x_1^*(t) - y_1(t)| + a_{12}(t)|x_2^*(t) - y_2(t)| \\
&\quad + a_{11}(t)\Bigg|\int_{t-\tau_{11}}^{t}\{x_1^*(u)[r_1(u) - a_{11}(u)x_1^*(u-\tau_{11}) - a_{12}(u)x_2^*(u)] \\
&\quad -y_1(u)[r_1(u) - a_{11}(u)y_1(u-\tau_{11}) - a_{12}(u)y_2(u)]\}\,du\Bigg| \\
&= -a_{11}(t)|x_1^*(t) - y_1(t)| + a_{12}(t)|x_2^*(t) - y_2(t)| \\
&\quad + a_{11}(t)\Bigg|\int_{t-\tau_{11}}^{t}\{(x_1^*(u) - y_1(u))[r_1(u) \\
&\quad -a_{11}(u)y_1(u-\tau_{11}) - a_{12}(u)y_2(u)] \\
&\quad - a_{11}(u)x_1^*(u)(x_1^*(u-\tau_{11}) - y_1(u-\tau_{11})) \\
&\quad -a_{12}(u)x_1^*(u)(x_2^*(u) - y_2(u))\}\,du\Bigg|.
\end{aligned} \tag{3.7}$$

It follows from (3.4) and (3.7) that, for $t \ge T + 2\tau$,

$$\begin{aligned}
&D^+V_{11}(t) \le -a_{11}(t)|x_1^*(t) - y_1(t)| + a_{12}(t)|x_2^*(t) - y_2(t)| \\
&+a_{11}(t)\int_{t-\tau_{11}}^{t}\{[r_1(u) + a_{11}(u)M_1 + a_{12}(u)M_2]|x_1^*(u) - y_1(u)| \\
&+a_{11}(u)M_1|x_1^*(u-\tau_{11}) - y_1(u-\tau_{11})| + a_{12}(u)M_1|x_2^*(u) - y_2(u)|\}\,du.
\end{aligned} \tag{3.8}$$

Define

$$\begin{aligned}
V_{12}(t) &= \int_t^{t+\tau_{11}}\int_{s-\tau_{11}}^{t} a_{11}(s)[r_1(u) + a_{11}(u)M_1 + a_{12}(u)M_2]|x_1^*(u) - y_1(u)|\,du\,ds \\
&\quad +M_1\int_t^{t+\tau_{11}}\int_{s-\tau_{11}}^{t} a_{11}(s)a_{11}(u)|x_1^*(u-\tau_{11}) - y_1(u-\tau_{11})|\,du\,ds \\
&\quad +M_1\int_t^{t+\tau_{11}}\int_{s-\tau_{11}}^{t} a_{11}(s)a_{12}(u)|x_2^*(u) - y_2(u)|\,du\,ds.
\end{aligned} \tag{3.9}$$

Then, it follows from (3.8) and (3.9) that, for $t \geq T + 2\tau$,

$$\begin{aligned} D^+V_{11}(t) + \dot{V}_{12}(t) &\leq -a_{11}(t)\left|x_1^*(t) - y_1(t)\right| + a_{12}(t)\left|x_2^*(t) - y_2(t)\right| \\ &+ \left[r_1(t) + a_{11}(t) M_1 + a_{12}(t) M_2\right] \int_t^{t+\tau_{11}} a_{11}(s)\, ds \left|x_1^*(t) - y_1(t)\right| \\ &+ M_1 a_{11}(t) \int_t^{t+\tau_{11}} a_{11}(s)\, ds \left|x_1^*(t-\tau_{11}) - y_1(t-\tau_{11})\right| \\ &+ M_1 a_{12}(t) \int_t^{t+\tau_{11}} a_{11}(s)\, ds \left|x_2^*(t) - y_2(t)\right|. \end{aligned} \tag{3.10}$$

We now define

$$V_1(t) = V_{11}(t) + V_{12}(t) + V_{13}(t), \tag{3.11}$$

in which

$$V_{13}(t) = M_1 \int_{t-\tau_{11}}^{t} \int_{l+\tau_{11}}^{l+2\tau_{11}} a_{11}(s)\, a_{11}(l+\tau_{11}) \left|x_1^*(l) - y_1(l)\right| ds\, dl. \tag{3.12}$$

Then, it follows from (3.10), (3.11), and (3.12) that, for $t \geq T + 2\tau$,

$$\begin{aligned} D^+V_1(t) &\leq -a_{11}(t)\left|x_1^*(t) - y_1(t)\right| + a_{12}(t)\left|x_2^*(t) - y_2(t)\right| \\ &+ \left(r_1(t) + a_{11}(t) M_1 + a_{12}(t) M_2\right) \int_t^{t+\tau_{11}} a_{11}(s)\, ds \left|x_1^*(t) - y_1(t)\right| \\ &+ M_1 a_{11}(t+\tau_{11}) \int_{t+\tau_{11}}^{t+2\tau_{11}} a_{11}(s)\, ds \left|x_1^*(t) - y_1(t)\right| \\ &+ M_1 a_{12}(t) \int_t^{t+\tau_{11}} a_{11}(s)\, ds \left|x_2^*(t) - y_2(t)\right|. \end{aligned} \tag{3.13}$$

Next, let

$$V_2(t) = V_{21}(t) + V_{22}(t) + V_{23}(t), \tag{3.14}$$

where

$$\begin{aligned} V_{21}(t) &= \left|\ln x_2^*(t) - \ln y_2(t)\right|, \\ V_{22}(t) &= \int_{t-\tau_{21}}^{t} a_{21}(s+\tau_{21}) \left|x_1^*(s) - y_1(s)\right| ds \\ &+ \int_t^{t+\tau_{22}} \int_{s-\tau_{22}}^{t} a_{22}(s) \left(r_2(u) + \sum_{k=1}^{3} a_{2k}(u) M_k\right) \left|x_2^*(u) - y_2(u)\right| du\, ds \\ &+ M_2 \int_t^{t+\tau_{22}} \int_{s-\tau_{22}}^{t} a_{22}(s) \sum_{k=1}^{2} a_{2k}(u) \left|x_k^*(u-\tau_{2k}) - y_k(u-\tau_{2k})\right| du\, ds \\ &+ M_2 \int_t^{t+\tau_{22}} \int_{s-\tau_{22}}^{t} a_{22}(s)\, a_{23}(u) \left|x_3^*(u) - y_3(u)\right| du\, ds, \\ V_{23}(t) &= M_2 \sum_{k=1}^{2} \int_{t-\tau_{2k}}^{t} \int_{l+\tau_{2k}}^{l+\tau_{2k}+\tau_{22}} a_{22}(s)\, a_{2k}(l+\tau_{2k}) \left|x_k^*(l) - y_k(l)\right| ds\, dl. \end{aligned} \tag{3.15}$$

Calculating the upper right derivative of $V_2(t)$ along the solution of (1.1), we derive that, for $t \geq T + 2\tau$,

$$\begin{aligned} D^+V_2(t) &\leq -a_{22}(t)\left|x_2^*(t) - y_2(t)\right| + a_{21}(t+\tau_{21})\left|x_1^*(t) - y_1(t)\right| \\ &+ a_{23}(t)\left|x_3^*(t) - y_3(t)\right| + \left(r_2(t) + \sum_{k=1}^{3} a_{2k}(t) M_k\right) \int_t^{t+\tau_{22}} a_{22}(s)\, ds \left|x_2^*(t) - y_2(t)\right| \\ &+ M_2 \sum_{k=1}^{2} a_{2k}(t+\tau_{2k}) \int_{t+\tau_{2k}}^{t+\tau_{22}+\tau_{2k}} a_{22}(s)\, ds \left|x_k^*(t) - y_k(t)\right| \\ &+ M_2 a_{23}(t) \int_t^{t+\tau_{22}} a_{22}(s)\, ds \left|x_3^*(t) - y_3(t)\right|. \end{aligned} \tag{3.17}$$

Finally, we define

$$V_3(t) = V_{31}(t) + V_{32}(t) + V_{33}(t), \tag{3.17}$$

where

$$\begin{aligned}
V_{31}(t) &= \left|\ln x_3^*(t) - \ln y_3(t)\right|, \\
V_{32}(t) &= \int_{t-\tau_{32}}^{t} a_{32}(s+\tau_{32})\left|x_2^*(s) - y_2(s)\right| ds \qquad (3.18)\\
&\quad + \int_t^{t+\tau_{33}} \int_{s-\tau_{33}}^{t} a_{33}(s)\left(r_3(u) + \sum_{k=2}^{3} a_{3k}(u) M_k\right)\left|x_3^*(u) - y_3(u)\right| du\, ds \\
&\quad + M_3 \int_t^{t+\tau_{33}} \int_{s-\tau_{33}}^{t} a_{33}(s) \sum_{k=2}^{3} \\
&\quad \times a_{3k}(u)\left|x_k^*(u-\tau_{3k}) - y_k(u-\tau_{3k})\right| du\, ds, \qquad (3.18)(\text{cont.})\\
V_{33}(t) &= M_3 \sum_{k=2}^{3} \int_{t-\tau_{3k}}^{t} \int_{l+\tau_{3k}}^{l+\tau_{33}+\tau_{3k}} a_{33}(s)\, a_{3k}(l+\tau_{3k})\left|x_k^*(l) - y_k(l)\right| ds\, dl.
\end{aligned}$$

Calculating the upper right derivative of $V_3(t)$ along the solution of (1.1), it follows that, for $t \geq T + 2\tau$,

$$\begin{aligned}
D^+V_3(t) &\leq -a_{33}(t)\left|x_3^*(t) - y_3(t)\right| + a_{32}(t+\tau_{32})\left|x_2^*(t) - y_2(t)\right| \\
&\quad + \left(r_3(t) + \sum_{k=2}^{3} a_{3k}(t) M_k\right) \int_t^{t+\tau_{33}} a_{33}(s)\, ds \left|x_3^*(t) - y_3(t)\right| \\
&\quad + M_3 \sum_{k=2}^{3} a_{3k}(t+\tau_{3k}) \int_{t+\tau_{3k}}^{t+\tau_{33}+\tau_{3k}} a_{33}(s)\, ds \left|x_k^*(t) - y_k(t)\right|.
\end{aligned} \tag{3.19}$$

Now, we define a Lyapunov functional $V(t)$ as

$$V(t) = V_1(t) + V_2(t) + V_3(t). \tag{3.20}$$

Then, it follows from (3.13), (3.16), (3.19), and (3.20) that, for $t \geq T + 2\tau$,

$$D^+V(t) \leq -\sum_{i=1}^{3} A_i(t)\left|x_i^*(t) - y_i(t)\right|, \tag{3.21}$$

where $A_i(t)$ is defined in (3.3).

By Hypotheses (H3), there exist constants $\alpha_i > 0$ $(i = 1, 2, 3)$ and $T^* \geq T + 2\tau$, such that

$$A_i(t) \geq \alpha_i > 0, \qquad \text{for } t \geq T^*. \tag{3.22}$$

Integrating both sides of (3.21) on interval $[T^*, t]$, we derive

$$V(t) + \sum_{i=1}^{3} \int_{T^*}^{t} A_i(s)\left|x_i^*(s) - y_i(s)\right| ds \leq V(T^*), \qquad \text{for } t \geq T^*. \tag{3.23}$$

It follows from (3.22) and (3.23) that

$$V(t) + \sum_{i=1}^{3} \alpha_i \int_{T^*}^{t} \left|x_i^*(s) - y_i(s)\right| ds \leq V(T^*), \qquad \text{for } t \geq T^*. \tag{3.24}$$

Therefore, $V(t)$ is bounded on $[T^*, \infty)$ and also

$$\int_{T^*}^{\infty} |x_i^*(s) - y_i(s)|\, ds < \infty, \qquad i = 1,2,3. \tag{3.25}$$

By Lemma 3.1, $|x_i^*(t) - y_i(t)|$ $(i = 1,2,3)$ are bounded on $[T^*, \infty)$.

On the other hand, it is easy to see that $\dot{x}_i^*(t)$ and $\dot{y}_i(t)$ $(i = 1,2,3)$ are bounded for $t \geq T^*$. Therefore, $|x_i^*(t) - y_i(t)| (i = 1,2,3)$ are uniformly continuous on $[T^*, \infty)$. By Barbalat's lemma (see [8, Lemmas 1.2.2 and 1.2.3]), one can conclude that

$$\lim_{t\to\infty} |x_i^*(t) - y_i(t)| = 0, \qquad i = 1,2,3.$$

The proof is complete. ∎

REMARK 1. If $\tau_{ij} \equiv 0$, $i,j = 1,2,3$, then, system (1.1) reduces to an instantaneous system, i.e., one without delay

$$\begin{aligned}
\dot{x}_1 &= x_1(t)\left(r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t)\right),\\
\dot{x}_2 &= x_2(t)\left(-r_2(t) + a_{21}(t)x_1(t) - a_{22}(t)x_2(t) - a_{23}(t)x_3(t)\right),\\
\dot{x}_3 &= x_3(t)\left(-r_3(t) + a_{32}(t)x_2(t) - a_{33}(t)x_3(t)\right).
\end{aligned} \tag{3.26}$$

On substituting $\tau_{ij} \equiv 0$ $(i,j = 1,2,3)$ into (H3), Theorem 3.1 yields that system (3.26) has a unique positive ω-periodic solution which is globally asymptotically stable provided that (H1),(H2) and the following hold,

$$\begin{aligned}
\liminf_{t\to\infty} \left[a_{11}(t) - a_{21}(t)\right] &> 0,\\
\liminf_{t\to\infty} \left[a_{22}(t) - a_{12}(t) - a_{32}(t)\right] &> 0,\\
\liminf_{t\to\infty} \left[a_{33}(t) - a_{23}(t)\right] &> 0.
\end{aligned} \tag{3.27}$$

Therefore, we have the following corollary to Theorem 3.1.

COROLLARY 3.1. *System (1.1),(1.2) has a unique positive ω-periodic solution, which is globally stable, if the corresponding instantaneous system (without time delay) has a unique and globally stable positive ω-periodic solution and τ_{ij} $(i,j = 1,2,3)$ are sufficiently small satisfying (H3).*

Finally, we give two examples to illustrate the feasibility of our main results.

EXAMPLE 1. We consider the following Lotka-Volterra food-chain model

$$\begin{aligned}
\dot{x}_1(t) =& x_1(t)\left[30 + \sin t - 10x_1\left(t - \frac{1}{10^5}\right) - (2+\cos t)x_2(t)\right],\\
\dot{x}_2(t) =& x_2(t)\left[-\frac{3}{10} - \frac{1}{100}\sin t + (8+\cos t)x_1\left(t - \frac{1}{10^5}\right)\right.\\
&\left. -10x_2\left(t - \frac{1}{10^5}\right) - (2+\sin t)x_3(t)\right],\\
\dot{x}_3(t) =& x_3(t)\left[-\frac{2}{10} - \frac{1}{100}\sin t + (5+\sin t)x_2\left(t - \frac{1}{10^5}\right) - 4x_3\left(t - \frac{1}{10^5}\right)\right].
\end{aligned} \tag{3.28}$$

It is easy to examine that the coefficients in system (3.28) satisfy all the assumptions in Theorems 2.1 and 3.1. Thus, by Theorem 3.1, system (3.28) has a unique positive 2π-periodic solution which is globally stable. Using Shampine's and Thompson's program **dde23** in solving DDEs [20], numerical simulation shows that system (3.28) has a unique 2π-periodic solution which is globally asymptotically stable (see Figure 1).

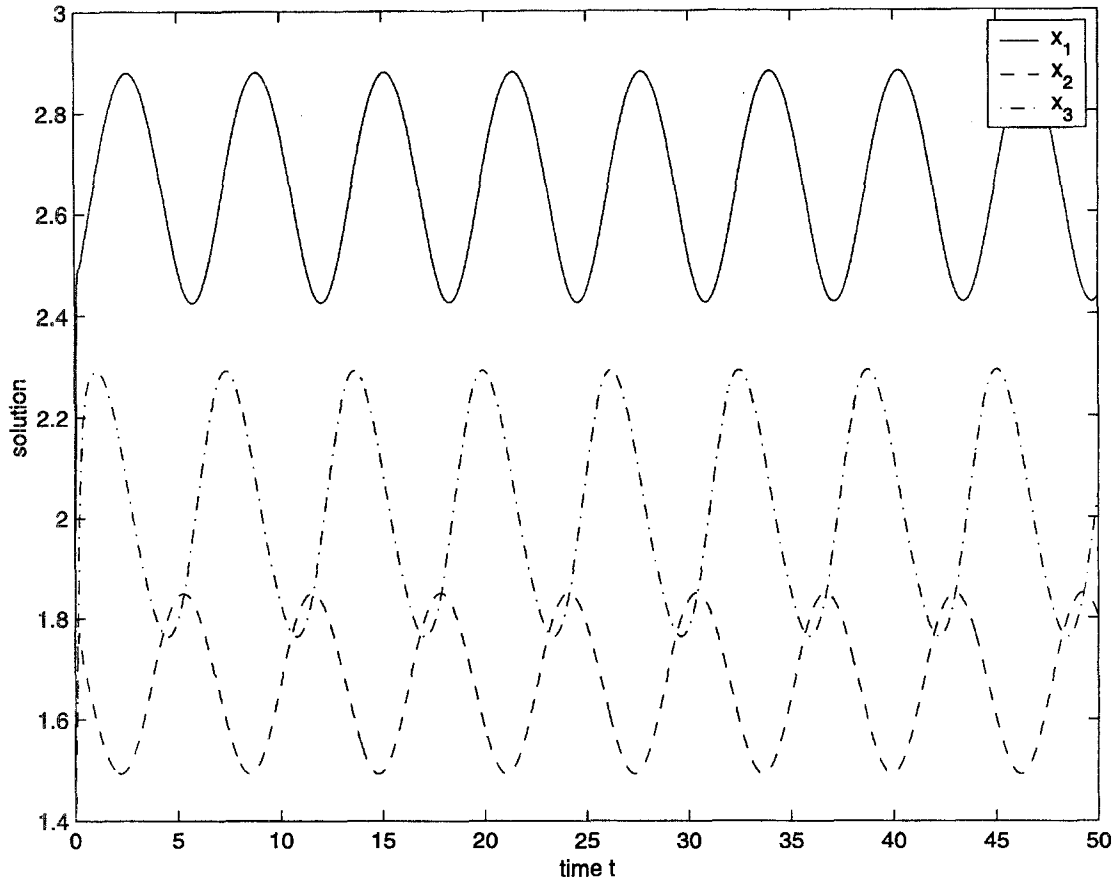

Figure 1. The existence of 2π-periodic solution of system (3.28) with initial conditions $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \equiv (1.5, 1.5, 1, 5)$.

EXAMPLE 2. We consider another delayed periodic Lotka-Volterra food-chain system,

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[20 - \sin t - (4 - \sin t)\, x_1\left(t - \frac{1}{10^3}\right) - 6x_2(t)\right], \\ \dot{x}_2(t) &= x_2(t)\left[-2 + \sin t + 2x_1\left(t - \frac{1}{10^3}\right) - 5x_2\left(t - \frac{1}{10^3}\right) - 2x_3(t)\right], \\ \dot{x}_3(t) &= x_3(t)\left[-\frac{2}{10} + \frac{2}{100}\cos t + (1.5 + \sin t)\, x_2\left(t - \frac{1}{10^3}\right) - \frac{1}{10}x_3\left(t - \frac{1}{10^3}\right)\right]. \end{aligned} \tag{3.29}$$

It is easy to examine that (H2) doesn't hold for system (3.29). Numerical simulation shows that system (3.29) still has at least one positive 2π-periodic solution (see Figure 2).

4. DISCUSSION

In this paper, we have investigated the combined effects of periodicity of the environment and time delays on the dynamics of Lotka-Volterra food-chain predator-prey systems. By using the effective and powerful continuation theorem of coincidence degree theory and by constructing a suitable Lyapunov functional, we have discussed the existence, uniqueness and global stability of periodic solutions for a delayed three-species periodic simple food-chain system. By Theorem 2.1, we see that system (1.1) with initial conditions (1.2) will have at least one periodic solution if the prey intrinsic growth rate, the conversion rates of the predator and the top predator are high and the death rates of the predator and the top predator are low. We have shown in Theorem 3.1 that *small delays are negligible* for the global asymptotic stability of positive periodic solutions of the delayed three-species periodic Lotka-Volterra food-chain systems provided that the delayed

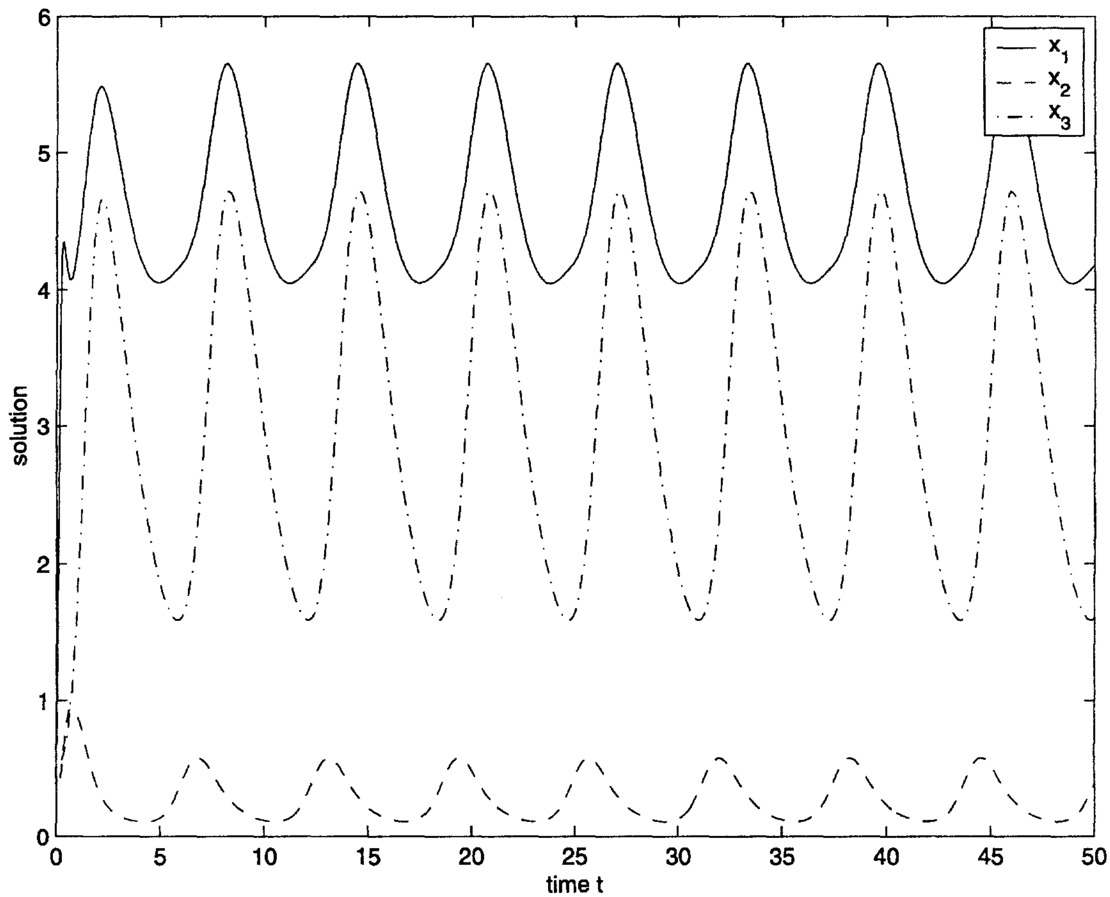

Figure 2. The 2π-periodic solution of system (3.29) with initial conditions $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \equiv (0.5, 0.5, 0, 5)$.

negative feedbacks dominate other interspecific interaction effects with or without delays. We would like to mention here that Example 2 shows that our results in Theorems 2.1 and 3.1 have room for improvement. By using a similar technique, it is interesting to discuss the existence, uniqueness and global stability of positive periodic solutions to an n-species periodic Lotka-Volterra predator-prey system or competition system with time delays. We leave these for future work.

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