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Global asymptotic stability in *n*-species nonautonomous Lotka–Volterra competitive systems with infinite delays

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Abstract

By means of Lyapunov functional, we have succeeded in establishing the global asymptotic stability of the positive solutions of a delayed *n*-species nonautonomous Lotka–Volterra type competitive system without dominating instantaneous negative feedbacks. As a corollary, we show that the global asymptotic stability of the positive solution is maintained provided that the delayed negative feedbacks dominate other interspecific interaction effects with delays and the mean delays are sufficiently small. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

The classical Lotka–Volterra type systems are very important in the models of multi-species population dynamics. There are considerable works on the study of the global asymptotic stability of Lotka–Volterra type systems with time delays that have been developed in [1–11]. In addition to these, the books of Gopalsamy [12] and Kuang [13] are good sources for these topics of

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Lotka–Volterra type systems with time delays. Recently, by constructing appropriate Lyapunov functionals, Bereketoglu and Gyori [10] and Teng and Yu [11] have studied the global asymptotic stability of nonautonomous Lotka–Volterra type systems with finite and infinite delays, respectively. The systems they considered contain instantaneous delay-independent terms which dominate other intraspecific and interspecific interaction effects with delays. However, when such delay-independent terms are not present, the Lyapunov functionals in [10,11] are not valid. In view of the fact that in real-life interactions, instantaneous responses are rare or weak relative to delayed responses, more realistic models should consist of delay differential systems without instantaneous (negative) feedbacks. Most of the known convergence results for delayed systems require dominating or strong instantaneous negative feedbacks or some restrictions on initial conditions. In order to justify the common belief that "small delays are negligible in some modelling process as far as stabilities are concerned", it is thus important and urgent to show under some reasonable assumptions that the global stability of a Lotka–Volterra system or other general systems persist when time delays are small enough. This is indeed a long standing question in the qualitative analysis of systems of delay differential equations. Kuang [5] presented a partial answer to this open question for Lotka–Volterra type systems with a saturated equilibrium.

Motivated by the works of Bereketoglu and Gyori [10], Teng and Yu [11] and Kuang [5], in this paper, we consider delayed nonautonomous Lotka–Volterra type systems without saturated steady state solutions and without dominating instantaneous negative feedbacks. For n-species Lotka–Volterra competitive system with infinite delays, this results in the following delayed system:

$$\dot{x}_i(t) = x_i(t)\left(r_i(t) - a_i(t)x_i(t) - \sum_{j=1}^{n} a_{ij}(t)\int_0^{\infty} K_{ij}(s)x_j(t-s)\,\mathrm{d}s\right),\quad i = 1,2,\ldots,n, \tag{1.1}$$

where $x_i(t)$ denotes the density of the ith population, respectively, $i = 1,2,\ldots,n$.

In this paper, for system (1.1) we always assume that for all $i,j = 1,2,\ldots,n$:

(H1) $r_i(t),\ a_i(t),\ a_{ij}(t)$ are continuous, bounded functions defined on $[0,+\infty)$, $r_i(t) > 0,\ a_i(t) \geqslant 0,\ a_{ij}(t) \geqslant 0\ (i \neq j),\ a_{ii}(t) > 0$ for all $t \in [0,\infty)$;

(H2) $K_{ij}(t) : [0,\infty) \mapsto [0,\infty)$ is piecewise continuous and normalized function such that

$$\int_0^{\infty} K_{ij}(s)\,\mathrm{d}s = 1, \quad \sigma_{ij} = \int_0^{\infty} sK_{ij}(s)\,\mathrm{d}s < \infty. \tag{1.2}$$

Motivated by application of system (1.1) in population dynamics, we assume that solutions of system (1.1) satisfy the initial conditions

$$x_i(\theta) = \phi_i(\theta), \qquad \theta \in (-\infty, 0], \quad \phi_i(0) > 0, \quad i = 1, 2, \ldots, n, \tag{1.3}$$

where each ϕ_i is given nonnegative and bounded continuous function on $(-\infty, 0]$. It is well known that by the fundamental theory of functional differential equations [14], system (1.1) has a unique solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ satisfying the initial conditions (1.3). It is easy to verify that solutions of system (1.1) corresponding to initial conditions (1.3) are defined on $[0, +\infty)$ and remain positive for all $t \geqslant 0$. In this paper, the solution of system (1.1) satisfying the initial conditions (1.3) is said to be positive.

If $f(t)$ is a continuous bounded function defined on $[0, +\infty)$, we set

$$f^L = \inf_{t \geqslant 0} f(t), \quad f^M = \sup_{t \geqslant 0} f(t). \tag{1.4}$$

In the following we say a positive solution of system (1.1) is globally asymptotically stable if it attracts all other positive solutions of the system.

The organization of this paper is as follows. In the following section, we present the boundedness of the positive solution of system (1.1) with initial conditions (1.3), and sufficient conditions are derived for the positive solution of system (1.1) with initial conditions (1.3) to be globally asymptotically stable.

2. Main results

Lemma 2.1. *Let* $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ *denote any positive solution of system* (1.1) *with initial conditions* (1.3). *Then there exists a* $T > 0$ *such that*

$$0 < x_i(t) \leqslant M_i \quad (i = 1, 2, \ldots, n) \quad \textit{for } t \geqslant T, \tag{2.1}$$

where

$$M_i = \frac{r_i^M}{a_{ii}^L \alpha_{ii}} e^{r_i^M \sigma_{ii}}, \quad i = 1, 2, \ldots, n, \tag{2.2}$$

and

$$\alpha_{ii} = \int_0^{\sigma_{ii}} K_{ii}(s)\, ds, \quad i = 1, 2, \ldots, n. \tag{2.3}$$

Proof. Suppose $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ is a solution of system (1.1) which satisfies (1.3). According to the ith equation of system (1.1), it follows from the positivity of the solution of (1.1) and the nonnegativity of the initial values that

$$\frac{dx_i(t)}{dt} \leqslant x_i(t)\left(r_i^M - a_{ii}^L \int_0^{\infty} K_{ii}(s)x_i(t-s)\,ds\right)$$
$$\leqslant x_i(t)\left(r_i^M - a_{ii}^L \int_0^{\sigma_{ii}} K_{ii}(s)x_i(t-s)\,ds\right). \tag{2.4}$$

Take $M_i^* = r_i^M/a_{ii}^L\alpha_{ii}$. Suppose $x_i(t)$ is not oscillatory about M_i^*; that is, there exists a $T_{i1} > 0$ such that

$$x_i(t) > M_i^* \quad \text{for } t > T_{i1}, \tag{2.5}$$

or

$$x_i(t) < M_i^* \quad \text{for } t > T_{i1}. \tag{2.6}$$

If (2.6) holds, then (2.1) follows. Suppose (2.5) holds, (2.4) implies that

$$\frac{dx_i(t)}{dt} < 0 \quad \text{for } t > T_{i1} + \sigma_{ii},$$

which implies that there exists a constant x_i^* such that

$$\lim_{t\to\infty} x_i(t) = x_i^*. \tag{2.7}$$

If $x_i^* > M_i^* e^{r_i^M \sigma_{ii}}$, then there exists a $T_{i2} \geqslant T_{i1} + \sigma_{ii}$ such that if $t \geqslant T_{i2}$, $x_i(t) \geqslant M_i^* e^{r_i^M \sigma_{ii}}$. It then follows from (2.4) that for $t \geqslant T_{i2} + \sigma_{ii}$

$$\frac{dx_i(t)}{dt} \leqslant r_i^M(1 - e^{r_i^M \sigma_{ii}})x_i(t).$$

This will lead to a contradiction. Therefore, we have $x_i^* \leqslant M_i^* e^{r_i^M \sigma_{ii}}$. It follows from (2.7) that there must exist a $T_{i3} \geqslant T_{i1}$ such that

$$x_i(t) \leqslant M_i^* e^{r_i^M \sigma_{ii}} = M_i \quad \text{for all } t \geqslant T_{i3}.$$

Suppose now that $x_i(t)$ is oscillatory about M_i^*. Let $x_i(T_{i0})$ denote an arbitrary local maximum of $x_i(t)$. Then it is easy to see from (2.4) that

$$0 = \frac{dx_i(T_{i0})}{dt} \leqslant x_i(T_{i0})\left(r_i^M - a_{ii}^L \int_0^{\sigma_{ii}} K_{ii}(s)x_i(T_{i0}-s)\,ds\right). \tag{2.8}$$

This leads to

$$\int_0^{\sigma_{ii}} K_{ii}(s)x_i(T_{i0}-s)\,ds \leqslant \frac{r_i^M}{a_{ii}^L}. \tag{2.9}$$

Hence, there must exist a $T_{i4} \in [T_{i0} - \sigma_{ii}, T_{i0})$ such that

$$x_i(T_{i4}) \leqslant M_i^*. \tag{2.10}$$

Integrating both sides of (2.4) on the interval $[T_{i4}, T_{i0}]$, we have

$$\ln\left[\frac{x_i(T_{i0})}{x_i(T_{i4})}\right] \leqslant \int_{T_{i4}}^{T_{i0}} \left(r_i^M - a_{ii}^L \int_0^{\sigma_{ii}} K_{ii}(s) x_i(t-s)\,\mathrm{d}s\right)\mathrm{d}t.$$

It follows that

$$\ln\left[\frac{x_i(T_{i0})}{x_i(T_{i4})}\right] \leqslant \int_{T_{i4}}^{T_{i0}} r_i^M\,\mathrm{d}t = r_i^M (T_{i0} - T_{i4}). \tag{2.11}$$

From (2.10) and (2.11), we have

$$x_i(T_{i0}) \leqslant M_i^* \mathrm{e}^{r_i^{M(T_{i0}-T_{i4})}} \leqslant M_i^* \mathrm{e}^{r_i^M \sigma_{ii}} = M_i.$$

Since $x_i(T_{i0})$ is an arbitrary local maximum of $x_i(t)$, therefore, we can conclude that there exists a $T_i > 0$ such that $x_i(t) \leqslant M_i$ for all $t \geqslant T_i$. We set $T = \max_{1 \leqslant i \leqslant n} T_i$. This completes the proof. □

In the following, by improving the method given in [10,11], we further discuss the global asymptotical stability of the positive solutions of system (1.1). Now we formulate our main result below.

Theorem 2.1. *In addition to* (H1) *and* (H2), *assume further that*

(H3) *there exist constants* $c_i > 0,\ i = 1, 2, \ldots, n$, *such that for* $i = 1, 2, \ldots, n$

$$\begin{aligned}\liminf_{t\to\infty} \Bigg[& c_i(a_i(t) + a_{ii}(t)) - \sum_{j=1, j\neq i}^{n} c_j \int_0^\infty K_{ji}(s) a_{ji}(t+s)\,\mathrm{d}s \\ & - c_i\left(r_i(t) + 2a_i(t)M_i + \sum_{j=1}^{n} a_{ij}(t) M_j\right) \int_0^\infty \int_{t-s}^{t} K_{ii}(s) a_{ii}(v+s)\,\mathrm{d}v\,\mathrm{d}s \\ & - \sum_{j=1}^{n} c_j M_j \int_0^\infty \int_0^\infty \int_{t+r-s}^{t+r} K_{jj}(s) a_{jj}(v+s) K_{ji}(r) a_{ji}(t+r)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}r \Bigg] > 0.\end{aligned}$$

Then for any positive solutions $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ *and* $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))$ *of system* (1.1), *one has*

$$\lim_{t\to\infty} |x_i(t) - y_i(t)| = 0, \quad i = 1, 2, \ldots, n. \tag{2.12}$$

Proof. For two arbitrary nontrival solutions $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ and $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))$ of system (1.1), we have from Lemma 2.1 that there exist positive constants T and M_i $(i = 1, 2, \ldots, n)$ such that for all $t \geqslant T$,

$$0 < x_i(t) \leqslant M_i, \quad 0 < y_i(t) \leqslant M_i, \qquad i = 1, 2, \ldots, n. \tag{2.13}$$

For $i = 1, 2, \ldots, n$, we let

$$V_{i1}(t) = |\ln x_i(t) - \ln y_i(t)|. \tag{2.14}$$

Calculating the upper right derivative of $V_{i1}(t)$ along the solution of (1.1), it follows that

$$\begin{aligned} D^+ V_{i1}(t) &= \left(\frac{\dot{x}_i(t)}{x_i(t)} - \frac{\dot{y}_i(t)}{y_i(t)} \right) \operatorname{sgn}(x_i(t) - y_i(t)) \\ &= \operatorname{sgn}(x_i(t) - y_i(t)) \Bigg[- a_i(t)(x_i(t) - y_i(t)) \\ &\quad - \sum_{j=1}^{n} a_{ij}(t) \int_0^\infty K_{ij}(s)(x_j(t-s) - y_j(t-s))\,\mathrm{d}s \Bigg] \\ &= \operatorname{sgn}(x_i(t) - y_i(t)) \Bigg[- (a_i(t) + a_{ii}(t))(x_i(t) - y_i(t)) \\ &\quad - \sum_{j=1, j \neq i}^{n} a_{ij}(t) \int_0^\infty K_{ij}(s)(x_j(t-s) - y_j(t-s))\,\mathrm{d}s \\ &\quad + a_{ii}(t) \int_0^t \int_{t-s}^t K_{ii}(s)(\dot{x}_i(u) - \dot{y}_i(u))\,\mathrm{d}u\,\mathrm{d}s + W_i(t) \Bigg], \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} W_i(t) &= a_{ii}(t) \int_t^\infty K_{ii}(s)(x_i(t) - x_i(t-s))\,\mathrm{d}s - a_{ii}(t) \\ &\quad \times \int_t^\infty K_{ii}(s)(y_i(t) - y_i(t-s))\,\mathrm{d}s. \end{aligned}$$

On substituting (1.1) into (2.15), we derive

$$\begin{aligned}
D^+V_{i1}(t) = {} & \mathrm{sgn}(x_i(t)-y_i(t))\Bigg[-(a_i(t)+a_{ii}(t))(x_i(t)-y_i(t)) \\
& - \sum_{j=1,j\neq i}^{n} a_{ij}(t)\int_0^{\infty} K_{ij}(s)(x_j(t-s)-y_j(t-s))\,\mathrm{d}s \\
& + a_{ii}(t)\int_0^t \int_{t-s}^t K_{ii}(s)\Bigg\{x_i(u)\Bigg[r_i(u)-a_i(u)x_i(u) \\
& - \sum_{j=1}^{n} a_{ij}(u)\int_0^{\infty} K_{ij}(r)x_j(u-r)\,\mathrm{d}r\Bigg] - y_i(u)\Bigg[r_i(u)-a_i(u)y_i(u) \\
& - \sum_{j=1}^{n} a_{ij}(u)\int_0^{\infty} K_{ij}(r)y_j(u-r)\,\mathrm{d}r\Bigg]\Bigg\}\,\mathrm{d}u\,\mathrm{d}s + W_i(t)\Bigg] \\
= {} & \mathrm{sgn}(x_i(t)-y_i(t))\Bigg[-(a_i(t)+a_{ii}(t))(x_i(t)-y_i(t)) \\
& - \sum_{j=1,j\neq i}^{n} a_{ij}(t)\int_0^{\infty} K_{ij}(s)(x_j(t-s)-y_j(t-s))\,\mathrm{d}s \\
& + a_{ii}(t)\int_0^t \int_{t-s}^t K_{ii}(s)\Bigg\{(x_i(u)-y_i(u))\Bigg[r_i(u)-a_i(u)y_i(u) \\
& - \sum_{j=1}^{n} a_{ij}(u)\int_0^{\infty} K_{ij}(r)y_j(u-r)\,\mathrm{d}r\Bigg] \\
& + x_i(u)\Bigg[-a_i(u)(x_i(u)-y_i(u)) \\
& - \sum_{j=1}^{n} a_{ij}(u)\int_0^{\infty} K_{ij}(r)(x_j(u-r)-y_j(u-r))\,\mathrm{d}r\Bigg]\Bigg\}\,\mathrm{d}u\,\mathrm{d}s \\
& + W_i(t)\Bigg] \\
\leqslant {} & -(a_i(t)+a_{ii}(t))|x_i(t)-y_i(t)| \\
& + \sum_{j=1,j\neq i}^{n} a_{ij}(t)\int_0^{\infty} K_{ij}(s)|x_j(t-s)-y_j(t-s)|\,\mathrm{d}s \\
& + a_{ii}(t)\int_0^{\infty} \int_{t-s}^t K_{ii}(s)\Bigg\{|x_i(u)-y_i(u)|\Bigg[r_i(u)+a_i(u)y_i(u) \\
& + \sum_{j=1}^{n} a_{ij}(u)\int_0^{\infty} K_{ij}(r)y_j(u-r)\,\mathrm{d}r\Bigg] + x_i(u)\Bigg[a_i(u)|x_i(u)-y_i(u)| \\
& + \sum_{j=1}^{n} a_{ij}(u)\int_0^{\infty} K_{ij}(r)|x_j(u-r)-y_j(u-r)|\,\mathrm{d}r\Bigg]\Bigg\}\,\mathrm{d}u\,\mathrm{d}s \\
& + |W_i(t)|.
\end{aligned} \tag{2.16}$$

Define

$$
\begin{aligned}
V_{i2}(t) = {} & \sum_{j=1, j\neq i}^{n} \int_0^{\infty} \int_{t-s}^{t} K_{ij}(s) a_{ij}(u+s) |x_j(u) - y_j(u)| \, \mathrm{d}u \, \mathrm{d}s \\
& + \int_0^{\infty} \int_{t-s}^{t} \int_{v}^{t} K_{ii}(s) a_{ii}(v+s) \Bigg[r_i(u) + a_i(u)(x_i(u) + y_i(u)) \\
& + \sum_{j=1}^{n} a_{ij}(u) \int_0^{\infty} K_{ij}(r) y_j(u-r) \, \mathrm{d}r \Bigg] |x_i(u) - y_i(u)| \, \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}s \\
& + \sum_{j=1}^{n} \int_0^{\infty} \int_{t-s}^{t} \int_{v}^{t} K_{ii}(s) a_{ii}(v+s) a_{ij}(u) x_i(u) \Bigg[\int_0^{\infty} K_{ij}(r) |x_j(u-r) \\
& - y_j(u-r)| \, \mathrm{d}r \Bigg] \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}s. \qquad (2.17)
\end{aligned}
$$

It then follows from (2.16) and (2.17) that

$$
\begin{aligned}
D^+ V_{i1}(t) + \dot{V}_{i2}(t) \leqslant {} & -(a_i(t) + a_{ii}(t)) |x_i(t) - y_i(t)| \\
& + \sum_{j=1, j\neq i}^{n} \int_0^{\infty} K_{ij}(s) a_{ij}(t+s) \, \mathrm{d}s \, |x_j(t) - y_j(t)| \\
& + \int_0^{\infty} \int_{t-s}^{t} K_{ii}(s) a_{ii}(v+s) \, \mathrm{d}v \, \mathrm{d}s \Bigg[r_i(t) + a_i(t)(x_i(t) + y_i(t)) \\
& + \sum_{j=1}^{n} a_{ij}(t) \int_0^{\infty} K_{ij}(r) y_j(t-r) \, \mathrm{d}r \Bigg] |x_i(t) - y_i(t)| \\
& + \sum_{j=1}^{n} a_{ij}(t) x_i(t) \int_0^{\infty} \int_{t-s}^{t} K_{ii}(s) a_{ii}(v+s) \, \mathrm{d}v \, \mathrm{d}s \\
& \times \int_0^{\infty} K_{ij}(r) |x_j(t-r) - y_j(t-r)| \, \mathrm{d}r + |W_i(t)|. \qquad (2.18)
\end{aligned}
$$

We now define

$$
V_i(t) = V_{i1}(t) + V_{i2}(t) + V_{i3}(t), \qquad (2.19)
$$

in which

$$V_{i3}(t) = \sum_{j=1}^{n} \int_0^\infty \int_{t-r}^{t} \int_0^\infty \int_{u+r-s}^{u+r} K_{ii}(s) a_{ii}(v+s) K_{ij}(r) \times a_{ij}(u+r) x_i(u+r) |x_j(u) - y_j(u)| \, \mathrm{d}v \, \mathrm{d}s \, \mathrm{d}u \, \mathrm{d}r. \tag{2.20}$$

Then it follows from (2.18), (2.19) and (2.20) that

$$\begin{aligned} D^+ V_i(t) \leqslant & -(a_i(t) + a_{ii}(t))|x_i(t) - y_i(t)| \\ & + \sum_{j=1, j\neq i}^{n} \int_0^\infty K_{ij}(s) a_{ij}(t+s) \, \mathrm{d}s \, |x_j(t) - y_j(t)| \\ & + \int_0^\infty \int_{t-s}^{t} K_{ii}(s) a_{ii}(v+s) \, \mathrm{d}v \, \mathrm{d}s \Bigg[r_i(t) + a_i(t)(x_i(t) + y_i(t)) \\ & + \sum_{j=1}^{n} a_{ij}(t) \int_0^\infty K_{ij}(r) y_j(t-r) \, \mathrm{d}r \Bigg] |x_i(t) - y_i(t)| \\ & + \sum_{j=1}^{n} \int_0^\infty \int_0^\infty \int_{t+r-s}^{t+r} K_{ii}(s) a_{ii}(v+s) \\ & \times K_{ij}(r) a_{ij}(t+r) x_i(t+r) \, \mathrm{d}v \, \mathrm{d}s \, \mathrm{d}r \, |x_j(t) - y_j(t)| + |W_i(t)| \\ = & -(a_i(t) + a_{ii}(t))|x_i(t) - y_i(t)| \\ & + \sum_{j=1, j\neq i}^{n} \int_0^\infty K_{ij}(s) a_{ij}(t+s) \, \mathrm{d}s \, |x_j(t) - y_j(t)| \\ & + [r_i(t) + a_i(t)(x_i(t) + y_i(t))] \int_0^\infty \int_{t-s}^{t} K_{ii}(s) a_{ii}(v+s) \, \mathrm{d}v \, \mathrm{d}s \\ & \times |x_i(t) - y_i(t)| + \int_0^\infty \int_{t-s}^{t} K_{ii}(s) a_{ii}(v+s) \, \mathrm{d}v \, \mathrm{d}s \\ & \times \sum_{j=1}^{n} a_{ij}(t) \int_0^{t-T} [K_{ij}(r) y_j(t-r) \, \mathrm{d}r] |x_i(t) - y_i(t)| \\ & + \sum_{j=1}^{n} \int_0^\infty \int_0^\infty \int_{t+r-s}^{t+r} K_{ii}(s) a_{ii}(v+s) \\ & \times K_{ij}(r) a_{ij}(t+r) x_i(t+r) \, \mathrm{d}v \, \mathrm{d}s \, \mathrm{d}r \, |x_j(t) - y_j(t)| + |W_i(t)| + B_i(t), \end{aligned} \tag{2.21}$$

where

$$B_i(t) = \int_0^\infty \int_{t-s}^{t} K_{ii}(s) a_{ii}(v+s) \, \mathrm{d}v \, \mathrm{d}s \sum_{j=1}^{n} a_{ij}(t) \times \int_{t-T}^{\infty} K_{ij}(r) y_j(t-r) \, \mathrm{d}r \, |x_i(t) - y_i(t)|.$$

By (2.13), $x_i(t)\leqslant M_i,\ y_j(t)\leqslant M_j\ (i,j=1,2,\ldots,n)$ for $t\geqslant T$. This, together with (2.21), for $t\geqslant T$, leads to

$$\begin{aligned}D^+V_i(t)\leqslant &-(a_i(t)+a_{ii}(t))|x_i(t)-y_i(t)|+\sum_{j=1,j\neq i}^{n}\int_0^\infty K_{ij}(s)a_{ij}(t+s)\,\mathrm{d}s\,|x_j(t)\\ &-y_j(t)|+\left[r_i(t)+2a_i(t)M_i+\sum_{j=1}^{n}a_{ij}(t)M_j\right]\\ &\times\int_0^\infty\int_{t-s}^{t}K_{ii}(s)a_{ii}(v+s)\,\mathrm{d}v\,\mathrm{d}s\,|x_i(t)-y_i(t)|\\ &+M_i\sum_{j=1}^{n}\int_0^\infty\int_0^\infty\int_{t+r-s}^{t+r}K_{ii}(s)a_{ii}(v+s)K_{ij}(r)a_{ij}(t+r)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}r\\ &\times|x_j(t)-y_j(t)|+|W_i(t)|+B_i(t).\end{aligned}\tag{2.22}$$

Now we define a Lyapunov functional $V(t)$ as

$$V(t)=\sum_{i=1}^{n}c_iV_i(t).\tag{2.23}$$

It then follows from (2.22) and (2.23) that for $t\geqslant T$

$$D^+V(t)\leqslant -\sum_{i=1}^{n}A_i(t)|x_i(t)-y_i(t)|+W(t)+B(t),\tag{2.24}$$

where

$$\begin{aligned}A_i(t)=\;&c_i(a_i(t)+a_{ii}(t))-\sum_{j=1,j\neq i}^{n}c_j\int_0^\infty K_{ji}(s)a_{ji}(t+s)\,\mathrm{d}s\\ &-c_i\left(r_i(t)+2a_i(t)M_i+\sum_{j=1}^{n}a_{ij}(t)M_j\right)\int_0^\infty\int_{t-s}^{t}K_{ii}(s)a_{ii}(v+s)\,\mathrm{d}v\,\mathrm{d}s\\ &-\sum_{j=1}^{n}c_jM_j\int_0^\infty\int_0^\infty\int_{t+r-s}^{t+r}K_{jj}(s)a_{jj}(v+s)K_{ji}(r)a_{ji}(t+r)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}r,\\ &i=1,2,\ldots,n,\\ W(t)=\;&\sum_{i=1}^{n}c_i|W_i(t)|,\\ B(t)=\;&\sum_{i=1}^{n}c_iB_i(t).\end{aligned}$$

By the hypotheses in (H3), there exist constants $\alpha_i>0$ $(i=1,2,\ldots,n)$ and $T_1^*\geqslant T$ such that

$$A_i(t)\geqslant\alpha_i>0\quad\text{for } t\geqslant T_1^*.\tag{2.25}$$

Noting that

$$W(t) \leqslant \sum_{i=1}^{n} c_i a_{ii}^M \left[|x_i(t) - y_i(t)| + \sup_{-\infty < s \leqslant 0} |x_i(s) - y_i(s)| \right] \int_t^{\infty} K_{ii}(s)\,\mathrm{d}s,$$

$$B(t) \leqslant \sum_{i=1}^{n} c_i a_{ii}^M \sigma_{ii} |x_i(t) - y_i(t)| \sum_{j=1}^{n} a_{ij}^M \sup_{-\infty < s \leqslant T} y_j(s) \int_{t-T}^{\infty} K_{ij}(s)\,\mathrm{d}s. \tag{2.26}$$

By assumption (H2), $\int_t^{\infty} K_{ii}(s)\,\mathrm{d}s \to 0$ and $\int_{t-T}^{\infty} K_{ij}(s)\,\mathrm{d}s \to 0$ as $t \to \infty$, $i, j = 1, 2, \ldots, n$. It follows from (2.26) that there exist δ_i, η_i, $0 < \delta_i < \frac{1}{2}\alpha_i$, $0 < \eta_i < \frac{1}{2}\alpha_i$ and a $T^* \geqslant T_1^*$ such that for $t \geqslant T^*$

$$\begin{aligned} W(t) &\leqslant \sum_{i=1}^{n} \delta_i |x_i(t) - y_i(t)| + Z(t), \\ B(t) &\leqslant \sum_{i=1}^{n} \eta_i |x_i(t) - y_i(t)|, \end{aligned} \tag{2.27}$$

where

$$Z(t) = \sum_{i=1}^{n} c_i a_{ii}^M \sup_{-\infty < s \leqslant 0} |x_i(s) - y_i(s)| \int_t^{\infty} K_{ii}(s)\,\mathrm{d}s.$$

Integrating both sides of (2.24) on interval $[T^*, t]$,

$$\begin{aligned} &V(t) + \sum_{i=1}^{n} \int_{T^*}^{t} A_i(s) |x_i(s) - y_i(s)|\,\mathrm{d}s \\ &\quad \leqslant V(T^*) + \int_{T^*}^{t} W(s)\,\mathrm{d}s + \int_{T^*}^{t} B(s)\,\mathrm{d}s \quad \text{for } t \geqslant T^*. \end{aligned} \tag{2.28}$$

It follows from (2.25), (2.27) and (2.28) that

$$\begin{aligned} &V(t) + \sum_{i=1}^{n} (\alpha_i - \delta_i - \eta_i) \int_{T^*}^{t} |x_i(s) - y_i(s)|\,\mathrm{d}s \\ &\quad \leqslant V(T^*) + \int_{T^*}^{t} Z(s)\,\mathrm{d}s \quad \text{for } t \geqslant T^*. \end{aligned} \tag{2.29}$$

Note that

$$\begin{aligned} \int_{T^*}^{t} Z(s)\,\mathrm{d}s &= \sum_{i=1}^{n} c_i a_{ii}^M \sup_{-\infty < s \leqslant 0} |x_i(s) - y_i(s)| \int_{T^*}^{t} \int_s^{\infty} K_{ii}(r)\,\mathrm{d}r\,\mathrm{d}s \\ &\leqslant \sum_{i=1}^{n} c_i a_{ii}^M \sup_{-\infty < s \leqslant 0} |x_i(s) - y_i(s)| \int_0^{\infty} s K_{ii}(s)\,\mathrm{d}s. \end{aligned} \tag{2.30}$$

It therefore follows from (2.29) and (2.30) that for $t \geqslant T^*$

$$V(t) + \sum_{i=1}^{n}(\alpha_i - \delta_i - \eta_i)\int_{T^*}^{t} |x_i(s) - y_i(s)|\,\mathrm{d}s \leqslant C, \tag{2.31}$$

for some constant $C > 0$. Therefore, $V(t)$ is bounded on $[T^*, \infty)$ and also

$$\int_{T^*}^{\infty} |x_i(s) - y_i(s)|\,\mathrm{d}s < \infty, \quad i = 1, 2, \ldots, n. \tag{2.32}$$

By Lemma 2.1, $|x_i(t) - y_i(t)|$ is bounded on $[T^*, \infty),\ i = 1, 2, \ldots, n$.

On the other hand, it is easy to see that $\dot{x}_i(t)$ and $\dot{y}_i(t)$ $(i = 1, 2, \ldots, n)$ are bounded for $t \geqslant T^*$. Therefore, $|x_i(t) - y_i(t)|,\ i = 1, 2, \ldots, n$, is uniformly continuous on $[T^*, \infty)$. By Barbalat's lemma (see [12]), one can conclude that

$$\lim_{t\to\infty} |x_i(t) - y_i(t)| = 0, \quad i = 1, 2, \ldots, n. \tag{2.33}$$

The proof of the theorem is complete. □

Remark 1. By Theorem 2.1 and the notation in (1.4), one can get a set of easily verifiable sufficient conditions that guarantee the global asymptotic stability of the positive solution of system (1.1). The following result gives an estimate on the size of delays in terms of the coefficients and the upper bounds $r_i^M,\ a_{ij}^M$ and $M_i,\ i, j = 1, 2, \ldots, n$.

Corollary 2.1. *Assume* (H1)–(H2) *hold. Then the positive solution of system* (1.1) *with initial conditions* (1.3) *is globally asymptotically stable provided that*

(H4) *there exist* $c_i > 0,\ i = 1, 2, \ldots, n$, *such that the following hold*:

$$\liminf_{t\to\infty}\Bigg[c_i(a_i(t) + a_{ii}(t)) - \sum_{j=1, j\neq i}^{n} c_j \int_0^{\infty} K_{ji}(s) a_{ji}(t+s)\,\mathrm{d}s - c_i a_{ii}^M \sigma_{ii}\left(r_i^M + 2a_i^M M_i + \sum_{j=1}^{n} a_{ij}^M M_j\right) - \sum_{j=1}^{n} c_j a_{jj}^M a_{ji}^M M_j \sigma_{jj}\Bigg] > 0.$$

Remark 2. If $K_{ij}(s) \equiv \delta(s)$, the Kronecker delta function, $i, j = 1, 2, \ldots, n$, then $\sigma_{ij} = 0$ and system (1.1) reduces to an instantaneous system i.e. one without delay

$$\dot{x}_i(t) = x_i(t)(r_i(t) - a_i(t)x_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t)), \quad i = 1, 2, \ldots, n. \tag{2.34}$$

On substituting $K_{ij}(s) = \delta(s)$ and $\sigma_{ij} = 0$ $(i, j = 1, 2, \dots, n)$ into (H4), Corollary 2.1 yields that the positive solution of system (2.34) is globally asymptotically stable provided that

(H5) there exist positive constants c_i $(i = 1, 2, \dots, n)$ such that

$$\liminf_{t\to\infty}\left[c_i(a_i(t) + a_{ii}(t)) - \sum_{j=1, j\neq i}^{n} c_j a_{ji}(t)\right] > 0.$$

We therefore have the following corollary to Corollary 2.1.

Corollary 2.2. *The positive solution of system* (1.1) *with initial conditions* (1.3) *is globally asymptotically stable if the positive solution of the corresponding instantaneous system* (*without time delay*) *is globally asymptotically stable and* σ_{ij} *is sufficiently small satisfying* (H4).

In this paper, we have shown that *small delays are negligible* for the global asymptotic stability of the nonautonomous n-species Lotka–Volterra competitive systems with infinite delays provided that the delayed negative feedbacks dominate other interspecific interaction effects with delays. We note that Theorem 2.1 cannot be obtained by [10, Theorem 1], and [11, Theorem 3], when $a_i(t) \equiv 0$, $i = 1, 2, \dots, n$.

Finally, we give a suitable example to illustrate the feasibility of our main result.

Example. We consider the following delayed system

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left[2 + \sin t - 6\int_0^\infty \alpha_{11}e^{-\alpha_{11}s}x_1(t-s)\,ds\right.\\ &\qquad\left. - (2 + \sin t)\int_0^\infty \alpha_{12}e^{-\alpha_{12}s}x_2(t-s)\,ds\right],\\ \dot{x}_2(t) &= x_2(t)\left[2 + \cos t - (2 + \sin t)\int_0^\infty \alpha_{21}e^{-\alpha_{21}s}x_1(t-s)\,ds\right.\\ &\qquad\left. - 9\int_0^\infty \alpha_{22}e^{-\alpha_{22}s}x_2(t-s)\,ds\right]\end{aligned} \tag{2.35}$$

with $\alpha_{ij} > 0,\ i, j = 1, 2$. In this case, $K_{ij}(s) = \alpha_{ij}e^{-\alpha_{ij}s}$ and it follows directly that $K_{ij}(s)$ satisfies (1.2) and that $\sigma_{ij} = 1/\alpha_{ij}$.

We set $c_1 = c_2 = 1$. Then it is easy to examine that

$$A_1(t) = 4 - \frac{\alpha_{21}}{1+\alpha_{21}^2}(\alpha_{21}\sin t + \cos t) - 6\sigma_{11}[12M_1 + (2+\sin t)(1+M_2)] - 9M_2\sigma_{22}\left[2 + \frac{\alpha_{21}}{1+\alpha_{21}^2}(\alpha_{21}\sin t + \cos t)\right],$$

$$A_2(t) = 7 - \frac{\alpha_{12}}{1+\alpha_{12}^2}(\alpha_{12}\sin t + \cos t) - 6M_1\sigma_{11}\left[2 + \frac{\alpha_{12}}{1+\alpha_{12}^2}(\alpha_{12}\sin t + \cos t)\right] - 9\sigma_{22}[2 + \cos t + (2+\sin t)M_1 + 18M_2],$$

where

$$M_1 = \frac{e}{2(e-1)}\mathrm{e}^{3\sigma_{11}}, \quad M_2 = \frac{e}{3(e-1)}\mathrm{e}^{3\sigma_{22}}.$$

Obviously, $A_i(t) > 0$ $(i = 1,2)$ provided that σ_{ii} $(i = 1,2)$ are sufficiently small. Therefore, by Theorem 2.1, if $A_i(t) > 0$, $i = 1,2$, then the positive solution of system (2.35) is globally asymptotically stable. We note that this property cannot be obtained by [10, Theorem 1], and [11, Theorem 3], because $a_i(t) \equiv 0$.

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