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Travelling wave and convergence in stage-structured reaction–diffusion competitive models with nonlocal delays

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Abstract

In this paper, we first investigate a stage-structured competitive model with time delays, harvesting, and nonlocal spatial effect. By using an iterative technique recently developed by Wu and Zou (Wu J, Zou X. Travelling wave fronts of reaction–diffusion systems with delay. J Dynam Differen Equat 2001;13:651–87), sufficient conditions are established for the existence of travelling front solution connecting the two boundary equilibria in the case when there is no positive equilibrium. The travelling wave front corresponds to an invasion by a stronger species which drives the weaker species to extinction. Secondly, we consider a stage-structured competitive model with time delays and nonlocal spatial effect when the domain is finite. We prove the global stability of each of the nonnegative equilibria and demonstrate that the more complex model studied here admits three possible long term behaviors: *coexistence, bistability* and *dominance* as is the case for the standard Lotka–Voltera competitive model.

1. Introduction

Population models with stage structure are of current research interest in mathematical biology. They can exhibit phenomena similar to those of partial differential equations and many important physiological parameters can be incorporated [4]. Moreover, they are often much simpler than the corresponding models governed by partial differential equations. There has been much work on modelling stage-structured population models (see, for example, [1–13,18–22,24,25]). In [1], the work of Aiello and Freedman on a single species growth model with stage structure, represents a mathematically careful and biologically meaningful approach. In [1], a model of single species population growth incorporating stage structure as a reasonable generalization of the classical logistic model was derived and investigated. This model assumes an average age to maturity which appears as a constant time delay reflecting a delayed birth of immatures and a reduced survival of immatures to their maturity. The model takes the form

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$$\dot{u}_{\mathrm{i}}(t) = \alpha u_{\mathrm{m}}(t) - \gamma u_{\mathrm{i}}(t) - \alpha \mathrm{e}^{-\gamma\tau} u_{\mathrm{m}}(t-\tau),$$
$$\dot{u}_{\mathrm{m}}(t) = \alpha \mathrm{e}^{-\gamma\tau} u_{\mathrm{m}}(t-\tau) - \beta u_{\mathrm{m}}^2(t), \quad t > \tau, \tag{1.1}$$

where $u_{\mathrm{i}}(t)$ denotes the immature population density, $u_{\mathrm{m}}(t)$ represents the mature population density, $\alpha > 0$ represents the birth rate, $\gamma > 0$ is the immature death rate, $\beta > 0$ is the mature death and overcrowding rate, τ is the time to maturity. The term $\alpha \mathrm{e}^{-\gamma\tau} u_{\mathrm{m}}(t-\tau)$ represents the immatures who were born at time $t-\tau$ and survive at time t (with the immature death rate γ), and therefore represents the transformation of immatures to matures.

In [6], Gourley and Kuang studied a diffusion version of the system (1.1) to allow for individuals moving around. When motion is allowed for, the derivation of the time-delayed term has to be altered to take proper account of motion. Let $u_{\mathrm{i}}(t,x)$ and $u_{\mathrm{m}}(t,x)$ be the densities of the immature and the mature population at time t and location x. They argued that the delayed term $\alpha \mathrm{e}^{-\gamma\tau} u_{\mathrm{m}}(t-\tau)$ appearing twice in system (1.1), and representing the rate of leaving the immature and entering the mature class, must be replaced by

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_i \tau}} \mathrm{e}^{-(x-y)^2/4d_i\tau} \mathrm{e}^{-\gamma\tau} \alpha u_{\mathrm{m}}(t-\tau, y)\,\mathrm{d}y, \tag{1.2}$$

where $d_i > 0$ is the diffusivity of the immature species. Expression (1.2) allows for the fact that an individual that enters the mature class at location x will most likely have been born at some other point y. In (1.2), $\alpha u_{\mathrm{m}}(t-\tau, y)$ represents the number born at time $t-\tau$ and location y, $\mathrm{e}^{-\gamma\tau} \alpha u_{\mathrm{m}}(t-\tau, y)$ represents the number born at time $t-\tau$ and location y and is still alive at time t, the term

$$\frac{1}{\sqrt{4\pi d_i \tau}} \mathrm{e}^{-(x-y)^2/4d_i\tau} \mathrm{e}^{-\gamma\tau} \alpha u_{\mathrm{m}}(t-\tau, y)\,\mathrm{d}y$$

represents the number born at time $t-\tau$ and location y and still alive at time t and now at location x. Finally, the integral (1.2) totals up the contributions from all parts of the domain. Based on the assumptions above, Gourley and Kuang [6] discussed the following reaction–diffusion single-species population model with stage structure:

$$\frac{\partial u_{\mathrm{i}}}{\partial t} = d_i \frac{\partial^2 u_{\mathrm{i}}}{\partial x^2} + \alpha u_{\mathrm{m}} - \gamma u_{\mathrm{i}} - \alpha \mathrm{e}^{-\gamma\tau} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_i \tau}} \mathrm{e}^{-(x-y)^2/4d_i\tau} u_{\mathrm{m}}(t-\tau, y)\,\mathrm{d}y,$$
$$\frac{\partial u_{\mathrm{m}}}{\partial t} = d_{\mathrm{m}} \frac{\partial^2 u_{\mathrm{m}}}{\partial x^2} + \alpha \mathrm{e}^{-\gamma\tau} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_i \tau}} \mathrm{e}^{-(x-y)^2/4d_i\tau} u_{\mathrm{m}}(t-\tau, y)\,\mathrm{d}y - \beta u_{\mathrm{m}}^2 \tag{1.3}$$

for $t > 0$ and $x \in (-\infty, \infty)$. In the case of finite domain, through the use of a sub/supersolution pair method, it is shown in [6] that system (1.3) continue to generate simple global dynamics (the positive solution of system (1.3) uniformly converges to the positive equilibrium). In the case of infinite domain, they also shown the possibility of travelling wavefront solutions of the scalar equation for the mature population.

Motivated by the work of Gourley and Kuang [6], in the present paper we are concerned with the following reaction–diffusion competitive model with stage structure, harvesting, and nonlocal spatial effect:

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= D_1 \frac{\partial^2 u_1}{\partial x^2} + \alpha_1 \mathrm{e}^{-\gamma_1\tau_1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_1 \tau_1}} \mathrm{e}^{-(x-y)^2/4d_1\tau_1} u_1(t-\tau_1, y)\,\mathrm{d}y \\
&\quad - \beta_1 u_1^2(t,x) - E_1 u_1(t,x) - a_1 u_1(t,x) u_2(t,x), \\
\frac{\partial v_1}{\partial t} &= d_1 \frac{\partial^2 v_1}{\partial x^2} + \alpha_1 u_1 - \gamma_1 v_1 - \alpha_1 \mathrm{e}^{-\gamma_1\tau_1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_1 \tau_1}} \mathrm{e}^{-(x-y)^2/4d_1\tau_1} u_1(t-\tau_1, y)\,\mathrm{d}y, \\
\frac{\partial u_2}{\partial t} &= D_2 \frac{\partial^2 u_2}{\partial x^2} + \alpha_2 \mathrm{e}^{-\gamma_2\tau_2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_2 \tau_2}} \mathrm{e}^{-(x-y)^2/4d_2\tau_2} u_2(t-\tau_2, y)\,\mathrm{d}y \\
&\quad - \beta_2 u_2^2(t,x) - E_2 u_2(t,x) - a_2 u_1(t,x) u_2(t,x), \\
\frac{\partial v_2}{\partial t} &= d_2 \frac{\partial^2 v_2}{\partial x^2} + \alpha_2 u_2 - \gamma_2 v_2 - \alpha_2 \mathrm{e}^{-\gamma_2\tau_2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_2 \tau_2}} \mathrm{e}^{-(x-y)^2/4d_2\tau_2} u_2(t-\tau_2, y)\,\mathrm{d}y
\end{aligned} \tag{1.4}$$

for $t > 0$, $x \in (-\infty, \infty)$.

In system (1.4), $u_1(t,x)$ and $u_2(t,x)$ represent the densities of the mature populations of two species at time t and location x, respectively; $v_1(t,x)$ and $v_2(t,x)$ represent the densities of the immature populations of two species at time t and location x, respectively. The parameters a_i, d_i, D_i, E_i, α_i, β_i, γ_i, τ_i $(i = 1, 2)$ are positive constants, where a_1 and a_2 are the rate of competition between the two mature competitors. It is assumed in (1.4) that the competition happens only between the mature individuals. E_1 and E_2 denote, respectively, the harvesting effort on the two mature populations.

This paper is organized as follows. In the next section, we first introduce some notations and terminology, and show the existence of a travelling wavefront solution to the two coupled equations of the mature populations (the first and the third equations in system (1.4)) by using the technique developed by Wu and Zou [23]. In Section 3, in the case of the domain being finite, by using the coupled lower–upper solution technique developed by Redlinger [17] we investigate the global convergence of the positive solutions to a stage-structured reaction–diffusion competition model with delays and nonlocal spatial effect under homogeneous Neumann boundary conditions.

2. Existence of travelling wavefronts

In this section, we discuss the existence of travelling wavefronts of the equations describing the competition between the adult members of the two species in (1.4), i.e., we study the following subsystem:

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= D_1\frac{\partial^2 u_1}{\partial x^2} + \alpha_1 e^{-\gamma_1\tau_1}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_1\tau_1}}e^{-(x-y)^2/4d_1\tau_1}u_1(t-\tau_1,y)\,dy\\ &\quad - \beta_1 u_1^2(t,x) - E_1u_1(t,x) - a_1u_1(t,x)u_2(t,x),\\ \frac{\partial u_2}{\partial t} &= D_2\frac{\partial^2 u_2}{\partial x^2} + \alpha_2 e^{-\gamma_2\tau_2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_2\tau_2}}e^{-(x-y)^2/4d_2\tau_2}u_2(t-\tau_2,y)\,dy\\ &\quad - \beta_2 u_2^2(t,x) - E_2u_2(t,x) - a_2u_1(t,x)u_2(t,x)\end{aligned} \tag{2.1}$$

for $t>0$, $x\in(-\infty,\infty)$.

System (2.1) always has a trivial equilibrium $E_0(0,0)$. If $\alpha_1e^{-\gamma_1\tau_1}>E_1$, $\alpha_2e^{-\gamma_2\tau_2}>E_2$, then system (2.1) has two semi-trivial equilibria $E_1^*(k_1,0)$ and $E_2^*(0,k_2)$, where

$$k_1=\frac{\alpha_1e^{-\gamma_1\tau_1}-E_1}{\beta_1},\quad k_2=\frac{\alpha_2e^{-\gamma_2\tau_2}-E_2}{\beta_2}. \tag{2.2}$$

If $\beta_2(\alpha_1e^{-\gamma_1\tau_1}-E_1)>a_1(\alpha_2e^{-\gamma_2\tau_2}-E_2)>0$, $\beta_1(\alpha_2e^{-\gamma_2\tau_2}-E_2)>a_2(\alpha_1e^{-\gamma_1\tau_1}-E_1)>0$, then system (2.1) has a unique positive equilibrium $E^*(u_1^*,u_2^*)$, where

$$u_1^*=\frac{\beta_2(\alpha_1e^{-\gamma_1\tau_1}-E_1)-a_1(\alpha_2e^{-\gamma_2\tau_2}-E_2)}{\beta_1\beta_2-a_1a_2},$$
$$u_2^*=\frac{\beta_1(\alpha_2e^{-\gamma_2\tau_2}-E_2)-a_2(\alpha_1e^{-\gamma_1\tau_1}-E_1)}{\beta_1\beta_2-a_1a_2}.$$

In the following, we first discuss the local stability of the nonnegative equilibria E_1^* and E_2^* of system (2.1).

Lemma 2.1. *If the following hold*:

(A1) $\beta_2(\alpha_1e^{-\gamma_1\tau_1}-E_1)>a_1(\alpha_2e^{-\gamma_2\tau_2}-E_2)$,
(A2) $\beta_1(\alpha_2e^{-\gamma_2\tau_2}-E_2)<a_2(\alpha_1e^{-\gamma_1\tau_1}-E_1)$.

Then the nonnegative equilibrium $E_1^*(k_1,0)$ is locally stable and $E_2^*(0,k_2)$ is unstable.

Proof. For the nonnegative equilibrium $E_1^*(k_1,0)$, we find, after some algebra, that the associated linearized system has nontrivial solutions of the form $(c_1,c_2)\exp(\sigma t+ikx)$ if and only if the following holds:

$$\begin{vmatrix} f_1(\sigma,k^2) & \frac{a_1}{\beta_1}(\alpha_1e^{-\gamma_1\tau_1}-E_1)\\ 0 & f_2(\sigma,k^2)\end{vmatrix}=0,$$

where

$$f_1(\sigma,k^2)=\sigma-\alpha_1e^{-\gamma_1\tau_1}e^{-\sigma\tau_1}e^{-d_1k^2\tau_1}+D_1k^2+2\alpha_1e^{-\gamma_1\tau_1}-E_1,$$
$$f_2(\sigma,k^2)=\sigma-\alpha_2e^{-\gamma_2\tau_2}e^{-\sigma\tau_2}e^{-d_2k^2\tau_2}+D_2k^2+E_2+\frac{a_2}{\beta_1}(\alpha_1e^{-\gamma_1\tau_1}-E_1).$$

In what follows, by a *root* we mean a value $\hat{\sigma}(k^2)$ satisfying $f_i(\hat{\sigma}, k^2) = 0$. Clearly, when $\tau_1 = 0$, equation $f_1(\sigma, k^2) = 0$ has only one real negative root $-D_1k^2 - (\alpha_1 - E_1)$. We now claim for any $\tau_1 > 0$, the roots of $f_1(\sigma, k^2) = 0$ have only negative real parts. Suppose that $\mathrm{Re}\,\sigma \geqslant 0$. Then it follows that:

$$\mathrm{Re}\,\sigma = \alpha_1 e^{-\gamma_1\tau_1} e^{-\tau_1 \mathrm{Re}\,\sigma} \cos(\tau_1 \mathrm{Im}\,\sigma) e^{-d_1k^2\tau_1} - [D_1k^2 + 2\alpha_1 e^{-\gamma_1\tau_1} - E_1] \leqslant -D_1k^2 - [\alpha_1 e^{-\gamma_1\tau_1} - E] < 0,$$

which is a contradiction. Therefore, we have $\mathrm{Re}\,\sigma < 0$ for all $\tau_1 \geqslant 0$.

We now claim all the roots of $f_2(\sigma, k^2) = 0$ have negative real parts. Suppose otherwise that $\mathrm{Re}\,\sigma \geqslant 0$. Then it follows that:

$$\begin{aligned}\mathrm{Re}\,\sigma &= \alpha_2 e^{-\gamma_2\tau_2} e^{-\tau_2 \mathrm{Re}\,\sigma} \cos(\tau_2 \mathrm{Im}\,\sigma) e^{-d_2k^2\tau_2} - \left[D_2k^2 + E_2 + \frac{a_2}{\beta_1}(\alpha_1 e^{-\gamma_1\tau_1} - E_1)\right] \\ &\leqslant -D_2k^2 + \alpha_2 e^{-\gamma_2\tau_2} - E_2 - \frac{a_2}{\beta_1}(\alpha_1 e^{-\gamma_1\tau_1} - E_1) < 0\end{aligned}$$

since (A2) holds. This is a contradiction and hence, we have $\mathrm{Re}\,\sigma < 0$ for all $\tau_2 \geqslant 0$. Accordingly, it follows by standard argument that $E_1^*(k_1, 0)$ is locally stable.

In a similar way, one can verify that the nonnegative equilibrium $E_2^*(0, k_2)$ is locally unstable if (A1) holds. This completes the proof. □

Remark. Clearly, if (A1)–(A2) hold, then the positive equilibrium $E^*(u_1^*, u_2^*)$ is not feasible. However, if $\beta_2(\alpha_1 e^{-\gamma_1\tau_1} - E_1) > a_1(\alpha_2 e^{-\gamma_2\tau_2} - E_2)$, $\beta_1(\alpha_2 e^{-\gamma_2\tau_2} - E_2) > a_2(\alpha_1 e^{-\gamma_1\tau_1} - E_1)$, we can also discuss the local stability of the positive equilibrium of system (2.1) using similar arguments to those described in the next section, we omit it here.

In order to investigate the existence of travelling wavefronts of system (2.1), in the following we first summarize some basic notations and concepts.

Consider the following reaction–diffusion system with time delay:

$$\frac{\partial u(t,x)}{\partial t} = D\frac{\partial^2 u(t,x)}{\partial x^2} + f(u_t(x)), \tag{2.3}$$

where $t \geqslant 0$, $x \in R$, $u \in R^2$, $D = \mathrm{diag}(D_1, D_2)$ with $D_i > 0$ $(i = 1, 2)$; f: $C([-\tau, 0]; R^2) \to R^2$ is continuous, and $u_t(x)$ is an element in $C([-\tau, 0]; R^2) \to R^2$ parameterized by $x \in R$ and is given by

$$u_t(x)(s) = u(t + s, x), \quad s \in [-\tau, 0], \quad t \geqslant 0, \quad x \in R.$$

A travelling wave solution of (2.3) is a solution of the form $u(t, x) = \phi(x + ct)$, where $\phi \in C^2(R, R^2)$ and $c > 0$ is a constant corresponding to the wave speed. On substituting $u(t, x) = \phi(z) \in R^2, z = x + ct$ with $c \geqslant 0$, we derive from (2.3) that

$$D\phi''(z) - c\phi'(z) + f_c(\phi_z) = 0, \quad z \in R, \tag{2.4}$$

where $\phi_z(\zeta) = \phi(\zeta + z)$, and the function f_c: $X_c := C([-c\tau, 0]; R^2) \to R^2$ is defined by

$$f_c(\psi) = f(\psi^c), \quad \psi^c(s) = \psi(cs), \quad s \in [-\tau, 0].$$

If, for some $c > 0$, (2.4) has a monotone (componentwise) solution defined on R satisfying

$$\lim_{z\to-\infty} \phi(z) = u_-, \quad \lim_{z\to+\infty} \phi(z) = u_+, \tag{2.5}$$

where u_- and u_+ are equilibria of (2.3). Then $u(t, x) = \phi(x + ct)$ is called a wave front of system (2.3) with speed c. Without loss of generality, we can assume $u_- = \mathbf{0}$, $u_+ = \mathbf{K}$, and seek for travelling wave front solution connecting these two equilibria.

We look for wave front solutions to system (2.3) in the following *profile set*:

$$\Gamma^* = \left\{ \phi \in C(R, R^2) : \begin{array}{l} \text{(i) } \phi \text{ is nondecreasing in } R, \\ \text{(ii) } \lim_{z\to-\infty} \phi(z) = \mathbf{0}, \lim_{z\to+\infty} \phi(z) = \mathbf{K} \end{array} \right\}.$$

We also need the following definition of upper and lower solutions to system (2.4).

Definition 2.1. A continuous function $\bar{\phi} : R \to R^2$ is called an upper solution of system (2.4) if $\bar{\phi}$ is twice differentiable almost everywhere in R and satisfies

$$D\bar{\phi}''(z) - c\bar{\phi}'(z) + f_c(\bar{\phi}_z) \leqslant 0. \tag{2.6}$$

A lower solution of (2.4) is defined in a similar way by reversing the inequality in (2.6).

We are now in a position to discuss the existence of travelling wavefronts of system (2.1) by using the upper–lower solution method and an iteration scheme developed by Wu and Zou [23].

To seek a travelling wave front solution of system (2.1), we set $u_1(t,x)=\phi_1(z)$, $u_2(t,x)=\phi_2(z)$, where $z=x+ct$ and $c>0$ is the wave speed. Then system (2.1) becomes

$$\begin{aligned} &D_1\phi_1''(z)-c\phi_1'(z)+\alpha_1\mathrm{e}^{-\gamma_1\tau_1}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_1\tau_1}}\mathrm{e}^{-y^2/4d_1\tau_1}\phi_1(z-c\tau_1-y)\,\mathrm{d}y\\ &\qquad-\beta_1\phi_1^2(z)-E_1\phi_1(z)-a_1\phi_1(z)\phi_2(z)=0,\\ &D_2\phi_2''(z)-c\phi_2'(z)+\alpha_2\mathrm{e}^{-\gamma_2\tau_2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_2\tau_2}}\mathrm{e}^{-y^2/4d_2\tau_2}\phi_2(z-c\tau_2-y)\,\mathrm{d}y\\ &\qquad-\beta_2\phi_2^2(z)-E_2\phi_2(z)-a_2\phi_1(z)\phi_2(z)=0, \end{aligned} \tag{2.7}$$

which will be solved subject to the following conditions:

$$\begin{aligned} &\lim_{z\to-\infty}\phi_1(z)=0,\quad \lim_{z\to+\infty}\phi_1(z)=k_1,\\ &\lim_{z\to-\infty}\phi_2(z)=k_2,\quad \lim_{z\to+\infty}\phi_2(z)=0. \end{aligned} \tag{2.8}$$

The solution of (2.7) and (2.8) corresponds to the travelling wave fronts of (2.1) connecting two equilibria $(0,k_2)$ and $(k_1,0)$.

Now, by making change of variables $\tilde\phi_1=\phi_1$, $\tilde\phi_2=k_2-\phi_2$ and dropping the tildes, (2.7) and (2.8) become, respectively,

$$\begin{aligned} &D_1\phi_1''(z)-c\phi_1'(z)+f_{c1}(\phi_z)=0,\\ &D_2\phi_2''(z)-c\phi_2'(z)+f_{c2}(\phi_z)=0, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\lim_{z\to-\infty}(\phi_1(z),\phi_2(z))=(0,0):=\mathbf{0},\\ &\lim_{z\to+\infty}(\phi_1(z),\phi_2(z))=(k_1,k_2):=\mathbf{K}, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} f_{c1}(\phi_z)&=\alpha_1\mathrm{e}^{-\gamma_1\tau_1}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_1\tau_1}}\mathrm{e}^{-y^2/4d_1\tau_1}\phi_1(z-c\tau_1-y)\,\mathrm{d}y\\ &\quad-\beta_1\phi_1^2(z)-E_1\phi_1(z)-a_1k_2\phi_1(z)+a_1\phi_1(z)\phi_2(z),\\ f_{c2}(\phi_z)&=\alpha_2\mathrm{e}^{-\gamma_2\tau_2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_2\tau_2}}\mathrm{e}^{-y^2/4d_2\tau_2}\phi_2(z-c\tau_2-y)\,\mathrm{d}y\\ &\quad+\beta_2\phi_2^2(z)-(2\beta_2k_2+E_2)\phi_2(z)+a_2k_2\phi_1(z)-a_2\phi_1(z)\phi_2(z). \end{aligned} \tag{2.11}$$

Denote

$$\begin{aligned} \Delta_1(\lambda,c)&=\alpha_1\mathrm{e}^{-\gamma_1\tau_1}\mathrm{e}^{d_1\tau_1\lambda^2-c\tau_1\lambda}-(E_1+a_1k_2+c\lambda-D_1\lambda^2),\\ \Delta_2(\lambda,c)&=\alpha_2\mathrm{e}^{-\gamma_2\tau_2}\mathrm{e}^{d_2\tau_2\lambda^2-c\tau_2\lambda}-(E_2+a_2k_1+c\lambda-D_2\lambda^2). \end{aligned} \tag{2.12}$$

In the following we discuss the property of the roots of equations $\Delta_1(\lambda,c)=0$ and $\Delta_2(\lambda,c)=0$.

Lemma 2.2. *Let (A1)–(A2) hold and $\Delta_1(\lambda,c)$ and $\Delta_2(\lambda,c)$ be defined by (2.12).*

(i) *There exists a $c^*(\tau_1,a_1,\alpha_1,\gamma_1,E_1,d_1,D_1,k_2)>0$ such that for $c>c^*$ the equation $\Delta_1(\lambda,c)=0$ has two positive real roots $0<\lambda_1<\lambda_2$ and*

$$\Delta_1(\lambda,c)\begin{cases}>0 & \text{for } \lambda<\lambda_1,\\ <0 & \text{for } \lambda\in(\lambda_1,\lambda_2),\\ >0 & \text{for } \lambda>\lambda_2.\end{cases} \tag{2.13}$$

(ii) *The equation* $\Delta_2(\lambda, c) = 0$ *has a negative real root* λ_3 *and a positive real root* λ_4 *and*

$$\Delta_2(\lambda, c)\begin{cases} > 0 & \text{for } \lambda < \lambda_3, \\ < 0 & \text{for } \lambda \in (\lambda_3, \lambda_4), \\ > 0 & \text{for } \lambda > \lambda_4. \end{cases} \tag{2.14}$$

Proof

(i) We note that if $c = 0$, then

$$\Delta_1(\lambda, 0) = \alpha_1 e^{-\gamma_1\tau_1} e^{d_1\tau_1\lambda^2} - (E_1 + a_1k_2 - D_1\lambda^2).$$

By (A1) we have $\alpha_1 e^{-\gamma_1\tau_1} > E_1 + a_1k_2$. Hence, $\Delta_1(\lambda, 0) = 0$ has no real roots. By continuity, for sufficiently small $c > 0$, $\Delta_1(\lambda, c) = 0$ still has no real roots.We now consider the critical situation: the curves $y_1(\lambda) = \alpha_1 e^{-\gamma_1\tau_1} e^{d_1\tau_1\lambda^2 - c\tau_1\lambda}$ and $y_2(\lambda) = E_1 + a_1k_2 + c\lambda - D_1\lambda^2$ touch. In this case, we are concerned with

$$\Delta_1(\lambda, c) = 0, \quad \frac{\partial}{\partial\lambda}\Delta_1(\lambda, c) = 0. \tag{2.15}$$

It follows from (2.15) that:

$$g(\lambda) = 2d_1D_1\tau_1\lambda^3 - (D_1c\tau_1 + 2d_1c\tau_1)\lambda^2 + [c^2\tau_1 - 2d_1\tau_1(E_1 + a_1k_2) - 2D_1]\lambda + c\tau_1(E_1 + a_1k_2) + c = 0.$$

Clearly, the equation $g(\lambda) = 0$ has a real negative root. Denote

$$\lambda_0 = \frac{c + \sqrt{c^2 + 4D_1(E_1 + a_1k_2)}}{2D_1}.$$

It is easy to verify that $g(\lambda_0) < 0$. We therefore know that $g(\lambda) = 0$ has two positive real roots λ_1^* and λ_2^* satisfying $0 < \lambda_1^* < \lambda_0 < \lambda_2^*$. We note that $y_2(\lambda_2^*) < y_2(\lambda_0) = 0$. Hence, only λ_1^* is a repeated root of $\Delta_1(\lambda, c) = 0$. On substituting λ_1^* into $\Delta_1(\lambda, c) = 0$, one can get $c^* > 0$ implicitly. Therefore, if $c > c^*$, then the equation $\Delta_1(\lambda, c) = 0$ will have two real positive roots satisfying (2.13).

(ii) Denote $h_1(\lambda) = \alpha_2 e^{-\gamma_2\tau_2} e^{d_2\tau_2\lambda^2 - c\tau_2\lambda}$, $h_2(\lambda) = E_2 + a_2k_1 + c\lambda - D_2\lambda^2$. By (A2) we have $h_1(0) = \alpha_2 e^{-\gamma_2\tau_2} < h_2(0) = E_2 + a_2k_1$. Therefore, $\Delta_2(\lambda, c) = 0$ admits one real negative root and one real positive root satisfying (2.14). The proof is complete. □

Taking

$$\delta_1 > a_1k_2 + 2\beta_1k_1 + E_1, \quad \delta_2 > a_2k_1 + 2\beta_2k_2 + E_2, \tag{2.16}$$

we define

$$H^*(\phi)(z) = f_c(\phi_z) + \delta\phi(z), \quad \phi \in C(R, R^2), \quad z \in R. \tag{2.17}$$

where $\delta = (\delta_1, \delta_2)$. The operator H^* has the following properties:

Lemma 2.3. *For any* $\phi \in \Gamma^*$, *we have*

(i) $H^*(\phi)(z) \geqslant 0$ *for all* $z \in R$;
(ii) $H^*(\phi)(z)$ *is nondecreasing in* $z \in R$;
(iii) $H^*(\psi)(z) \leqslant H^*(\phi)(z)$ *for all* $z \in R$, *provided that* ϕ, $\psi \in C(R; R^2)$ *is such that* $0 \leqslant \psi(z) \leqslant \phi(z) \leqslant K$ *for* $z \in R$.

Proof. Let $H^*(\phi)(z) = (H_1^*(\phi)(z), H_2^*(\phi)(z))$.

(i) Noting that for $\forall z \in R$,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_1\tau_1}} e^{-y^2/4d_1\tau_1}\phi_1(z - c\tau_1 - y)\,dy \geqslant 0,$$

we derive that

$$\begin{aligned} H_1^*(\phi)(z) &= f_{c1}(\phi_z) + \delta_1\phi_1(z) \geqslant \delta_1\phi_1(z) - \beta_1\phi_1^2(z) - E_1\phi_1(z) - a_1k_2\phi_1(z) + a_1\phi_1(z)\phi_2(z) \\ &\geqslant \delta_1\phi_1(z) - \beta_1k_1\phi_1(z) - E_1\phi_1(z) - a_1k_2\phi_1(z) = (\delta_1 - \beta_1k_1 - a_1k_2 - E_1)\phi_1(z) \geqslant 0. \end{aligned}$$

In a similar way we can obtain that $H_2^*(\phi)(z) \geqslant 0$.

(ii) Let $z_1 > z_2$. It follows that:

$$\begin{aligned} H_1^*(\phi)(z_1) - H_1^*(\phi)(z_2) &= f_{c1}(\phi_{z_1}) - f_{c1}(\phi_{z_2}) + \delta_1(\phi_1(z_1) - \phi_1(z_2)) \\ &= \delta_1(\phi_1(z_1) - \phi_1(z_2)) + \alpha_1 e^{-\gamma_1\tau_1} \\ &\quad \times \int_{-\infty}^{\infty}\left\{\frac{1}{\sqrt{4\pi d_1\tau_1}}e^{-y^2/4d_1\tau_1} \times [\phi_1(z_1 - c\tau_1 - y) - \phi_1(z_2 - c\tau_1 - y)]\right\}dy \\ &\quad - \beta_1[\phi_1^2(z_1) - \phi_1^2(z_2)] - (E_1 + a_1k_2)[\phi_1(z_1) - \phi_1(z_2)] + a_1[\phi_1(z_1)\phi_2(z_1) - \phi_1(z_2)\phi_2(z_2)] \\ &\geqslant \delta_1(\phi_1(z_1) - \phi_1(z_2)) - \beta_1(\phi_1(z_1) + \phi_1(z_2))(\phi_1(z_1) - \phi_1(z_2)) \\ &\quad - (E_1 + a_1k_2)[\phi_1(z_1) - \phi_1(z_2)] \\ &\geqslant (\delta_1 - 2\beta_1k_1 - E_1 - a_1k_2)(\phi_1(z_1) - \phi_1(z_2)) \geqslant 0. \end{aligned}$$

Similarly, we can derive $H_1^*(\phi)(z_1) - H_1^*(\phi)(z_2) \geqslant 0$. Therefore, $H^*(\phi)(z)$ is nondecreasing in $z \in R$.

(iii) Let $\phi, \psi \in \Gamma^*$ be such that $0 \leqslant \psi(z) \leqslant \phi(z) \leqslant K$ for $z \in R$. It follows that:

$$\begin{aligned} H_1^*(\phi)(z) - H_1^*(\phi)(z) &= \alpha_1 e^{-\gamma_1\tau_1}\int_{-\infty}^{\infty}\left\{\frac{1}{\sqrt{4\pi d_1\tau_1}}e^{-y^2/4d_1\tau_1} \times [\phi_1(z - c\tau_1 - y) - \psi_1(z - c\tau - y)]\right\}dy \\ &\quad - \beta_1[\phi_1^2(z) - \psi_1^2(z)] - E_1(\phi_1(z) - \psi_1(z)) - a_1k_2(\phi_1(z) - \psi_1(z)) \\ &\quad + a_1(\phi_1(z)\phi_2(z) - \psi_1(z)\psi_2(z)) + \delta_1(\phi_1(z) - \psi_1(z)) \\ &\geqslant -\beta_1[\phi_1^2(z) - \psi_1^2(z)] - E_1(\phi_1(z) - \psi_1(z)) - a_1k_2(\phi_1(z) - \psi_1(z)) + \delta_1(\phi_1(z) - \psi_1(z)) \\ &\geqslant (\delta_1 - 2\beta_1k_1 - E_1 - a_1k_2)(\phi_1(z) - \psi_1(z)) \geqslant 0. \end{aligned}$$

In a similar way, we can derive $H_2^*(\phi)(z) \geqslant H_2^*(\psi)(z)$ for $z \in R$. This completes the proof. □

We note that (2.9) is equivalent to the following system:

$$\begin{aligned} &D_1\phi_1''(z) - c\phi_1'(z) - \delta_1\phi_1(z) + H_1^*(\phi)(z) = 0, \\ &D_2\phi_2''(z) - c\phi_2'(z) - \delta_2\phi_2(z) + H_2^*(\phi)(z) = 0. \end{aligned} \tag{2.18}$$

We now assume that an upper solution $\bar{\phi} \in \Gamma^*$ and a lower solution $\underline{\phi}$ (which is not necessarily in Γ^*) of (2.18) are given (we shall construct such a pair later in this section) so that

(B1) $\mathbf{0} \leqslant \phi(z) \leqslant \bar{\phi}(z) \leqslant \mathbf{K}$ for all $z \in R$;
(B2) $\underline{\phi}(z) \neq 0$.

Consider the following iteration scheme:

$$cx_n'(z) = Dx_n''(z) - \delta x_n(z) + H^*(x_{n-1})(z), \quad z \in R, \quad n = 1, 2, \ldots \tag{2.19}$$

with the boundary conditions

$$\lim_{z\to-\infty} x_n(z) = \mathbf{0}, \quad \lim_{z\to+\infty} x_n(z) = \mathbf{K}, \tag{2.20}$$

where $x_0 = \bar{\phi}$, $D = (D_1, D_2)$, $\delta = (\delta_1\ \delta_2)$, $x_n(z) = (x_{n1}(z), x_{n2}(z))$.

Solving (2.19) and (2.20) for $n = 1, 2, \ldots$, we get a sequence of functions $\{x_n\}_{n=1}^{\infty}$, given by

$$\begin{aligned} &x_{0i}(z) = \bar{\phi}_i(z), \ z \in R, \\ &x_{ni}(z) = \frac{1}{D_i(\lambda_{2i} - \lambda_{1i})}\left[\int_{-\infty}^{t} e^{\lambda_{1i}(z-s)}H_i^*(x_{n-1})(s)\,ds + \int_{t}^{\infty} e^{\lambda_{2i}(z-s)}H_i^*(x_{n-1})(s)\,ds\right], \end{aligned} \tag{2.21}$$

where $i = 1, 2$; $n = 1, 2, \ldots$; $z \in R$, and

$$\lambda_{1i} = \frac{c - \sqrt{c^2 + 4\delta_i D_i}}{2D_i}, \quad \lambda_{2i} = \frac{c + \sqrt{c^2 + 4\delta_i D_i}}{2D_i}. \tag{2.22}$$

Using Lemma 2.3, we can establish the following result (see, [23], Lemmas 3.3–3.4 and Proposition 3.5).

Theorem 2.1. *The sequence of functions* $\{x_n\}_{n=1}^{\infty}$ *satisfies*

(a) $x_n \in \Gamma^*$ *for all* $n = 1, 2, \ldots$;
(b) $\underline{\phi}(z) \leqslant x_n(z) \leqslant x_{n-1}(z) \leqslant \bar{\phi}(z)$ *for all* $n = 1, 2, \ldots$ *and* $z \in R$;
(c) *each* x_n *is an upper solution of* (2.19);
(d) $x(z) := lim_{n\to\infty} x_n(z)$ *is nondecreasing and is a solution of* (2.19) *and* (2.20).

From Theorem 2.1, we see that the existence of travelling wave fronts for system (2.1) follows from the existence of a pair of upper and lower solutions of (2.19) (or (2.9)) satisfying (B1)–(B2). We now construct such a pair of upper and lower solutions for (2.18) (or (2.9)). We shall use the same upper–lower solutions as in [3]. However, here we need to tackle in detail the nonlocal terms

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_i \tau_i}} e^{-y^2/4d_i\tau_i} \phi_i(z - c\tau_i - y)\,dy \quad (i = 1, 2).$$

Define

$$\bar{\phi}_1(z) = \begin{cases} k_1 e^{\lambda_1 z}, & z \leqslant 0, \\ k_1, & z > 0, \end{cases} \quad \bar{\phi}_2(z) = \begin{cases} k_2 e^{\lambda_1 z}, & z \leqslant 0, \\ k_2, & z > 0, \end{cases}$$

We now claim that if $\Delta_2(\lambda_1, c) \leqslant 2(\beta_2 k_2 - a_2 k_1)$, $\bar{\phi}(z) = (\bar{\phi}_1(z), \bar{\phi}_2(z))$ is an upper solution of (2.9).

Clearly, $\bar{\phi}(z) \in \Gamma^*$.

(i) If $z > 0$, then $\bar{\phi}_1(z) = k_1$, $\bar{\phi}_2(z) = k_2$. Noting that $\bar{\phi}_1(z) \leqslant k_1$ for $z \leqslant 0$, it follows that:

$$\begin{aligned} D_1\bar{\phi}_1''(z) - c\bar{\phi}_1'(z) + f_{c1}(\bar{\phi}_z) &\leqslant \alpha_1 e^{-\gamma_1\tau_1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_1\tau_1}} e^{-y^2/4d_1\tau_1} \bar{\phi}_1(z - c\tau_1 - y)\,dy - \beta_1 k_1^2 - E_1 k_1 - a_1 k_1 k_2 + a_1 k_1 k_2 \\ &\leqslant k_1 \alpha_1 e^{-\gamma_1\tau_1} - \beta_1 k_1^2 - E_1 k_1 = 0. \end{aligned}$$

Similarly, we have

$$D_2\bar{\phi}_2''(z) - c\bar{\phi}_2'(z) + f_{c2}(\bar{\phi}_z) \leqslant 0.$$

(ii) For $z \leqslant 0$, we have $\bar{\phi}_1(z) = k_1 e^{\lambda_1 z}$, $\bar{\phi}_2(z) = k_2 e^{\lambda_1 z}$. On substituting into (2.9), we derive that

$$\begin{aligned} D_1\bar{\phi}_1''(z) - c\bar{\phi}_1'(z) + f_{c1}(\bar{\phi}_z) &= \alpha_1 e^{-\gamma_1\tau_1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_1\tau_1}} e^{-y^2/4d_1\tau_1} \bar{\phi}_1(z - c\tau_1 - y)\,dy + D_1 k_1 \lambda_1^2 e^{\lambda_1 z} - c k_1 \lambda_1 e^{\lambda_1 z} - \beta_1 k_1^2 e^{2\lambda_1 z} \\ &\quad - E_1 k_1 e^{\lambda_1 z} - a_1 k_1 k_2 e^{\lambda_1 z} + a_1 k_1 k_2 e^{2\lambda_1 z}. \end{aligned}$$

We note that if $z - c\tau_1 \leqslant y$, then $\bar{\phi}_1(z - c\tau_1 - y) = k_1 e^{\lambda_1(z - c\tau_1 - y)}$; if $z - c\tau_1 > y$, then $\bar{\phi}_1(z - c\tau_1 - y) = k_1 \leqslant k_1 e^{\lambda_1(z - c\tau_1 - y)}$. It therefore follows that:

$$\begin{aligned} D_1\bar{\phi}_1''(z) - c\bar{\phi}_1'(z) + f_{c1}(\bar{\phi}_z) &\leqslant \alpha_1 k_1 e^{-\gamma_1\tau_1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_1\tau_1}} e^{-y^2/4d_1\tau_1} e^{\lambda_1(z - c\tau_1 - y)}\,dy + D_1 k_1 \lambda_1^2 e^{\lambda_1 z} - c k_1 \lambda_1 e^{\lambda_1 z} - \beta_1 k_1^2 e^{2\lambda_1 z} \\ &\quad - E_1 k_1 e^{\lambda_1 z} - a_1 k_1 k_2 e^{\lambda_1 z} + a_1 k_1 k_2 e^{2\lambda_1 z} \\ &= \alpha_1 k_1 e^{-\gamma_1\tau_1} e^{d_1\tau_1\lambda_1^2 + \lambda_1(z - c\tau_1)} + D_1 k_1 \lambda_1^2 e^{\lambda_1 z} - c k_1 \lambda_1 e^{\lambda_1 z} - \beta_1 k_1^2 e^{2\lambda_1 z} - E_1 k_1 e^{\lambda_1 z} - a_1 k_1 k_2 e^{\lambda_1 z} \\ &\quad + a_1 k_1 k_2 e^{2\lambda_1 z} \\ &= k_1 e^{\lambda_1 z}[\alpha_1 e^{-\gamma_1\tau_1} e^{d_1\tau_1\lambda_1^2 - c\tau_1\lambda_1} + D_1\lambda_1^2 - c\lambda_1 - E_1 - a_1 k_2] + k_1 e^{2\lambda_1 z}(a_1 k_2 - \beta_1 k_1) \\ &= k_1 e^{\lambda_1 z} \Delta_1(\lambda_1, c) + k_1 e^{2\lambda_1 z}(a_1 k_2 - \beta_1 k_1) = k_1 e^{2\lambda_1 z}(a_1 k_2 - \beta_1 k_1). \end{aligned}$$

By (A2) we see that $a_1 k_2 < \beta_1 k_1$. We therefore have

$$D_1\bar{\phi}_1''(z) - c\bar{\phi}_1'(z) + f_{c1}(\bar{\phi}_z) \leqslant 0.$$

In a similar way, for $\bar{\phi}_2(z)$, we derive that

$$\begin{aligned}
D_2\bar{\phi}_2''(z) - c\bar{\phi}_2'(z) + f_{c2}(\bar{\phi}_z) &= \alpha_2 e^{-\gamma_2\tau_2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_2\tau_2}}e^{-y^2/4d_2\tau_2}\bar{\phi}_2(z-c\tau_2-y)\,dy + D_2k_2\lambda_1^2e^{\lambda_1 z} - ck_2\lambda_1e^{\lambda_1 z} + \beta_2k_2^2e^{2\lambda_1 z}\\
&\quad - (2\beta_2k_2+E_2)k_2e^{\lambda_1 z} + a_2k_2k_1e^{\lambda_1 z} - a_2k_1k_2e^{2\lambda_1 z}\\
&\leqslant \alpha_2 e^{-\gamma_2\tau_2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_2\tau_2}}e^{-y^2/4d_2\tau_2}e^{\lambda_1(z-c\tau_2-y)}\,dy + D_2k_2\lambda_1^2e^{\lambda_1 z} - ck_2\lambda_1e^{\lambda_1 z} + \beta_2k_2^2e^{2\lambda_1 z}\\
&\quad - (2\beta_2k_2+E_2)k_2e^{\lambda_1 z} + a_2k_2k_1e^{\lambda_1 z} - a_2k_1k_2e^{2\lambda_1 z}\\
&= k_2e^{\lambda_1 z}[\alpha_2e^{-\gamma_2\tau_2}e^{d_2\tau_2\lambda_1^2-c\tau_2\lambda_1} + D_2\lambda_1^2 - c\lambda_1 - 2\beta_2k_2 - E_2 + a_2k_1] + k_2e^{2\lambda_1 z}(\beta_2k_2 - a_2k_1)\\
&= k_2e^{\lambda_1 z}(\Delta_2(\lambda_1,c) - 2\beta_2k_2 + 2a_2k_1) + k_2e^{2\lambda_1 z}(\beta_2k_2 - a_2k_1).
\end{aligned}$$

Noting that $\beta_2k_2 - a_2k_1 < 0$, if $\Delta_2(\lambda_1,c) \leqslant 2(\beta_2k_2 - a_2k_1)$, then we have

$$D_2\bar{\phi}_2''(z) - c\bar{\phi}_2'(z) + f_{c2}(\bar{\phi}_z) \leqslant 0.$$

Hence, if $\Delta_2(\lambda_1,c) \leqslant 2(\beta_2k_2 - a_2k_1)$, $\bar{\phi}(z) = (\bar{\phi}_1(z), \bar{\phi}_2(z))$ is an upper solution of (2.9).

We now construct a lower solution of (2.9). Choose $\varepsilon > 0$ sufficiently small satisfying $\lambda_1 < \lambda_1 + \varepsilon < \lambda_2$. Let $M > 1$ be a constant to be chosen later. Define

$$\underline{\phi}_1(z) = \begin{cases}(1 - Me^{\varepsilon z})e^{\lambda_1 z}, & z < z_1,\\ 0, & z \geqslant z_1,\end{cases} \qquad \underline{\phi}_2(z) = 0,$$

where $z_1 = -(1/\varepsilon)\ln M < 0$. Clearly, $\underline{\phi}(z) \geqslant 0$ for all $z \in R$.

For $z > z_1$, $\underline{\phi}_1(z) = 0$, and hence

$$D_1\underline{\phi}_1''(z) - c\underline{\phi}_1'(z) + f_{c1}(\underline{\phi}_z) = \alpha_1 e^{-\gamma_1\tau_1}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_1\tau_1}}e^{-y^2/4d_1\tau_1}\underline{\phi}_1(z-c\tau_1-y)\,dy \geqslant 0.$$

For $z < z_1$,

$$\begin{aligned}
\underline{\phi}_1(z) &= [1 - Me^{\varepsilon z}]e^{\lambda_1 z},\\
\underline{\phi}_1'(z) &= [\lambda_1 - M(\lambda_1+\varepsilon)e^{\varepsilon z}]e^{\lambda_1 z},\\
\underline{\phi}_1''(z) &= [\lambda_1^2 - M(\lambda_1+\varepsilon)^2e^{\varepsilon z}]e^{\lambda_1 z}.
\end{aligned}$$

Therefore, it follows that:

$$\begin{aligned}
D_1\underline{\phi}_1''(z) - c\underline{\phi}_1'(z) + f_{c1}(\underline{\phi}_z) &= D_1[\lambda_1^2 - M(\lambda_1+\varepsilon)^2e^{\varepsilon z}]e^{\lambda_1 z} - c[\lambda_1 - M(\lambda_1+\varepsilon)e^{\varepsilon z}]e^{\lambda_1 z}\\
&\quad + \alpha_1 e^{-\gamma_1\tau_1}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_1\tau_1}}e^{-y^2/4d_1\tau_1}\underline{\phi}_1(z-c\tau_1-y)\,dy - \beta_1[1 - Me^{\varepsilon z}]^2e^{2\lambda_1 z}\\
&\quad - (E_1 + a_1k_2)[1 - Me^{\varepsilon z}]e^{\lambda_1 z}.
\end{aligned}$$

We note that if $z - c\tau_1 - z_1 < y$, $\underline{\phi}_1(z-c\tau_1-y) = [1 - Me^{\varepsilon(z-c\tau_1-y)}]e^{\lambda_1(z-c\tau_1-y)}$; if $z - c\tau_1 - z_1 \geqslant y$, $\underline{\phi}_1(z-c\tau_1-y) = 0 \geqslant [1 - Me^{\varepsilon(z-c\tau_1-y)}]e^{\lambda_1(z-c\tau_1-y)}$. We therefore have

$$\begin{aligned}
D_1\underline{\phi}_1''(z) - c\underline{\phi}_1'(z) + f_{c1}(\underline{\phi}_z) &\geqslant D_1[\lambda_1^2 - M(\lambda_1+\varepsilon)^2e^{\varepsilon z}]e^{\lambda_1 z} - c[\lambda_1 - M(\lambda_1+\varepsilon)e^{\varepsilon z}]e^{\lambda_1 z}\\
&\quad + \alpha_1 e^{-\gamma_1\tau_1}\int_{-\infty}^{\infty}\left\{\frac{1}{\sqrt{4\pi d_1\tau_1}}e^{-y^2/4d_1\tau_1}\times[1 - Me^{\varepsilon(z-c\tau_1-y)}]e^{\lambda_1(z-c\tau_1-y)}\right\}dy\\
&\quad - \beta_1[1 - Me^{\varepsilon z}]^2e^{2\lambda_1 z} - (E_1 + a_1k_2)[1 - Me^{\varepsilon z}]e^{\lambda_1 z}\\
&= e^{\lambda_1 z}[D_1\lambda_1^2 - c\lambda_1 + \alpha_1e^{-\gamma_1\tau_1}e^{d_1\tau_1\lambda_1^2 - c\tau_1\lambda_1} - E_1 - a_1k_2] - Me^{\lambda_1 z}e^{\varepsilon z}[D_1(\lambda_1+\varepsilon)^2 - c(\lambda_1+\varepsilon)\\
&\quad + \alpha_1e^{-\gamma_1\tau_1}e^{d_1\tau_1(\lambda_1+\varepsilon)^2 - c\tau_1(\lambda_1+\varepsilon)} - E_1 - a_1k_2] - \beta_1[1 - Me^{\varepsilon z}]^2e^{2\lambda_1 z}\\
&= e^{\lambda_1 z}\Delta_1(\lambda_1,c) - Me^{\lambda_1 z}e^{\varepsilon z}\Delta_1(\lambda_1+\varepsilon,c) - \beta_1[1 - Me^{\varepsilon z}]^2e^{2\lambda_1 z}\\
&= -Me^{\lambda_1 z}e^{\varepsilon z}\Delta_1(\lambda_1+\varepsilon,c) - \beta_1[1 - Me^{\varepsilon z}]^2e^{2\lambda_1 z}.
\end{aligned}$$

Since $z < z_1 < 0$ and $\varepsilon < \lambda_1$, we have $e^{\lambda_1 z} < e^{\varepsilon z}$, and

$$(1 - Me^{\varepsilon z})^2 \leqslant (1 + Me^{\varepsilon z})^2 \leqslant (1 + Me^{\varepsilon z_1})^2 = (1+1)^2 = 4.$$

We therefore derive that

$$D_1\underline{\phi}_1''(z) - c\underline{\phi}_1'(z) + f_{c1}(\underline{\phi}_z) \geqslant e^{(\lambda_1+\varepsilon)z}\{-M\Delta_1(\lambda_1+\varepsilon,c) - 4\beta_1\} = -e^{(\lambda_1+\varepsilon)z}\Delta_1(\lambda_1+\varepsilon,c)\left\{M - \frac{4\beta_1}{-\Delta_1(\lambda_1,c)}\right\}.$$

By the choice of $\varepsilon > 0$, we have $\Delta_1(\lambda_1+\varepsilon,c) < 0$. If we choose

$$M > \frac{4\beta_1}{-\Delta_1(\lambda_1,c)},$$

then it follows that:

$$D_1\underline{\phi}_1''(z) - c\underline{\phi}_1'(z) + f_{c1}(\underline{\phi}_z) \geqslant 0.$$

For $\underline{\phi}_2(z)$, the proof is trivial. Therefore, for such a $M > 1$, $\underline{\phi}(z)$ is a lower solution of (2.9).

From what has been discussed above, we can conclude the following result.

Theorem 2.2. *In addition to (A1)–(A2), assume that* $\Delta_2(\lambda_1,c) \leqslant 2(\beta_2 k_2 - a_2 k_1)$. *For every* $c > c^*$, *regardless of the value of* $\tau_1 \geqslant 0$, $\tau_2 \geqslant 0$, *(2.1) always has a travelling wave front with speed c connecting two nonnegative uniform equilibria* $E_1^*(k_1,0)$ *and* $E_2^*(0,k_2)$.

3. Global convergence

In this section, based on the work of Gourley and Kuang [6], we shall extend system (1.4) to a stage structured competitive model with nonlocal spatial effect in the case of the domain being finite with homogeneous Neumann boundary conditions. Such boundary conditions model a closed environment with reflecting boundaries, i.e., individuals cannot leave the domain. Referring to [6] we see that, for the case of a finite spatial domain $0 \leqslant x \leqslant \pi$, the delayed term $\alpha e^{-\gamma\tau}u_m(t-\tau)$ in system (1.1) detailed in Section 1 cannot be replaced by expression (1.3) because the arguments that led to that expression assumed the domain to be infinite. Thus, in [6], the second equation of (1.3) was replaced by

$$\frac{\partial u_m}{\partial t} = d_m\frac{\partial^2 u_m}{\partial x^2} + \alpha e^{-\gamma\tau}\int_0^\pi G(x,y,\tau)u_m(t-\tau,y)\,dy - \beta u_m^2, \quad t > 0,\ \ 0 < x < \pi \tag{3.1}$$

with boundary conditions

$$\frac{\partial u_m}{\partial x}(t,0) = \frac{\partial u_m}{\partial x}(t,\pi) = 0, \quad t > 0, \tag{3.2}$$

and initial conditions

$$u_m(t,x) = \phi(t,x) \quad \text{for} \quad (t,x) \in [-\tau,0]\times[0,\pi]. \tag{3.3}$$

In (3.1), the kernel $G(x,y,t)$ is of the form

$$G(x,y,t) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-d_i n^2 t}\cos nx\cos ny, \tag{3.4}$$

which is the solution of

$$\frac{\partial G}{\partial t} = d_i\frac{\partial^2 G}{\partial x^2}, \quad 0 < x < \pi \tag{3.5}$$

subject to

$$\frac{\partial G}{\partial x} = 0 \text{ at } x = 0,\pi \quad \text{and} \quad G(x,y,0) = \delta(x-y). \tag{3.6}$$

Gourley and Kuang [6] discussed the global convergence of positive solutions of problem ((3.1)–(3.3)). For convenience of use, we restate their result here.

Theorem 3.1. *Let the initial function ϕ be Hölder continuous in $[-\tau,0]\times[0,\pi]$ and satisfy $\phi \geqslant 0$ with $\phi \neq 0$. Assume that $\alpha,\beta>0$ and $\gamma,\tau \geqslant 0$. Then problem (3.1,3.2,3.3) has a unique positive solution $u_m(t,x)$ satisfying*

$$\lim_{t\to+\infty} u_{\mathrm{m}}(t,x)=\frac{\alpha}{\beta}\mathrm{e}^{-\gamma\tau}$$

uniformly for $x\in[0,\pi]$.

Motivated by the work of Gourley and Kuang [6], in this section, we shall study the following stage-structured competition model with nonlocal spatial effect:

$$\begin{aligned}
\frac{\partial u_1}{\partial t}&=D_1\frac{\partial^2 u_1}{\partial x^2}+\alpha_1\mathrm{e}^{-\gamma_1\tau_1}\int_0^\pi G_1(x,y,\tau_1)u_1(t-\tau_1,y)\,\mathrm{d}y-\beta_1u_1^2(t,x)-a_1u_1(t,x)u_2(t,x),\\
\frac{\partial v_1}{\partial t}&=d_1\frac{\partial^2 v_1}{\partial x^2}+\alpha_1u_1-\gamma_1v_1-\alpha_1\mathrm{e}^{-\gamma_1\tau_1}\int_0^\pi G_1(x,y,\tau_1)u_1(t-\tau_1,y)\,\mathrm{d}y,\\
\frac{\partial u_2}{\partial t}&=D_2\frac{\partial^2 u_2}{\partial x^2}+\alpha_2\mathrm{e}^{-\gamma_2\tau_2}\int_0^\pi G_2(x,y,\tau_2)u_2(t-\tau_2,y)\,\mathrm{d}y-\beta_2u_2^2(t,x)-a_2u_1(t,x)u_2(t,x),\\
\frac{\partial v_2}{\partial t}&=d_2\frac{\partial^2 v_2}{\partial x^2}+\alpha_2u_2-\gamma_2v_2-\alpha_2\mathrm{e}^{-\gamma_2\tau_2}\int_0^\pi G_2(x,y,\tau_2)u_2(t-\tau_2,y)\,\mathrm{d}y
\end{aligned}\tag{3.7}$$

for $t>0$, $x\in(0,\pi)$, where $\tau=\max\{\tau_1,\tau_2\}$, with boundary conditions

$$\begin{aligned}
&\frac{\partial u_1}{\partial x}(t,0)=\frac{\partial u_2}{\partial x}(t,0)=\frac{\partial v_1}{\partial x}(t,0)=\frac{\partial v_2}{\partial x}(t,0)=0,\\
&\frac{\partial u_1}{\partial x}(t,\pi)=\frac{\partial u_2}{\partial x}(t,\pi)=\frac{\partial v_1}{\partial x}(t,\pi)=\frac{\partial v_2}{\partial x}(t,\pi)=0
\end{aligned}\tag{3.8}$$

for $t>0$, and the initial conditions

$$u_{\mathrm{i}}(t,x)=\phi_i(t,x),\quad v_i(t,x)=\psi_i(t,x)\qquad (i=1,2)\ \text{for}\ (t,x)\in[-\tau,0]\times[0,\pi].\tag{3.9}$$

In (3.7), $G_1(x,y,t)$ and $G_2(x,y,t)$ are solutions of problem (3.5) and (3.6) with $d_i=d_1$ and $d_i=d_2$, respectively. The parameters a_i, d_i, D_i, E_i, α_i, β_i, γ_i, and τ_i are positive constants, $i=1,2$.

In order to discuss the global convergence of problem ((3.7)–(3.9)), we first consider the following subsystem of system (3.7) for the adult members of the two species in competition:

$$\begin{aligned}
\frac{\partial u_1}{\partial t}&=D_1\frac{\partial^2 u_1}{\partial x^2}+\alpha_1\mathrm{e}^{-\gamma_1\tau_1}\int_0^\pi G_1(x,y,\tau_1)u_1(t-\tau_1,y)\,\mathrm{d}y-\beta_1u_1^2(t,x)-a_1u_1(t,x)u_2(t,x),\\
\frac{\partial u_2}{\partial t}&=D_2\frac{\partial^2 u_2}{\partial x^2}+\alpha_2\mathrm{e}^{-\gamma_2\tau_2}\int_0^\pi G_2(x,y,\tau_2)u_2(t-\tau_2,y)\,\mathrm{d}y-\beta_2u_2^2(t,x)-a_2u_1(t,x)u_2(t,x),
\end{aligned}\tag{3.10}$$

for $t>0$, $x\in(0,\pi)$ with boundary conditions

$$\frac{\partial u_1}{\partial x}(t,0)=\frac{\partial u_2}{\partial x}(t,0)=\frac{\partial u_1}{\partial x}(t,\pi)=\frac{\partial u_2}{\partial x}(t,\pi)=0,\tag{3.11}$$

for $t>0$ and the initial conditions

$$u_1(t,x)=\phi_1(t,x),\quad u_2(t,x)=\phi_2(t,x),\tag{3.12}$$

for $(t,x)\in[-\tau,0]\times[0,\pi]$.

System (3.10) always has a trivial uniform equilibrium $E_0^*(0,0)$ and two nonnegative semi-trivial uniform equilibria $E_1^*(\alpha_1\mathrm{e}^{-\gamma_1\tau_1}/\beta_1,0)$ and $E_2^*(0,\alpha_2\mathrm{e}^{-\gamma_2\tau_2}/\beta_2)$. If the following hold:

(C1) $\beta_2\alpha_1\mathrm{e}^{-\gamma_1\tau_1}>a_1\alpha_2\mathrm{e}^{-\gamma_2\tau_2}$,
(C2) $\beta_1\alpha_2\mathrm{e}^{-\gamma_2\tau_2}>a_2\alpha_1\mathrm{e}^{-\gamma_1\tau_1}$,

then system (3.10) also admits a unique positive uniform equilibrium $E^*(u_1^*,u_2^*)$, where

$$\begin{aligned}
u_1^*&=\frac{\beta_2\alpha_1\mathrm{e}^{-\gamma_1\tau_1}-a_1\alpha_2\mathrm{e}^{-\gamma_2\tau_2}}{\beta_1\beta_2-a_1a_2},\\
u_2^*&=\frac{\beta_1\alpha_2\mathrm{e}^{-\gamma_2\tau_2}-a_2\alpha_1\mathrm{e}^{-\gamma_1\tau_1}}{\beta_1\beta_2-a_1a_2}.
\end{aligned}$$

We note that if (C1)–(C2) hold, then $\beta_1\beta_2 > a_1a_2$. In this case it is easy to show that the equilibrium $E_0^*(0,0)$ is unstable. In the following, we first study the local stability of the positive equilibrium E^*.

Lemma 3.1. *(i) If (C1)–(C2) hold, then the positive equilibrium E^* of system (3.10) is locally stable.*

Proof. We find, after some algebra, that the linearized equations have nontrivial solutions of the form $(c_1,c_2)e^{\sigma t}\cos nx$ $(n=0,1,\ldots)$ if and only if the following holds:

$$B(\sigma) := \begin{vmatrix} g_1(\sigma,n^2) & a_1u_1^* \\ a_2u_2^* & g_2(\sigma,n^2) \end{vmatrix} = 0, \tag{3.13}$$

where

$$g_1(\sigma,n^2) = \sigma - \alpha_1 e^{-\gamma_1\tau_1}e^{-\sigma\tau_1}e^{-d_1n^2\tau_1} + D_1n^2 + 2\beta u_1^* + a_1u_2^*,$$
$$g_2(\sigma,n^2) = \sigma - \alpha_2 e^{-\gamma_2\tau_2}e^{-\sigma\tau_2}e^{-d_2n^2\tau_2} + D_2n^2 + 2\beta_2 u_2^* + a_2u_1^*.$$

We now claim that if (C1)–(C2) hold, then the positive equilibrium $E^*(u_1^*,u_2^*)$ is locally stable. In order to do so we therefore need to show the roots of $B(\sigma)=0$ must have negative real parts. Let $\sigma=\mu+\mathrm{i}v$, where μ and v are real numbers. Denote

$$A_1 = \mu - \alpha_1 e^{-\gamma_1\tau_1}e^{-d_1n^2\tau_1}e^{-\mu\tau_1}\cos(v\tau_1) + D_1n^2 + 2\beta_1u_1^* + a_1u_2^*,$$
$$B_1 = v + \alpha_1 e^{-\gamma_1\tau_1}e^{-d_1n^2\tau_1}e^{-\mu\tau_1}\sin(v\tau_1),$$
$$A_2 = \mu - \alpha_2 e^{-\gamma_2\tau_2}e^{-d_2n^2\tau_2}e^{-\mu\tau_2}\cos(v\tau_2) + D_2n^2 + 2\beta_2u_2^* + a_2u_1^*,$$
$$B_2 = v + \alpha_2 e^{-\gamma_2\tau_2}e^{-d_2n^2\tau_2}e^{-\mu\tau_2}\sin(v\tau_2).$$

On substituting $\sigma=\mu+iv$ into (3.13) gives

$$A_1A_2 - B_1B_2 = a_1a_2u_1^*u_2^* \text{ and } A_1B_2 + A_2B_1 = 0.$$

It therefore follows that:

$$(a_1a_2u_1^*u_2^*)^2 = (A_1A_2 - B_1B_2)^2 = (A_1A_2)^2 + (B_1B_2)^2 - 2A_1A_2B_1B_2 = (A_1A_2)^2 + (B_1B_2)^2 + 2(A_1B_2)^2,$$

which yields

$$(a_1a_2u_1^*u_2^*)^2 \geqslant (A_1A_2)^2.$$

If $\mu \geqslant 0$, then we have

$$A_1 = \mu - \alpha_1 e^{-\gamma_1\tau_1}e^{-d_1n^2\tau_1}e^{-\mu\tau_1}\cos(v\tau_1) + D_1n^2 + 2\beta_1u_1^* + a_1u_2^* > -\alpha_1e^{-\gamma_1\tau_1} + 2\beta_1u_1^* + a_1u_2^* = \beta_1u_1^*.$$

Similarly, $A_2 > \beta_2u_2^*$. And hence $A_1A_2 > a_1a_2u_1^*u_2^*$. It follows that:

$$(a_1a_2u_1^*u_2^*)^2 \geqslant (A_1A_2)^2 > (\beta_1\beta_2u_1^*u_2^*)^2,$$

which is a contradiction. Hence, $\mu<0$, i.e., the positive equilibrium $E^*(u_1^*,u_2^*)$ is locally stable. This completes the proof. □

Similarly, we can derive the following results for the local stability of the nonnegative equilibria E_1^* and E_2^*, respectively.

Lemma 3.2. *The nonnegative equilibrium E_1^* is locally stable if*

(D1) $\dfrac{\alpha_1e^{-\gamma_1\tau_1}}{\alpha_2e^{-\gamma_2\tau_2}} > \dfrac{\beta_1}{a_2}$ *and* $\dfrac{\alpha_1e^{-\gamma_1\tau_1}}{\alpha_2e^{-\gamma_2\tau_2}} > \dfrac{a_1}{\beta_2}$.

Lemma 3.3. *The nonnegative equilibrium E_2^* is locally stable if*

(D2) $\dfrac{\alpha_1e^{-\gamma_1\tau_1}}{\alpha_2e^{-\gamma_2\tau_2}} < \dfrac{\beta_1}{a_2}$ *and* $\dfrac{\alpha_1e^{-\gamma_1\tau_1}}{\alpha_2e^{-\gamma_2\tau_2}} < \dfrac{a_1}{\beta_2}$.

In order to study the global convergence of the positive solutions to problem (3.10)–(3.12), we need the following concept and results.

Definition 3.1. A pair of smooth functions $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ and $\hat{u} = (\hat{u}_1, \hat{u}_2)$ are called coupled upper–lower solutions of (3.10)–(3.12), if $\tilde{u}_i \geqslant \hat{u}_i$ $(i = 1, 2)$ in $[-\tau, \infty) \times [0, \pi]$, and if for all $\hat{u}_i \leqslant \phi_i \leqslant \tilde{u}_i$, the following differential inequalities hold:

$$\begin{aligned}
&\frac{\partial \tilde{u}_1}{\partial t} \geqslant D_1 \frac{\partial^2 \tilde{u}_1}{\partial x^2} + \alpha_1 e^{-\gamma_1\tau_1} \int_0^\pi G_1(x, y, \tau_1)\phi_1(t - \tau_1, y)\,dy - \beta_1 \tilde{u}_1^2(t, x) - a_1 \tilde{u}_1(t, x)\hat{u}_2(t, x),\\
&\frac{\partial \tilde{u}_2}{\partial t} \geqslant D_2 \frac{\partial^2 \tilde{u}_2}{\partial x^2} + \alpha_2 e^{-\gamma_2\tau_2} \int_0^\pi G_2(x, y, \tau_2)\phi_2(t - \tau_2, y)\,dy - \beta_2 \tilde{u}_2^2(t, x) - a_2 \hat{u}_1(t, x)\tilde{u}_2(t, x),\\
&\frac{\partial \hat{u}_1}{\partial t} \leqslant D_1 \frac{\partial^2 \hat{u}_1}{\partial x^2} + \alpha_1 e^{-\gamma_1\tau_1} \int_0^\pi G_1(x, y, \tau_1)\phi_1(t - \tau_1, y)\,dy - \beta_1 \hat{u}_1^2(t, x) - a_1 \hat{u}_1(t, x)\tilde{u}_2(t, x),\\
&\frac{\partial \hat{u}_2}{\partial t} \leqslant D_2 \frac{\partial^2 \hat{u}_2}{\partial x^2} + \alpha_2 e^{-\gamma_2\tau_2} \int_0^\pi G_2(x, y, \tau_2)\phi_2(t - \tau_2, y)\,dy - \beta_2 \hat{u}_2^2(t, x) - a_2 \tilde{u}_1(t, x)\hat{u}_2(t, x),\\
&\frac{\partial \hat{u}_i}{\partial x}(t, 0) \geqslant 0, \quad \frac{\partial \hat{u}_i}{\partial x}(t, \pi) \leqslant 0, \quad \frac{\partial \tilde{u}_i}{\partial x}(t, 0) \leqslant 0, \quad \frac{\partial \tilde{u}_i}{\partial x}(t, \pi) \geqslant 0 \quad (i = 1, 2), \;\; t > 0,\\
&\hat{u}_i(t, x) \leqslant \phi_i(t, x) \leqslant \tilde{u}_i(t, x) \quad (i = 1, 2), \;\; (t, x) \in [-\tau, 0] \times [0, \pi].
\end{aligned} \tag{3.14}$$

With the definition of a pair of coupled upper–lower solutions, we can state the following result from Redlinger [17].

Lemma 3.4. *If there exist a pair of upper–lower solutions* $\hat{u}$, $\tilde{u}$ *of system* (3.10), *then the problem* (3.10)–(3.12) *has a unique solution* $(u_1(t, x), u_2(t, x))$ *satisfying* $\hat{u}_i \leqslant u_i \leqslant \tilde{u}_i$ $(i = 1, 2)$ *in* $[-\tau, \infty) \times [0, \pi]$.

For given (ϕ_1, ϕ_2), choose positive constants K_1, K_2 such that

$$K_1 \geqslant \max\left\{\|\phi_1\|, \frac{\alpha_1 e^{-\gamma_1\tau_1}}{\beta_1}\right\}, \quad K_2 \geqslant \max\left\{\|\phi_2\|, \frac{\alpha_2 e^{-\gamma_2\tau_2}}{\beta_2}\right\}, \tag{3.15}$$

where $\|\phi_i\| = \max_{(t,x)\in[-\tau,0]\times[0,\pi]}|\phi_i(t, x)|$, $i = 1, 2$. Then it is easy to see that $(0, 0)$ and (K_1, K_2) are a pair of coupled lower–upper solutions of (3.10). By Lemma (3.4), we see that problem (3.10)–(3.12) admits a unique solution (u_1, u_2) satisfying $0 \leqslant u_i \leqslant K_i$ $(i = 1, 2)$. Moreover, by using the maximum principle, it is easy to show that (u_1, u_2) is positive in $(0, \infty) \times [0, \pi]$ if $\phi_i \geqslant 0$, $\phi_i(0, x) \neq 0$ $(i = 1, 2)$.

Lemma 3.5. *Consider the following equation*:

$$\frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} + a \int_0^\pi G(x, y, \tau)u(t - \tau, y)\,dy - bu(t, x) - cu^2(t, x) \tag{3.16}$$

for $t > 0$, $x \in (0, \pi)$ *with boundary conditions*

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, \pi) = 0, \quad t > 0 \tag{3.17}$$

and initial conditions

$$u(t, x) = \phi(t, x) \quad \text{for} \quad (t, x) \in [-\tau, 0] \times [0, \pi], \tag{3.18}$$

where $G(x, y, t)$ *is given by* (3.4), *the parameters* a, b, c, d *are positive constants*, $\tau \geqslant 0$ *is a constant. Let* ϕ *be Hölder continuous in* $[-\tau, 0] \times [0, \pi]$ *and satisfy* $\phi \geqslant 0$, $\phi(0, x) \neq 0$. *Assume* $u(t, x)$ *is the solution of problem* ((3.16)–(3.18)). *We have*

(i) *If* $a > b$, *then* $\lim_{t\to+\infty} u(t, x) = (a - b)/c$ *uniformly for* $x \in [0, \pi]$.
(ii) *If* $a < b$, *then* $\lim_{t\to+\infty} u(t, x) = 0$.

The proof of Lemma 3.5 is very similar to that of Theorem 3.1 developed by Gourley and Kuang [6]. We therefore omit it here.

We are now in a position to state and prove our result on the global convergence of positive solution of problem (3.10)–(3.12).

Theorem 3.2. *Let the initial functions* $\phi_1(t, x)$ *and* $\phi_2(t, x)$ *be Hölder continuous in* $[-\tau, 0] \times [0, \pi]$, *and* ϕ_1, $\phi_2 \geqslant 0$, $\phi_1(0, x) \neq 0$, $\phi_2(0, x) \neq 0$. *Let* (C1)–(C2) *hold. Then problem* (3.10)–(3.12) *has a unique positive solution* $(u_1(t, x), u_2(t, x))$ *satisfying*

$$\lim_{t\to+\infty} u_1(t,x) = u_1^*, \quad \lim_{t\to+\infty} u_2(t,x) = u_2^*$$

uniformly for $x \in [0,\pi]$.

Proof. Let $(u_1(t,x), u_2(t,x))$ be the positive regular solution of problem (3.10)–(3.12). Denote

$$U_1 = \limsup_{t\to+\infty} \max_{x\in[0,\pi]} u_1(t,x), \quad V_1 = \liminf_{t\to+\infty} \min_{x\in[0,\pi]} u_1(t,x),$$
$$U_2 = \limsup_{t\to+\infty} \max_{x\in[0,\pi]} u_2(t,x), \quad V_2 = \liminf_{t\to+\infty} \min_{x\in[0,\pi]} u_2(t,x).$$

In the following, we shall claim $U_1 = V_1 = u_1^*$, $U_2 = V_2 = u_2^*$.

Define $(\bar{u}_1^{(1)}(t,x), \bar{u}_2^{(1)}(t,x))$ by

$$\begin{aligned}
&\frac{\partial \bar{u}_1^{(1)}}{\partial t} = D_1 \frac{\partial^2 \bar{u}_1^{(1)}}{\partial x^2} + \alpha_1 e^{-\gamma_1\tau_1} \int_0^\pi G_1(x,y,\tau_1)\bar{u}_1^{(1)}(t-\tau_1,y)\,dy - \beta_1(\bar{u}_1^{(1)}(t,x))^2, \quad t>0,\ x\in(0,\pi),\\
&\frac{\partial \bar{u}_2^{(1)}}{\partial t} = D_2 \frac{\partial^2 \bar{u}_2^{(1)}}{\partial x^2} + \alpha_2 e^{-\gamma_2\tau_2} \int_0^\pi G_2(x,y,\tau_2)\bar{u}_2^{(1)}(t-\tau_2,y)\,dy - \beta_2(\bar{u}_2^{(1)}(t,x))^2, \quad t>0,\ x\in(0,\pi),\\
&\frac{\partial \bar{u}_1^{(1)}}{\partial x} = \frac{\partial \bar{u}_2^{(1)}}{\partial x} = 0, \quad t>0,\ x=0,\pi,\\
&\bar{u}_1^{(1)}(t,x) = K_1, \quad \bar{u}_2^{(1)}(t,x) = K_2, \quad (t,x)\in[-\tau,0]\times[0,\pi],
\end{aligned} \tag{3.19}$$

where K_1 and K_2 are defined in (3.15). Clearly, $(0,0)$ and $(\bar{u}_1^{(1)}, \bar{u}_2^{(1)})$ are a pair of lower–upper solutions of problem (3.10)–(3.12). By Lemma 3.4, we derive

$$0 < u_1(t,x) \leqslant \bar{u}_1^{(1)}(t,x), \quad 0 < u_2(t,x) \leqslant \bar{u}_2^{(1)}(t,x).$$

By Lemma 3.5, it follows from (3.19) that:

$$\lim_{t\to+\infty} \bar{u}_i^{(1)}(t,x) = \frac{\alpha_i e^{-\gamma_i\tau_i}}{\beta_i} \quad (i=1,2) \text{ uniformly for } x\in[0,\pi]. \tag{3.20}$$

Hence, for $\forall\varepsilon > 0$ sufficiently small, there is a $T_{11} > 0$ such that if $t > T_{11}$,

$$\max_{x\in[0,\pi]} \bar{u}_i^{(1)}(t,x) < M_1^{u_i} + \varepsilon \quad (i=1,2), \tag{3.21}$$

where $M_1^{u_i} = \alpha_i e^{-\gamma_i\tau_i}/\beta_i$ $(i=1,2)$.

Since this is true for any $\varepsilon > 0$ sufficiently small, we conclude that

$$U_i = \limsup_{t\to+\infty} \max_{x\in[0,\pi]} u_i(t,x) \leqslant \frac{\alpha_i e^{-\gamma_i\tau_i}}{\beta_i} := M_1^{u_i} \quad (i=1,2). \tag{3.22}$$

Let $\underline{u}_1^{(1)}(t,x), \underline{u}_2^{(1)}(t,x)$ be the solution of

$$\begin{aligned}
&\frac{\partial \underline{u}_1^{(1)}}{\partial t} = D_1 \frac{\partial^2 \underline{u}_1^{(1)}}{\partial x^2} + \alpha_1 e^{-\gamma_1\tau_1} \int_0^\pi G_1(x,y,\tau_1)\underline{u}_1^{(1)}(t-\tau_1,y)\,dy - \beta_1(\underline{u}_1^{(1)}(t,x))^2 - a_1\underline{u}_1^{(1)}(t,x)\bar{u}_2^{(1)}(t,x), \quad t>T_{11},\ x\in(0,\pi),\\
&\frac{\partial \underline{u}_2^{(1)}}{\partial t} = D_2 \frac{\partial^2 \underline{u}_2^{(1)}}{\partial x^2} + \alpha_2 e^{-\gamma_2\tau_2} \int_0^\pi G_2(x,y,\tau_2)\underline{u}_2^{(1)}(t-\tau_2,y)\,dy - \beta_2(\underline{u}_2^{(1)}(t,x))^2 - a_2\bar{u}_1^{(1)}(t,x)\underline{u}_2^{(1)}(t,x), \quad t>T_{11},\ x\in(0,\pi),\\
&\frac{\partial \underline{u}_1^{(1)}}{\partial x} = \frac{\partial \underline{u}_2^{(1)}}{\partial x} = 0, \quad t>T_{11},\ x=0,\pi,\\
&\underline{u}_1^{(1)}(t,x) = \frac{1}{2}u_1(t,x), \quad \underline{u}_2^{(1)}(t,x) = \frac{1}{2}u_2(t,x), \quad (t,x)\in[-\tau,T_{11}]\times[0,\pi].
\end{aligned} \tag{3.23}$$

Thus, $(\bar{u}_1^{(1)}, \bar{u}_2^{(1)})$ and $(\underline{u}_1^{(1)}, \underline{u}_2^{(1)})$ are a pair of upper–lower solutions of problem (3.10)–(3.12). By Lemma 3.5 we have

$$\underline{u}_1^{(1)}(t,x) \leqslant u_1(t,x) \leqslant \bar{u}_1^{(1)}(t,x), \quad \underline{u}_1^{(1)}(t,x) \leqslant u_1(t,x) \leqslant \bar{u}_1^{(1)}(t,x).$$

For $\varepsilon > 0$ sufficiently small satisfying $\alpha_1 e^{\gamma_1\tau_1} > a_1(M_1^{u_2}+\varepsilon)$, $\alpha_2 e^{-\gamma_2\tau_2} > a_2(M_1^{u_1}+\varepsilon)$, we consider the following auxiliary problem:

$$\begin{aligned}
&\frac{\partial\omega_1^{(1)}}{\partial t} = D_1\frac{\partial^2\omega_1^{(1)}}{\partial x^2} + \alpha_1 e^{-\gamma_1\tau_1}\int_0^\pi G_1(x,y,\tau_1)\omega_1^{(1)}(t-\tau_1,y)\,dy - \beta_1(\omega_1^{(1)}(t,x))^2 - a_1\omega_1^{(1)}(t,x)(M_1^{u_2}+\varepsilon), \quad t>T_{11},\ x\in(0,\pi),\\
&\frac{\partial\omega_2^{(1)}}{\partial t} = D_2\frac{\partial^2\omega_2^{(1)}}{\partial x^2} + \alpha_2 e^{-\gamma_2\tau_2}\int_0^\pi G_2(x,y,\tau_2)\omega_2^{(1)}(t-\tau_2,y)\,dy - \beta_2(\omega_2^{(1)}(t,x))^2 - a_2(M_1^{u_1}+\varepsilon)\omega_2^{(1)}(t,x), \quad t>T_{11},\ x\in(0,\pi),\\
&\frac{\partial\omega_1^{(1)}}{\partial x} = \frac{\partial\omega_2^{(1)}}{\partial x} = 0, \quad t>T_{11},\ x=0,\pi,\\
&\omega_1^{(1)}(t,x) = \frac{1}{2}u_1(t,x), \quad \omega_2^{(1)}(t,x) = \frac{1}{2}u_2(t,x), \quad (t,x)\in[-\tau,T_{11}]\times[0,\pi].
\end{aligned} \tag{3.24}$$

By Lemma 3.5 it follows from (3.24) that:

$$\lim_{t\to+\infty}\omega_1^{(1)}(t,x) = \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_1(M_1^{u_2}+\varepsilon)}{\beta_1}, \quad \lim_{t\to+\infty}\omega_2^{(1)}(t,x) = \frac{\alpha_2 e^{-\gamma_2\tau_2} - a_2(M_1^{u_1}+\varepsilon)}{\beta_2}$$

uniformly for $x\in[0,\pi]$.

By comparison, for $\varepsilon>0$ sufficiently small, there is a $T_{12}\geqslant T_{11}$ such that if $t>T_{12}$,

$$\min_{x\in[0,\pi]}\underline{u}_1^{(1)}(t,x) > \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_1 M_1^{u_2}}{\beta_1} - \varepsilon, \quad \min_{x\in[0,\pi]}\underline{u}_2^{(1)}(t,x) > \frac{\alpha_2 e^{-\gamma_2\tau_2} - a_2 M_1^{u_1}}{\beta_2} - \varepsilon.$$

Since this is true for arbitrary $\varepsilon>0$ sufficiently small, we can conclude that

$$\begin{aligned}
V_1 &= \liminf_{t\to+\infty}\min_{x\in[0,\pi]}u_1(t,x) \geqslant \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_1 M_1^{u_2}}{\beta_1} := N_1^{u_1},\\
V_2 &= \liminf_{t\to+\infty}\min_{x\in[0,\pi]}u_2(t,x) \geqslant \frac{\alpha_2 e^{-\gamma_2\tau_2} - a_2 M_1^{u_1}}{\beta_2} := N_1^{u_2}.
\end{aligned} \tag{3.25}$$

Let $(\bar{u}_1^{(2)}(t,x),\ \bar{u}_2^{(2)}(t,x))$ be the solution of the following problem:

$$\begin{aligned}
&\frac{\partial\bar{u}_1^{(2)}}{\partial t} = D_1\frac{\partial^2\bar{u}_1^{(2)}}{\partial x^2} + \alpha_1 e^{-\gamma_1\tau_1}\int_0^\pi G_1(x,y,\tau_1)\bar{u}_1^{(2)}(t-\tau_1,y)\,dy - \beta_1(\bar{u}_1^{(2)}(t,x))^2 - a_1\bar{u}_1^{(2)}(t,x)\underline{u}_2^{(1)}(t,x), \quad t>T_{12},\ x\in(0,\pi),\\
&\frac{\partial\bar{u}_2^{(2)}}{\partial t} = D_2\frac{\partial^2\bar{u}_2^{(2)}}{\partial x^2} + \alpha_2 e^{-\gamma_2\tau_2}\int_0^\pi G_2(x,y,\tau_2)\bar{u}_2^{(2)}(t-\tau_2,y)\,dy - \beta_2(\bar{u}_2^{(2)}(t,x))^2 - a_2\underline{u}_1^{(1)}(t,x)\bar{u}_2^{(2)}(t,x), \quad t>T_{12},\ x\in(0,\pi),\\
&\frac{\partial\bar{u}_1^{(2)}}{\partial x} = \frac{\partial\bar{u}_2^{(2)}}{\partial x} = 0, \quad t>T_{12},\ x=0,\pi,\\
&\bar{u}_1^{(2)}(t,x) = K_1, \quad \bar{u}_2^{(2)}(t,x) = K_2, \quad (t,x)\in[-\tau,T_{12}]\times[0,\pi].
\end{aligned} \tag{3.26}$$

We note that $(\underline{u}_1^{(1)},\underline{u}_2^{(1)})$ and $(\bar{u}_1^{(2)},\bar{u}_2^{(2)})$ are a pair of lower–upper solutions of problem (3.10)–(3.12). By Lemma 3.5 it follows that:

$$\underline{u}_1^{(1)}(t,x) \leqslant u_1(t,x) \leqslant \bar{u}_1^{(2)}(t,x), \quad \underline{u}_2^{(1)}(t,x) \leqslant u_2(t,x) \leqslant \bar{u}_2^{(2)}(t,x). \tag{3.27}$$

We now consider the following problem:

$$\begin{aligned}
&\frac{\partial v_1^{(2)}}{\partial t} = D_1\frac{\partial^2 v_1^{(2)}}{\partial x^2} + \alpha_1 e^{-\gamma_1\tau_1}\int_0^\pi G_1(x,y,\tau_1)v_1^{(2)}(t-\tau_1,y)\,dy - \beta_1(v_1^{(2)}(t,x))^2 - a_1 v_1^{(2)}(t,x)(N_1^{u_2}-\varepsilon), \quad t>T_{12},\ x\in(0,\pi),\\
&\frac{\partial v_2^{(2)}}{\partial t} = D_2\frac{\partial^2 v_2^{(2)}}{\partial x^2} + \alpha_2 e^{-\gamma_2\tau_2}\int_0^\pi G_2(x,y,\tau_2)v_2^{(2)}(t-\tau_2,y)\,dy - \beta_2(v_2^{(2)}(t,x))^2 - a_2(N_1^{u_1}-\varepsilon)v_2^{(2)}(t,x), \quad t>T_{12},\ x\in(0,\pi),\\
&\frac{\partial v_1^{(2)}}{\partial x} = \frac{\partial v_2^{(2)}}{\partial x} = 0, \quad t>T_{12},\ x=0,\pi,\\
&v_1^{(2)}(t,x) = K_1, \quad v_2^{(2)}(t,x) = K_2,\ (t,x)\in[-\tau,T_{12}]\times[0,\pi].
\end{aligned} \tag{3.28}$$

By Lemma 3.5 we have

$$\lim_{t\to+\infty} v_1^{(2)}(t,x) = \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_1(N_1^{u_2} - \varepsilon)}{\beta_1}, \quad \lim_{t\to+\infty} v_2^{(2)}(t,x) = \frac{\alpha_2 e^{-\gamma_2\tau_2} - a_2(N_1^{u_1} - \varepsilon)}{\beta_2}$$

uniformly for $x \in [0,\pi]$. By comparison, for $\varepsilon > 0$ sufficiently small, there is a $T_{21} > T_{12}$ such that if $t > T_{21}$,

$$\begin{aligned} &\max_{x\in[0,\pi]} \bar{u}_1^{(2)}(t,x) \leqslant \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_1 N_1^{u_2}}{\beta_1} + \varepsilon, \\ &\max_{x\in[0,\pi]} \bar{u}_2^{(2)}(t,x) \leqslant \frac{\alpha_2 e^{-\gamma_2\tau_2} - a_2 N_1^{u_1}}{\beta_2} + \varepsilon. \end{aligned} \tag{3.29}$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, we derive that

$$\begin{aligned} &U_1 = \limsup_{t\to+\infty} \max_{x\in[0,\pi]} u_1(t,x) \leqslant \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_1 N_1^{u_2}}{\beta_1} := M_2^{u_1}, \\ &U_2 = \limsup_{t\to+\infty} \max_{x\in[0,\pi]} u_2(t,x) \leqslant \frac{\alpha_2 e^{-\gamma_2\tau_2} - a_2 N_1^{u_1}}{\beta_2} := M_2^{u_2}. \end{aligned} \tag{3.30}$$

Let $(\underline{u}_1^{(2)}(t,x), \underline{u}_2^{(2)}(t,x))$ be the solution of the following problem:

$$\begin{aligned} &\frac{\partial \underline{u}_1^{(2)}}{\partial t} = D_1 \frac{\partial^2 \underline{u}_1^{(2)}}{\partial x^2} + \alpha_1 e^{-\gamma_1\tau_1} \int_0^\pi G_1(x,y,\tau_1)\underline{u}_1^{(2)}(t-\tau_1,y)\,dy - \beta_1(\underline{u}_1^{(2)}(t,x))^2 - a_1\underline{u}_1^{(2)}(t,x)\bar{u}_2^{(2)}(t,x), \quad t > T_{21},\ x \in (0,\pi), \\ &\frac{\partial \underline{u}_2^{(2)}}{\partial t} = D_2 \frac{\partial^2 \underline{u}_2^{(2)}}{\partial x^2} + \alpha_2 e^{-\gamma_2\tau_2} \int_0^\pi G_2(x,y,\tau_2)\underline{u}_2^{(2)}(t-\tau_2,y)\,dy - \beta_2(\underline{u}_2^{(2)}(t,x))^2 - a_2\bar{u}_1^{(2)}(t,x)\underline{u}_2^{(2)}(t,x), \quad t > T_{21},\ x \in (0,\pi), \\ &\frac{\partial \underline{u}_1^{(2)}}{\partial x} = \frac{\partial \underline{u}_2^{(2)}}{\partial x} = 0, \quad t > T_{21},\ x = 0,\pi, \\ &\underline{u}_1^{(2)}(t,x) = \frac{1}{2}u_1(t,x), \quad \underline{u}_2^{(2)}(t,x) = \frac{1}{2}u_2(t,x), \quad (t,x) \in [-\tau, T_{21}] \times [0,\pi]. \end{aligned} \tag{3.31}$$

For $\varepsilon > 0$ sufficiently small, we consider the following auxiliary problem:

$$\begin{aligned} &\frac{\partial \omega_1^{(2)}}{\partial t} = D_1 \frac{\partial^2 \omega_1^{(2)}}{\partial x^2} + \alpha_1 e^{-\gamma_1\tau_1} \int_0^\pi G_1(x,y,\tau_1)\omega_1^{(2)}(t-\tau_1,y)\,dy - \beta_1(\omega_1^{(2)}(t,x))^2 - a_1\omega_1^{(2)}(t,x)(M_2^{u_2} + \varepsilon), \quad t > T_{21},\ x \in (0,\pi), \\ &\frac{\partial \omega_2^{(2)}}{\partial t} = D_2 \frac{\partial^2 \omega_2^{(2)}}{\partial x^2} + \alpha_2 e^{-\gamma_2\tau_2} \int_0^\pi G_2(x,y,\tau_2)\omega_2^{(2)}(t-\tau_2,y)\,dy - \beta_2(\omega_2^{(2)}(t,x))^2 - a_2(M_2^{u_1} + \varepsilon)\omega_2^{(2)}(t,x), \quad t > T_{21},\ x \in (0,\pi), \\ &\frac{\partial \omega_1^{(2)}}{\partial x} = \frac{\partial \omega_2^{(2)}}{\partial x} = 0, \quad t > T_{21},\ x = 0,\pi, \\ &\omega_1^{(2)}(t,x) = \frac{1}{2}u_1(t,x), \quad \omega_2^{(2)}(t,x) = \frac{1}{2}u_2(t,x), \quad (t,x) \in [-\tau, T_{21}] \times [0,\pi]. \end{aligned} \tag{3.32}$$

By Lemma 3.5, we have

$$\lim_{t\to+\infty} \omega_1^{(2)}(t,x) = \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_1(M_2^{u_2} + \varepsilon)}{\beta_1}, \quad \lim_{t\to+\infty} \omega_2^{(2)}(t,x) = \frac{\alpha_2 e^{-\gamma_2\tau_2} - a_2(M_2^{u_1} + \varepsilon)}{\beta_2}$$

uniformly for $x \in [0,\pi]$. By comparison, for $\varepsilon > 0$ sufficiently small, there is a $T_{22} > T_{21}$ such that if $t > T_{22}$

$$\min_{x\in[0,\pi]} \underline{u}_1^{(2)}(t,x) \geqslant \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_1 M_2^{u_2}}{\beta_1} - \varepsilon, \quad \min_{x\in[0,\pi]} \underline{u}_2^{(2)}(t,x) \geqslant \frac{\alpha_2 e^{-\gamma_2\tau_2} - a_2 M_2^{u_1}}{\beta_2} - \varepsilon.$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, it therefore follows that:

$$\begin{aligned} &V_1 = \liminf_{t\to+\infty} \min_{x\in[0,\pi]} u_1(t,x) \geqslant \frac{\alpha_1 e^{-\gamma_1\tau_1} - a_1 M_2^{u_2}}{\beta_1} := N_2^{u_1}, \\ &V_2 = \liminf_{t\to+\infty} \min_{x\in[0,\pi]} u_2(t,x) \geqslant \frac{\alpha_2 e^{-\gamma_2\tau_2} - a_2 M_2^{u_1}}{\beta_2} := N_2^{u_2}. \end{aligned} \tag{3.33}$$

Continuing this process, we get four sequences $M_n^{u_1}$, $M_n^{u_2}$, $N_n^{u_1}$, $N_n^{u_2}$ $(n = 1, 2, \ldots)$ such that for $n \geqslant 2$,

$$\begin{aligned} &\frac{\alpha_1 e^{-\gamma_1\tau_1} - a_1 M_n^{u_2}}{\beta_1} = N_n^{u_1}, \\ &\frac{\alpha_2 e^{-\gamma_2\tau_2} - a_2 M_n^{u_1}}{\beta_2} = N_n^{u_2}, \\ &\frac{\alpha_1 e^{-\gamma_1\tau_1} - a_1 N_{n-1}^{u_2}}{\beta_1} = M_n^{u_1}, \\ &\frac{\alpha_2 e^{-\gamma_2\tau_2} - a_2 N_{n-1}^{u_1}}{\beta_2} = M_n^{u_2}. \end{aligned} \tag{3.34}$$

Clearly we have

$$N_n^{u_1} \leqslant V_1 \leqslant U_1 \leqslant M_n^{u_1}, \quad N_n^{u_2} \leqslant V_2 \leqslant U_2 \leqslant M_n^{u_2}. \tag{3.35}$$

By induction, it is easy to show that the sequences $M_n^{u_1}$, $M_n^{u_2}$ are monotonically decreasing, but the sequences $N_n^{u_1}$, $N_n^{u_2}$ are monotonically increasing. Therefore, we see that $\lim_{n\to\infty} M_n^{u_1}$, $\lim_{n\to\infty} M_n^{u_2}$, $\lim_{n\to\infty} N_n^{u_1}$, $\lim_{n\to\infty} N_n^{u_2}$ exist. We denote them by A_1, A_2, B_1, B_2, respectively.

Letting $n \to +\infty$ in (3.34), we derive that

$$\begin{cases} a_1 A_2 + \beta_1 B_1 = \alpha_1 e^{-\gamma_1\tau_1}, \\ a_2 A_1 + \beta_2 B_2 = \alpha_2 e^{-\gamma_2\tau_2}, \\ a_1 B_2 + \beta_1 A_1 = \alpha_1 e^{-\gamma_1\tau_1}, \\ a_2 B_1 + \beta_2 A_2 = \alpha_2 e^{-\gamma_2\tau_2}, \end{cases}$$

which yields

$$\begin{aligned} &\beta_1(A_1 - B_1) - a_1(A_2 - B_2) = 0, \\ &a_2(A_1 - B_1) - \beta_2(A_2 - B_2) = 0. \end{aligned} \tag{3.36}$$

Noting that $\beta_1\beta_2 > a_1 a_2$, system (3.36) has only zero solution with respect to $A_1 - B_1$, $A_2 - B_2$. It therefore follows that $A_1 = B_1$, $A_2 = B_2$. It is easy to obtain from (3.34) and (3.35) that $U_1 = V_1 = A_1 = B_1 = u_1^*$, $U_2 = V_2 = A_2 = B_2 = u_2^*$. This completes the proof. □

In a similar way, one can prove the following results.

Theorem 3.3. *Let the initial functions* $\phi_1(t,x)$ *and* $\phi_2(t,x)$ *are Hölder continuous in* $[-\tau, 0] \times [0, \pi]$, *and* $\phi_1, \phi_2 \geqslant 0$, $\phi_1(0,x) \neq 0$, $\phi_2(0,x) \neq 0$. *Let* (*D*1) *hold. Then system* 3.10,3.11,3.12 *has a unique positive solution* $(u_1(t,x), u_2(t,x))$ *satisfying*

$$\lim_{t\to+\infty} u_1(t,x) = \frac{\alpha_1 e^{-\gamma_1\tau_1}}{\beta_1}, \quad \lim_{t\to+\infty} u_2(t,x) = 0$$

uniformly for $x \in [0, \pi]$.

Theorem 3.4. *Let the initial functions* $\phi_1(t,x)$ *and* $\phi_2(t,x)$ *are Hölder continuous in* $[-\tau, 0] \times [0, \pi]$, *and* $\phi_1, \phi_2 \geqslant 0$, $\phi_1(0,x) \neq 0$, $\phi_2(0,x) \neq 0$. *Let* (*D*2) *hold. Then system* 3.10,3.11,3.12 *has a unique positive solution* $(u_1(t,x), u_2(t,x))$ *satisfying*

$$\lim_{t\to+\infty} u_1(t,x) = 0, \quad \lim_{t\to+\infty} u_2(t,x) = \frac{\alpha_2 e^{-\gamma_2\tau_2}}{\beta_2}$$

uniformly for $x \in [0, \pi]$.

Finally, let $(u_1(t,x), v_1(t,x), u_2(t,x), v_2(t,x))$ be the solution of problem (3.10)–(3.12). Using similar arguments to those in [6], we can derive that

$$\lim_{t\to+\infty} v_i(t,x) = \frac{\alpha_i}{\gamma_i}(1 - e^{-\gamma_i\tau_i}) \lim_{t\to+\infty} u_i(t,x)$$

uniformly for $x \in [0, \pi]$.

4. Discussion

In this paper, we have discussed stage-structured reaction–diffusion competition models with delays and spatially nonlocal effect in the case of the domain being finite and infinite, respectively. In the latter case, we have discussed the effect of time delays on the existence of travelling fronts of system (2.1) modelling the competition between the adult members of two species. Our approach was to use the upper–lower solution technique and iteration method recently developed by Wu and Zou [23]. By Theorem 2.2, we see that if (A1)–(A2) hold, travelling wave fronts exist connecting the two boundary equilibria of system (2.1). Ecologically, if the intra-specific competition rate and the harvesting rate of the mature individuals u_1 and the transformation rate of the species u_2 are low, and the transformation rate of the species u_1 and the harvesting rate and the intra-specific competition rate of the species u_2 are sufficiently high to satisfy (A1)–(A2), the weaker competitor u_2 will be driven to extinction by the stronger u_1 in a "travelling wave" of invasion.

We note that if the immatures are not moving and no harvesting effort is put on the two mature populations, then letting $d_i \to 0$, $E_i = 0$ $(i = 1, 2)$, system (2.1) becomes

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= D_1\frac{\partial^2 u_1}{\partial x^2} + \alpha_1 e^{-\gamma_1\tau_1}u_1(t-\tau_1,x) - \beta_1 u_1^2(t,x) - a_1u_1(t,x)u_2(t,x),\\ \frac{\partial u_2}{\partial t} &= D_2\frac{\partial^2 u_2}{\partial x^2} + \alpha_2 e^{-\gamma_2\tau_2}u_2(t-\tau_2,x) - \beta_2 u_2^2(t,x) - a_2u_1(t,x)u_2(t,x)\end{aligned} \tag{4.1}$$

for $t > 0$, $x \in (-\infty, \infty)$. In [3], Al-Omari and Gourley discussed the existence of travelling wave fronts of system (4.1). Sufficient conditions were derived for the existence of a travelling front connecting the boundary equilibria $(\alpha_1 e^{-\gamma_1\tau_1}/\beta_1, 0)$ and $(0, \alpha_2 e^{-\gamma_2\tau_2}/\beta_2)$ under the following assumptions:

(i) $a_2\alpha_1 e^{-\gamma_1\tau_1} > \beta_1\alpha_2 e^{-\gamma_2\tau_2}$, $a_1\alpha_2 e^{-\gamma_2\tau_2} < \beta_1\alpha_1 e^{-\gamma_1\tau_1}$,
(ii) $\Delta(\lambda_1) \leqslant 2(\alpha_2 e^{-\gamma_2\tau_2} - \frac{a_2\alpha_1}{\beta_1}e^{-\gamma_1\tau_1})$,

where $\Delta = \alpha_2 e^{-\gamma_2\tau_2}e^{-\lambda c\tau_2} - (a_2\alpha_1/\beta_1)e^{-\gamma_1\tau_1} - (c\lambda - D_2\lambda^2)$, and λ_1 is the smaller positive root of the equation

$$\alpha_1 e^{-\gamma_1\tau_1}e^{-\lambda c\tau_1} - \frac{a_1\alpha_2}{\beta_2}e^{-\gamma_2\tau_2} - (c\lambda - D_1\lambda^2) = 0.$$

Therefore, Theorem 2.2 may be viewed as an extension of the work developed by Al-Omari and Gourley in [3] in the sense that the nonlocal spatial effect induced by the movement of the mature population is considered.

The global convergence of reaction–diffusion systems with delays have been studied by many authors. However, most of the systems previously considered are mixed quasi-monotone, and most of the discussions are in the framework of semigroup theory of dynamical systems. Recently, there has been much work on the global stability of the positive steady-state of reaction–diffusion population models by using the technique of coupled lower–upper solutions and monotone iterations (see, for example, [14–16]). However, these do not take into account the nonlocal spatial effect. In the present paper, we have been interested in the global dynamics of stage-structured competition model with delays and nonlocal spatial effect. In the case of the domain being finite, we have shown that the global dynamics of problem (3.10)–(3.12) can be completely determined except in the case when the two boundary equilibria are both locally stable ($\beta_1\beta_2 < a_1a_2$). In this case the asymptotic behavior will depend on the initial conditions. If $\beta_1\beta_2 > a_1a_2$, via successive modification of upper–lower solutions, a set of easily verifiable sufficient conditions were derived for the global attractiveness of the nonnegative uniform equilibria of problem (3.10)–(3.12). By Theorem 3.2, when both boundary equilibria are locally unstable, we showed that the positive equilibrium of problem (3.10)–(3.12) is uniformly globally attractive. Ecologically, this means that if the intra-specific competitions dominate the inter-specific competitions, then the competitors will coexist. By Theorems 3.3 and 3.4, we see that the weaker competitor will be driven to extinction if (D1) or (D2) holds. By Theorems 3.2, 3.3 and 3.4, we have shown the following three typical dynamical behaviors are possible: (i) *coexistence*, (ii) *bistability*, (iii) *dominance*. Therefore, in this sense problem (3.10)–(3.12) has similar global dynamics to the classical Lotka–Volterra competition model without time delays.

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