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Periodic solutions of a Lotka–Volterra type multi-species population model with time delays

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A delayed periodic Lotka–Volterra type population model with m predators and n preys is investigated. By using Gaines and Mawhin's continuation theorem of coincidence degree theory and by constructing suitable Lyapunov functionals, sufficient conditions are derived for the existence, uniqueness and global stability of positive periodic solutions of the model. Numerical simulation is presented to illustrate the feasibility of our main results.

1 Introduction

The classical Lotka–Volterra type systems form a significant component of the models of multi-species population dynamics. There is now a considerable body of work on the study of the global asymptotic stability of Lotka–Volterra type systems with time delays that have been developed by many authors (see, for example, [1, 6, 7, 10, 13, 15, 17, 18, 19, 20, 21, 23]). So far, most of the work on the global dynamics of Lotka–Volterra type systems have been done for pure predator-prey interactions or pure competitive interactions. In view of the fact that in real-life interactions, there may be competitions among different prey (predator) species at the same trophic level (see, for example, [16, 22]), an alternative, and perhaps more realistic approach, is to incorporate interspecific competitions among different prey species at the same trophic level consuming common resource and to incorporate interspecific competitions among different predator species at the same trophic level consuming common preys in an ecological system.

We note that any biological or environmental parameters are naturally subject to fluctuation in time. As Cushing [4] pointed out that it is necessary and important to consider models with periodic ecological parameters or perturbations which might be quite naturally exposed (for example, those due to seasonal effects of weather, food supply, mating habits, hunting or harvesting seasons, etc.). Thus, the assumption of periodicity of the parameters is a way of incorporating the periodicity of the environment.

Time delays of one type or another have been incorporated into biological models by many researchers, we refer to the monographs of Gopalsamy [11] and Kuang [14] for general delayed biological systems and to Beretta and Kuang [2, 3], Gopalsamy [8, 9], and the references cited therein for studies on delayed biological systems. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the population to fluctuate. Time delay due to gestation is a common example, because generally the consumption of prey by the predator throughout its past history governs the present birth rate of the predator. Therefore, more realistic models of population interactions should take into account the periodicity of the changing environment and the effects of time delays.

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The main purpose of this paper is to study the combined effects of periodicity of ecological and environmental parameters and time delays due to gestations and negative feedbacks on the dynamics of predation-competition systems in which m different predator species at the same trophic level compete and feed on n common preys. To do so, we study the following delayed Lotka–Volterra type system

$$\begin{cases} \dot{x}_i(t) \;=\; x_i \left[r_i(t) - \sum_{k=1}^{n} a_{ik}(t) x_k(t-\tau_{ik}) - \sum_{l=1}^{m} b_{il}(t) y_l(t) \right], \\ \dot{y}_j(t) \;=\; y_j \left[-e_j(t) + \sum_{k=1}^{n} c_{jk}(t) x_k(t-\sigma_j) - \sum_{l=1}^{m} d_{jl}(t) y_l(t-\mu_{jl}) \right], \\ \qquad\qquad i \,=\, 1,2,\ldots,n, \quad j \,=\, 1,2,\ldots,m, \end{cases} \tag{1.1}$$

with initial conditions

$$\begin{aligned} x_i(\theta) \;&=\; \phi_i(\theta), \quad \theta \in [-\tau, 0], \quad \phi_i(0) \,>\, 0, \quad i \,=\, 1,2,\ldots,n, \\ y_j(\theta) \;&=\; \psi_j(\theta), \quad \theta \in [-\tau, 0], \quad \psi_j(0) \,>\, 0, \quad j \,=\, 1,2,\ldots,m, \end{aligned} \tag{1.2}$$

where $x_i(t)$ denotes the density of prey species X_i at time t, $y_j(t)$ denotes the density of predator species Y_j at time t, respectively. $\tau = \max\{\tau_{ik},\ \sigma_j,\ \mu_{jl},\ i,k = 1,2,\ldots,n,\ j,l = 1,2,\ldots,m\}$; ϕ_i and ψ_j are given nonnegative and bounded continuous functions on $[-\tau, 0]$. $r_i(t)$ is the intrinsic growth rate of species X_i, $i = 1,2,\ldots,n$; $e_j(t)$ is the death rate of species Y_j when all prey species and other predator species are absent, $j = 1,2,\ldots,m$; $a_{ii}(t)$ is the intra-specific competition rate of species X_i, $i = 1,2,\ldots,n$; $d_{ii}(t)$ is the intra-specific competition rate of species Y_j, $j = 1,2,\ldots,m$; $b_{ij}(t)$ is the capturing rate of the species Y_j upon species X_i, $i = 1,2,\ldots,n$; $j = 1,2,\ldots,m$; $c_{ji}(t)/b_{ij}(t)$ is the conversion rate of nutrients into the reproduction of the species Y_j, $i = 1,2,\ldots,n$; $j = 1,2,\ldots,m$; $a_{ik}(t)$ and $a_{ki}(t)$ are the rates of interspecific competition between species X_i and X_k, $i \neq k$, $i,k = 1,2,\ldots,n$; $d_{jl}(t)$ and $d_{lj}(t)$ are the rates of interspecific competition between species Y_j and Y_l, $j \neq l$, $j,l = 1,2,\ldots,m$. τ_{ii} and μ_{jj} are time delays due to negative feedbacks of species X_i and Y_j, respectively, σ_j is the constant delay due to gestation of species Y_j, $j = 1,2,\ldots,m$. τ_{ik} and μ_{jl} may be the delays required for maturity of species X_k and Y_l, $i \neq k$, $l \neq j$, $i,k = 1,2,\ldots,n$; $j,l = 1,2,\ldots,m$.

In this paper, for system (1.1) we always assume that for all $i,k = 1,2,\ldots,n$; $j,l = 1,2,\ldots,m$:

(H1) $a_{ik}(t)$, $b_{il}(t)$, $c_{jk}(t)$, $d_{jl}(t)$, $r_i(t)$ and $e_j(t)$ are continuously positive periodic functions with period ω.

It is well-known by the fundamental theory of functional differential equations [12] that system (1.1) has a unique solution $z(t) = (x_1(t),\ldots,x_n(t),y_1(t),\ldots,y_m(t))$ satisfying the initial conditions (1.2). It is easy to show that solutions of system (1.1) with initial conditions (1.2) are defined on $[0,+\infty)$ and remain positive for all $t \geq 0$. In this paper, the solution of system (1.1) satisfying the initial conditions (1.2) is said to be positive.

The organization of this paper is as follows. In the next section, by using Gaines and Mawhin's continuation theorem of coincidence degree theory, we establish sufficient conditions for the existence of positive ω-periodic solutions of system (1.1). In Section 3, by constructing suitable Lyapunov functionals, sufficient conditions are derived for the uniqueness and global stability of positive periodic solutions of system (1.1). Numerical simulation is presented to illustrate the validity of our main results. In Section 4, a brief discussion is given to conclude this work.

2 Existence of positive periodic solutions

In this section, by using Gaines and Mawhin's continuation theorem of coincidence degree theory, we show the existence of positive periodic solutions of system (1.1). To this end, we first introduce some concepts and notations which will be used in this section.

Let X and Y be real Banach spaces, let L: $\operatorname{Dom} L \subset X \to Y$ be a linear mapping, and let $N : X \to Y$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$, and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, then the restriction L_P of

L to $\operatorname{Dom} L \cap \operatorname{Ker} P : (I-P)X \to \operatorname{Im} L$ is invertible. Denote the inverse of L_P by K_P. If Ω is an open bounded subset of X, the mapping N will be called *L-compact* on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

For convenience of use, we introduce the continuation theorem (see [5, p. 40]) as follows.

Lemma 2.1 . *Let $\Omega \subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and let N be L-compact on $\overline{\Omega}$. Assume*

(i) *for each $\lambda \in (0,1), x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;*
(ii) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$;*
(iii) $\deg\{JQN,\, \Omega \cap \operatorname{Ker} L,\, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

In what follows we shall use the following notations:

$$\bar{f} \;=\; \frac{1}{\omega}\int_0^\omega f(t)\,dt\,, \quad f^L \;=\; \min_{t\in[0,\omega]} f(t)\,, \quad f^M \;=\; \max_{[0,\omega]} f(t)\,,$$

where f is a continuous ω-periodic function.

We are now in a position to state and prove our result on the existence of positive periodic solutions of system (1.1).

Theorem 2.2 *In addition to* (H1)*, assume further that for $i = 1,2,\ldots,n$, and $j = 1,2,\ldots,m$:*

$$(\mathbf{H2}) \qquad \overline{r}_i - \sum_{k=1,k\neq i}^{n} \frac{\overline{r}_k}{a_{kk}^L}\left(a_{ik}^M + \sum_{l=1}^{m}\frac{b_{il}^M c_{lk}^M}{d_{ll}^L}\right) \;>\; 0\,,$$

$$(\mathbf{H3}) \qquad \sum_{k=1}^{n}\frac{c_{jk}^L}{a_{kk}^M}\left(\overline{r}_k - \sum_{l=1,l\neq k}^{n}\frac{\overline{r}_l a_{kl}^M}{a_{ll}^L}\right) - \sum_{k=1}^{n}\sum_{l=1,l\neq j}^{m}\frac{\overline{r}_k c_{lk}^M}{a_{kk}^L d_{ll}^L}\left(d_{jl}^M + \sum_{i=1}^{n}\frac{b_{il}^M c_{ji}^L}{a_{ii}^M}\right) - \overline{e}_j \;>\; 0\,.$$

Then system (1.1) *has at least one positive ω-periodic solution.*

P r o o f. Let

$$u_i(t) \;=\; \ln x_i(t)\,, \quad v_j(t) \;=\; \ln y_j(t) \quad (i = 1,2,\ldots,n;\; j = 1,2,\ldots,m)\,. \tag{2.1}$$

On substituting (2.1) into (1.1), system (1.1) becomes

$$\begin{aligned} \frac{du_i(t)}{dt} \;&=\; r_i(t) - \sum_{k=1}^{n} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum_{l=1}^{m} b_{il}(t)e^{v_l(t)}\,,\\ \frac{dv_j(t)}{dt} \;&=\; -e_j(t) + \sum_{k=1}^{n} c_{jk}(t)e^{u_k(t-\sigma_j)} - \sum_{l=1}^{m} d_{jl}(t)e^{v_l(t-\mu_{jl})}\,,\\ & i \;=\; 1,2,\ldots,n\,, \quad j \;=\; 1,2,\ldots,m\,. \end{aligned} \tag{2.2}$$

Clearly, if system (2.2) has one ω-periodic solution $(u_1^*(t),\ldots,u_n^*(t),v_1^*(t),\ldots,v_m^*(t))^T$, then $(\exp[u_1^*(t)],\ldots,$ $\exp[u_n^*(t)],\exp[v_1^*(t)],\ldots,\exp[v_m^*(t)])^T$ is a positive ω-periodic solution of system (1.1). Therefore, to complete the proof, it suffices to prove that (2.2) admits at least one ω-periodic solution.

Take

$$\begin{aligned} &u(t) \;=\; (u_1(t),\ldots,u_n(t))^T, \quad v(t) \;=\; (v_1(t),\ldots,v_m(t))^T,\\ &X \;=\; Y \;=\; \bigl\{(u(t),v(t))^T \in C(\mathbb{R},\mathbb{R}^{m+n}) : u(t+\omega) = u(t),\; v(t+\omega) = v(t)\bigr\} \end{aligned}$$

and

$$\bigl\|(u_1(t),\ldots,u_n(t),v_1(t),\ldots,v_m(t))^T\bigr\| \;=\; \sum_{i=1}^{n}\max_{t\in[0,\omega]}|u_i(t)| + \sum_{j=1}^{m}\max_{t\in[0,\omega]}|v_j(t)|\,,$$

here $|\cdot|$ denotes the Euclidean norm. It is easy to see that both X and Y are Banach spaces with the norm $\|\cdot\|$.

Let

$$\begin{aligned}\Delta_i^{(1)}(t) &= r_i(t) - \sum_{k=1}^{n} a_{ik}(t) e^{u_k(t-\tau_{ik})} - \sum_{l=1}^{m} b_{il}(t) e^{v_l(t)},\\ \Delta_j^{(2)}(t) &= -e_j(t) + \sum_{k=1}^{n} c_{jk}(t) e^{u_k(t-\sigma_j)} - \sum_{l=1}^{m} d_{jl}(t) e^{v_l(t-\mu_{jl})},\\ &\qquad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m.\end{aligned}$$

Set

$$L : \mathrm{Dom}\, L \cap X \longrightarrow X, \quad L(u(t), v(t))^T = \left(\frac{du(t)}{dt}, \frac{dv(t)}{dt}\right)^T,$$

where

$$\begin{aligned}\frac{du(t)}{dt} &= \left(\frac{du_1(t)}{dt}, \ldots, \frac{du_n(t)}{dt}\right)^T, \quad \frac{dv(t)}{dt} = \left(\frac{dv_1(t)}{dt}, \ldots, \frac{dv_m(t)}{dt}\right)^T,\\ \mathrm{Dom}\, L &= \left\{(u_1(t), \ldots, u_n(t), v_1(t), \ldots, v_m(t))^T \in C^1(\mathbb{R}, \mathbb{R}^{m+n})\right\},\end{aligned}$$

and $N : X \to X$,

$$N(u_1, \ldots, u_n, v_1, \ldots, v_m)^T = \left(\Delta_1^{(1)}(t), \ldots, \Delta_n^{(1)}(t), \Delta_1^{(2)}(t), \ldots, \Delta_m^{(2)}(t)\right)^T.$$

Define

$$P\begin{bmatrix} u_1\\ \vdots\\ u_n\\ v_1\\ \vdots\\ v_m\end{bmatrix} = Q\begin{bmatrix} u_1\\ \vdots\\ u_n\\ v_1\\ \vdots\\ v_m\end{bmatrix} = \begin{bmatrix} \frac{1}{\omega}\int_0^\omega u_1(t)\,dt\\ \vdots\\ \frac{1}{\omega}\int_0^\omega u_n(t)\,dt\\ \frac{1}{\omega}\int_0^\omega v_1(t)\,dt\\ \vdots\\ \frac{1}{\omega}\int_0^\omega v_m(t)\,dt\end{bmatrix}, \quad \begin{bmatrix} u_1\\ \vdots\\ u_n\\ v_1\\ \vdots\\ v_m\end{bmatrix} \in X = Y.$$

It is easy to show that

$$\begin{aligned}\mathrm{Ker}\, L &= \{x \mid x \in X,\ x = h,\ h \in \mathbb{R}^{m+n}\},\\ \mathrm{Im}\, L &= \left\{y \mid y \in Y,\ \int_0^\omega y(t)\,dt = 0\right\} \text{ is closed in } Y,\end{aligned}$$

and

$$\dim \mathrm{Ker}\, L = \mathrm{codim}\, \mathrm{Im}\, L = m + n.$$

Therefore, L is a Fredholm mapping of index zero. It is easy to show that P and Q are continuous projectors such that

$$\mathrm{Im}\, P = \mathrm{Ker}\, L, \quad \mathrm{Ker}\, Q = \mathrm{Im}\, L = \mathrm{Im}\,(I - Q).$$

Furthermore, the inverse K_P of L_P exists and has the form K_P: $\mathrm{Im}\, L \to \mathrm{Dom}\, L \cap \mathrm{Ker}\, P$,

$$K_P(y) = \int_0^t y(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t y(s)\,ds\,dt.$$

Then $QN : X \to Y$ and $K_P(I-Q)N : X \to X$ are given by

$$QNx \;=\; \left(\frac{1}{\omega}\int_0^\omega \Delta_1^{(1)}(t)\,dt,\ \ldots,\frac{1}{\omega}\int_0^\omega \Delta_n^{(1)}(t)\,dt,\ \frac{1}{\omega}\int_0^\omega \Delta_1^{(2)}(t)\,dt,\ \ldots,\ \frac{1}{\omega}\int_0^\omega \Delta_m^{(2)}(t)\,dt\right)^T,$$

$$K_P(I-Q)Nx \;=\; \int_0^t Nx(s)\,ds - \frac{1}{\omega}\int_0^\omega\int_0^t Nx(s)\,ds\,dt - \left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^\omega Nx(s)\,ds\,.$$

Obviously, QN and $K_P(I-Q)N$ are continuous.

In order to apply Lemma 2.1, we need to search for an appropriate open, bounded subset Ω.

Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$\begin{aligned} \frac{du_i(t)}{dt} \;&=\; \lambda\left[r_i(t) - \sum_{k=1}^n a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum_{l=1}^m b_{il}(t)e^{v_l(t)}\right], \\ \frac{dv_j(t)}{dt} \;&=\; \lambda\left[-e_j(t) + \sum_{k=1}^n c_{jk}(t)e^{u_k(t-\sigma_j)} - \sum_{l=1}^m d_{jl}(t)e^{v_l(t-\mu_{jl})}\right], \\ &\qquad i \;=\; 1,2,\ldots,n, \quad j \;=\; 1,2,\ldots,m. \end{aligned} \tag{2.3}$$

Suppose that $(u_1(t),\ldots,u_n(t),v_1(t),\ldots,v_m(t))^T \in X$ is a solution of (2.3) for a certain $\lambda \in (0,1)$. Integrating (2.3) over the interval $[0,\omega]$ we derive

$$\sum_{k=1}^n \int_0^\omega a_{ik}(t)e^{u_k(t-\tau_{ik})}\,dt + \sum_{l=1}^m \int_0^\omega b_{il}(t)e^{v_l(t)}\,dt \;=\; \int_0^\omega r_i(t)\,dt\,, \quad i \;=\; 1,2,\ldots,n, \tag{2.4}$$

$$\begin{aligned} \int_0^\omega e_j(t)\,dt + \sum_{l=1}^m \int_0^\omega d_{jl}(t)e^{v_l(t-\mu_{jl})}\,dt \;&=\; \sum_{k=1}^n \int_0^\omega c_{jk}(t)e^{u_k(t-\sigma_j)}\,dt\,, \\ &\qquad j \;=\; 1,2,\ldots,m. \end{aligned} \tag{2.5}$$

It follows from (2.3) and (2.4) that

$$\begin{aligned} \int_0^\omega |u_i'(t)|\,dt \;<\; \int_0^\omega \left[r_i(t) + \sum_{k=1}^n a_{ik}(t)e^{u_k(t-\tau_{ik})} + \sum_{l=1}^m b_{il}(t)e^{v_l(t)}\right]dt \;&=\; 2\overline{r}_i\omega\,, \\ i \;&=\; 1,2,\ldots,n. \end{aligned} \tag{2.6}$$

Since $(u_1(t),\ldots,u_n(t),v_1(t),\ldots,v_m(t))^T \in X$, there exist $\xi_i, \eta_i, \gamma_j, \delta_j \in [0,\omega]$ such that

$$\begin{aligned} u_i(\xi_i) \;&=\; \min_{t\in[0,\omega]} u_i(t)\,, \quad u_i(\eta_i) \;=\; \max_{t\in[0,\omega]} u_i(t)\,, \quad i \;=\; 1,2,\ldots,n, \\ v_j(\gamma_j) \;&=\; \min_{t\in[0,\omega]} v_j(t)\,, \quad v_j(\delta_j) \;=\; \max_{t\in[0,\omega]} v_j(t)\,, \quad j \;=\; 1,2,\ldots,m. \end{aligned} \tag{2.7}$$

It follows from (2.4) that

$$\int_0^\omega a_{ii}(t)e^{u_i(t-\tau_{ii})}\,dt \;\le\; \int_0^\omega r_i(t)\,dt\,. \tag{2.8}$$

Noting that

$$\int_0^\omega e^{u_i(t-\tau_{ik})}\,dt \;=\; \int_0^\omega e^{u_i(t)}\,dt\,, \quad i,k \;=\; 1,2,\ldots,n, \tag{2.9}$$

we derive from (2.8) that

$$\int_0^\omega e^{u_i(t)}\,dt \;\le\; \frac{\overline{r}_i\omega}{a_{ii}^L}\,, \quad u_i(\xi_i) \;\le\; \ln\frac{\overline{r}_i}{\overline{a}_{ii}}\,, \quad i \;=\; 1,2,\ldots,n. \tag{2.10}$$

It follows from (2.6) and (2.10) that

$$u_i(t) \le u_i(\xi_i) + \int_0^\omega |u_i'(t)|\,dt \le \ln\frac{\overline{r}_i}{\overline{a}_{ii}} + 2\overline{r}_i\omega\,, \quad i = 1, 2, \dots, n\,. \tag{2.11}$$

We derive from (2.5) that

$$d_{jj}^L \int_0^\omega e^{v_j(t-\mu_{jj})}\,dt \le \sum_{k=1}^n c_{jk}^M \int_0^\omega e^{u_k(t-\sigma_j)}\,dt\,, \quad j = 1, 2, \dots, m\,. \tag{2.12}$$

Noting that

$$\begin{aligned} \int_0^\omega e^{u_i(t-\sigma_j)}\,dt &= \int_0^\omega e^{u_i(t)}\,dt\,, \quad i = 1, 2, \dots, n\,, \\ \int_0^\omega e^{v_j(t-\mu_{jl})}\,dt &= \int_0^\omega e^{v_j(t)}\,dt\,, \quad j, l = 1, 2, \dots, m\,, \end{aligned} \tag{2.13}$$

it follows from (2.10) and (2.12) that

$$\begin{aligned} \int_0^\omega e^{v_j(t)}\,dt &\le \frac{1}{d_{jj}^L}\sum_{k=1}^n c_{jk}^M \int_0^\omega e^{u_k(t)}\,dt \le \frac{\omega}{d_{jj}^L}\sum_{k=1}^n \frac{\overline{r}_k c_{jk}^M}{a_{kk}^L}\,, \\ v_j(\gamma_j) &\le \ln\left\{\frac{1}{d_{jj}^L}\sum_{k=1}^n \frac{\overline{r}_k c_{jk}^M}{a_{kk}^L}\right\}, \quad j = 1, 2, \dots, m\,. \end{aligned} \tag{2.14}$$

We derive from (2.3), (2.5), (2.10) and (2.13) that

$$\begin{aligned} \int_0^\omega |v_j'(t)|\,dt &< \int_0^\omega \left[e_j(t) + \sum_{k=1}^n c_{jk}(t)e^{u_k(t-\sigma_j)} + \sum_{l=1}^m d_{jl}(t)e^{v_l(t-\mu_{jl})}\right]dt \\ &= 2\sum_{k=1}^n \int_0^\omega c_{jk}(t)e^{u_k(t-\sigma_j)}\,dt \\ &\le 2\sum_{k=1}^n c_{jk}^M \int_0^\omega e^{u_k(t)}\,dt \\ &= 2\sum_{k=1}^n \frac{\overline{r}_k c_{jk}^M \omega}{a_{kk}^L}\,. \end{aligned} \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$\begin{aligned} v_j(t) \le v_j(\gamma_j) + \int_0^\omega |v_j'(t)|\,dt \le \ln\left\{\frac{1}{d_{jj}^L}\sum_{k=1}^n \frac{\overline{r}_k c_{jk}^M}{a_{kk}^L}\right\} + 2\sum_{k=1}^n \frac{\overline{r}_k c_{jk}^M \omega}{a_{kk}^L}\,, \\ j = 1, 2, \dots, m\,. \end{aligned} \tag{2.16}$$

From (2.4), (2.13) and (2.14), we obtain

$$\begin{aligned} \int_0^\omega a_{ii}(t)e^{u_i(t-\tau_{ii})}\,dt &\ge \overline{r}_i\omega - \sum_{l=1}^m b_{il}^M \int_0^\omega e^{v_l(t)}\,dt - \sum_{k=1,\ k\ne i}^n a_{ik}^M \int_0^\omega e^{u_k(t-\tau_{ik})}\,dt \\ &\ge \overline{r}_i\omega - \sum_{l=1}^m \sum_{k=1}^n \frac{b_{il}^M c_{lk}^M}{d_{ll}^L}\int_0^\omega e^{u_k(t)}\,dt - \sum_{k=1,\ k\ne i}^n a_{ik}^M \int_0^\omega e^{u_k(t)}\,dt \\ &= \overline{r}_i\omega - \sum_{l=1}^m \frac{b_{il}^M c_{li}^M}{d_{ll}^L}\int_0^\omega e^{u_i(t)}\,dt - \sum_{\substack{k=1\\ k\ne i}}^n \left(a_{ik}^M + \sum_{l=1}^m \frac{b_{il}^M c_{lk}^M}{d_{ll}^L}\right)\int_0^\omega e^{u_k(t)}\,dt\,, \end{aligned}$$

which, together with (2.9) and (2.10), implies

$$\int_0^\omega e^{u_i(t)}\,dt \;\geq\; \frac{\overline{r}_i\omega - \sum\limits_{k=1,k\neq i}^{n} \overline{r}_k\omega\left(a_{ik}^M + \sum\limits_{l=1}^{m} b_{il}^M c_{lk}^M/d_{ll}^L\right)/a_{kk}^L}{a_{ii}^M + \sum\limits_{l=1}^{m} b_{il}^M c_{li}^M/d_{ll}^L} \;:=\; a_i\omega\,, \qquad i = 1,2,\dots,n\,, \tag{2.17}$$

and

$$u_i(\eta_i) \;\geq\; \ln \frac{\overline{r}_i - \sum\limits_{k=1,k\neq i}^{n} \overline{r}_k\left(a_{ik}^M + \sum\limits_{l=1}^{m} b_{il}^M c_{lk}^M/d_{ll}^L\right)/a_{kk}^L}{a_{ii}^M + \sum\limits_{l=1}^{m} b_{il}^M c_{li}^M/d_{ll}^L} \;:=\; \ln a_i\,,\;\; i = 1,2,\dots,n\,. \tag{2.18}$$

It follows from (2.6) and (2.18) that

$$u_i(t) \;\geq\; u_i(\eta_i) - \int_0^\omega |u_i'(t)|\,dt \;>\; \ln a_i - 2\overline{r}_i\omega\,, \quad i = 1,2,\dots,n\,,$$

which, together with (2.11), leads to

$$\max_{t\in[0,\omega]} |u_i(t)| \;<\; \max\left\{\left|\ln\frac{\overline{r}_i}{\overline{a}_{ii}}\right| + 2\overline{r}_i\omega, |\ln a_i| + 2\overline{r}_i\omega\right\} \;:=\; R_i\,, \quad i = 1,2,\dots,n\,. \tag{2.19}$$

We derive from (2.5) and (2.13) that

$$d_{jj}^M\int_0^\omega e^{v_j(t)}dt \;\geq\; \sum_{k=1}^{n} c_{jk}^L\int_0^\omega e^{u_k(t)}dt - \sum_{l=1,\,l\neq j}^{m} d_{jl}^M\int_0^\omega e^{v_l(t)}dt - \overline{e}_j\omega\,. \tag{2.20}$$

It follows from (2.4) and (2.9) that

$$a_{ii}^M\int_0^\omega e^{u_i(t)}dt \;\geq\; \overline{r}_i\omega - \sum_{l=1}^{m} b_{il}^M\int_0^\omega e^{v_l(t)}dt - \sum_{k=1,\,k\neq i}^{n} a_{ik}^M\int_0^\omega e^{u_k(t)}dt\,. \tag{2.21}$$

We obtain from (2.20) and (2.21) that

$$\begin{aligned}
d_{jj}^M\int_0^\omega e^{v_j(t)}dt \;\geq\;& \sum_{k=1}^{n}\frac{c_{jk}^L}{a_{kk}^M}\left[\overline{r}_k\omega - \sum_{l=1}^{m} b_{kl}^M\int_0^\omega e^{v_l(t)}dt - \sum_{l=1,\,l\neq k}^{n} a_{kl}^M\int_0^\omega e^{u_l(t)}dt\right]\\
&- \sum_{l=1,\,l\neq j}^{m} d_{jl}^M\int_0^\omega e^{v_l(t)}dt - \overline{e}_j\omega\\
=\;& \sum_{k=1}^{n}\frac{\overline{r}_k c_{jk}^L\omega}{a_{kk}^M} - \sum_{k=1}^{n}\sum_{l=1}^{m}\frac{b_{kl}^M c_{jk}^L}{a_{kk}^M}\int_0^\omega e^{v_l(t)}dt\\
&- \sum_{k=1}^{n}\sum_{l=1,\,l\neq k}^{n}\frac{a_{kl}^M c_{jk}^L}{a_{kk}^M}\int_0^\omega e^{u_l(t)}dt - \sum_{l=1,\,l\neq j}^{m} d_{jl}^M\int_0^\omega e^{v_l(t)}dt - \overline{e}_j\omega\\
=\;& \sum_{k=1}^{n}\frac{c_{jk}^L}{a_{kk}^M}\left(\overline{r}_k\omega - \sum_{l=1,\,l\neq k}^{n} a_{kl}^M\int_0^\omega e^{u_l(t)}dt\right)\\
&- \sum_{k=1}^{n}\frac{b_{kj}^M c_{jk}^L}{a_{kk}^M}\int_0^\omega e^{v_j(t)}dt - \sum_{k=1}^{n}\sum_{l=1,\,l\neq j}^{m}\frac{b_{kl}^M c_{jk}^L}{a_{kk}^M}\int_0^\omega e^{v_l(t)}dt\\
&- \sum_{l=1,\,l\neq j}^{m} d_{jl}^M\int_0^\omega e^{v_l(t)}dt - \overline{e}_j\omega\,,
\end{aligned}$$

which, together with (2.10) and (2.14), implies that

$$\int_0^\omega e^{v_j(t)}dt \geq \frac{\sum\limits_{k=1}^{n}\frac{c_{jk}^L\omega}{a_{kk}^M}\left(\overline{r}_k-\sum\limits_{l=1,\,l\neq k}^{n}\frac{\overline{r}_l a_{kl}^M}{a_{ll}^L}\right)-\sum\limits_{k=1}^{n}\sum\limits_{l=1,\,l\neq j}^{m}\frac{\overline{r}_k c_{lk}^M\omega}{a_{kk}^L d_{ll}^L}\left(d_{jl}^M+\sum\limits_{i=1}^{n}\frac{b_{il}^M c_{ji}^L}{a_{ii}^M}\right)-\overline{e}_j\omega}{d_{jj}^M+\sum\limits_{k=1}^{n}\frac{b_{kj}^M c_{jk}^L}{a_{kk}^M}}$$
$$:= b_j\omega\,,$$

which yields

$$v_j(\delta_j) \;\geq\; \ln b_j\,,\quad j\;=\;1\,,2\,,\ldots\,,m\,. \tag{2.22}$$

It follows from (2.15) and (2.22) that

$$v_j(t)\;\geq\;v_j(\delta_j)-\int_0^\omega|v_j'(t)|\,dt\;>\;\ln b_j-2\sum_{k=1}^{n}\frac{\overline{r}_k c_{jk}^M\omega}{a_{kk}^L}\,,$$

which, together with (2.16), leads to

$$\max_{t\in[0,\omega]}|v_j(t)|\;<\;\max\left\{\left|\ln\left\{\frac{1}{d_{jj}^L}\sum_{k=1}^{n}\frac{\overline{r}_k c_{jk}^M}{a_{kk}^L}\right\}\right|+2\sum_{k=1}^{n}\frac{\overline{r}_k c_{jk}^M\omega}{a_{kk}^L},\;|\ln b_j|+2\sum_{k=1}^{n}\frac{\overline{r}_k c_{jk}^M\omega}{a_{kk}^L}\right\} \tag{2.23}$$
$$:=\;Q_j\,.$$

We note that R_i and Q_j in (2.19) and (2.23) are independent of λ. Denote $M = \sum_{i=1}^n R_i + \sum_{j=1}^m Q_j + M_0$, where M_0 is taken sufficiently large such that each solution $(u_1^*,\ldots,u_n^*,v_1^*,\ldots,v_m^*)^T$ of the following system of algebraic equations

$$\begin{cases}\overline{r}_i-\sum\limits_{k=1}^{n}\overline{a}_{ik}e^{u_k}-\sum\limits_{l=1}^{m}\overline{b}_{il}e^{v_l}\;=\;0\,, & i\;=\;1\,,2\,,\ldots\,,n\,,\\ -\overline{e}_j+\sum\limits_{k=1}^{n}\overline{c}_{jk}e^{u_k}-\sum\limits_{l=1}^{m}\overline{d}_{jl}e^{v_l}\;=\;0\,, & j\;=\;1\,,2\,,\ldots\,,m\,,\end{cases} \tag{2.24}$$

satisfies $\left\|(u_1^*,\ldots,u_n^*,v_1^*,\ldots,v_m^*)^T\right\| = \sum_{i=1}^n|u_i^*| + \sum_{j=1}^m|v_j^*| < M$ (if there exists) and the following

$$\sum_{i=1}^{n}\max\left\{\left|\ln\frac{\overline{r}_i}{\overline{a}_{ii}}\right|,\,|\ln\rho_i|\right\}+\sum_{j=1}^{m}\max\left\{\left|\ln\left\{\frac{1}{\overline{d}_{jj}}\sum_{k=1}^{n}\frac{\overline{r}_k\overline{c}_{jk}}{\overline{a}_{kk}}\right\}\right|,\,|\ln\zeta_j|\right\}\;<\;M\,, \tag{2.25}$$

where

$$\rho_i\;=\;\frac{\overline{r}_i-\sum\limits_{k=1,k\neq i}^{n}\overline{r}_k\left(\overline{a}_{ik}+\sum\limits_{l=1}^{m}\overline{b}_{il}\overline{c}_{lk}/\overline{d}_{ll}\right)\Big/\overline{a}_{kk}}{\overline{a}_{ii}+\sum\limits_{l=1}^{m}\overline{b}_{il}\overline{c}_{li}/\overline{d}_{ll}}\,,$$

$$\zeta_j\;=\;\frac{\sum\limits_{k=1}^{n}\frac{\overline{c}_{jk}}{\overline{a}_{kk}}\left(\overline{r}_k-\sum\limits_{\substack{l=1\\l\neq k}}^{n}\frac{\overline{r}_l\overline{a}_{kl}}{\overline{a}_{ll}}\right)-\sum\limits_{k=1}^{n}\sum\limits_{l=1,\,l\neq j}^{m}\frac{\overline{r}_k\overline{c}_{lk}}{\overline{a}_{kk}\overline{d}_{ll}}\left(\overline{d}_{jl}+\sum\limits_{i=1}^{n}\frac{\overline{b}_{il}\overline{c}_{ji}}{\overline{a}_{ii}}\right)-\overline{e}_j}{\overline{d}_{jj}+\sum\limits_{k=1}^{n}\frac{\overline{b}_{kj}\overline{c}_{jk}}{\overline{a}_{kk}}}\,, \tag{2.26}$$
$$i\;=\;1\,,2\,,\ldots\,,n\,,\quad j\;=\;1\,,2\,,\ldots\,,m\,.$$

Here, (H2) and (H3) are used.

We now take $\Omega = \{(u_1(t),\dots,u_n(t),v_1(t),\dots,v_m(t))^T \in X : \|(u_1,\dots,u_n,v_1,\dots,v_m)^T\| < M\}$. Thus the condition (i) in Lemma 2.1 is satisfied. When $(u_1(t),\dots,u_n(t),v_1(t),\dots,v_m(t))^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^{m+n}$, $(u_1,\dots,u_n,v_1,\dots,v_m)^T$ is a constant vector in $\mathbb{R}^{m+n}$ with $\sum_{i=1}^n |u_i| + \sum_{j=1}^m |v_j| = M$. Thus, if system (2.24) has at least one solution, then we have

$$QN\begin{bmatrix} u_1 \\ \vdots \\ u_n \\ v_1 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} \overline{r}_1 - \sum\limits_{k=1}^n \overline{a}_{1k}e^{u_k} - \sum\limits_{l=1}^m \overline{b}_{1l}e^{v_l} \\ \vdots \\ \overline{r}_n - \sum\limits_{k=1}^n \overline{a}_{nk}e^{u_k} - \sum\limits_{l=1}^m \overline{b}_{nl}e^{v_l} \\ -\overline{e}_1 + \sum\limits_{k=1}^n \overline{c}_{1k}e^{u_k} - \sum\limits_{l=1}^m \overline{d}_{1l}e^{v_l} \\ \vdots \\ -\overline{e}_m + \sum\limits_{k=1}^n \overline{c}_{mk}e^{u_k} - \sum\limits_{l=1}^m \overline{d}_{ml}e^{v_l} \end{bmatrix} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If system (2.24) does not have a solution, we can directly derive

$$QN(u_1,\dots,u_n,v_1,\dots,v_m)^T \neq (0,\dots,0,0,\dots,0)^T .$$

This proves that condition (ii) in Lemma 2.1 is satisfied.

In order to prove that condition (iii) in Lemma 2.1 holds, we define $\phi : \operatorname{Dom} L \times [0,1] \to X$ by

$$\phi(u_1,\dots,u_n,v_1,\dots,v_m,\mu) = \begin{bmatrix} \overline{r}_1 - \overline{a}_{11}e^{u_1} \\ \vdots \\ \overline{r}_n - \overline{a}_{nn}e^{u_n} \\ \sum\limits_{k=1}^n \overline{c}_{1k}e^{u_k} - \overline{d}_{11}e^{v_1} \\ \vdots \\ \sum\limits_{k=1}^n \overline{c}_{mk}e^{u_k} - \overline{d}_{mm}e^{v_m} \end{bmatrix} + \mu \begin{bmatrix} -\sum\limits_{k=2}^n \overline{a}_{1k}e^{u_k} - \sum\limits_{l=1}^m \overline{b}_{1l}e^{v_l} \\ \vdots \\ -\sum\limits_{k=1}^{n-1} \overline{a}_{nk}e^{u_k} - \sum\limits_{l=1}^m \overline{b}_{nl}e^{v_l} \\ -\overline{e}_1 - \sum\limits_{l=2}^m \overline{d}_{1l}e^{v_l} \\ \vdots \\ -\overline{e}_m - \sum\limits_{l=1}^{m-1} \overline{d}_{ml}e^{v_l} \end{bmatrix},$$

where $\mu \in [0,1]$ is a parameter. When $(u_1,\dots,u_n,v_1,\dots,v_m)^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^{m+n}$, $(u_1,\dots,u_n,v_1,\dots,v_m)^T$ is a constant vector in $\mathbb{R}^{m+n}$ with $\sum_{i=1}^n |u_i| + \sum_{j=1}^m |v_j| = M$. We now show that when $(u_1,\dots,u_n,v_1,\dots,v_m)^T \in \partial\Omega \cap \operatorname{Ker} L$, $\phi(u_1,\dots,u_n,v_1,\dots,v_m,\mu) \neq 0$. Otherwise, there is a constant vector $(u_1,\dots,u_n,v_1,\dots,v_m)^T$ with $\sum_{i=1}^n |u_i| + \sum_{j=1}^m |v_j| = M$ satisfying $\phi(u_1,\dots,u_n,v_1,\dots,v_m,\mu) = 0$, that is,

$$\overline{r}_i - \overline{a}_{ii}e^{u_i} - \mu \sum_{k=1,\,k\neq i}^n \overline{a}_{ik}e^{u_k} - \mu \sum_{l=1}^m \overline{b}_{il}e^{v_l} = 0, \quad i = 1,2,\dots,n,$$

$$\sum_{k=1}^n \overline{c}_{jk}e^{u_k} - \overline{d}_{jj}e^{v_j} - \mu\overline{e}_j - \mu \sum_{l=1,\,l\neq j}^m \overline{d}_{jl}e^{v_l} = 0, \quad j = 1,2,\dots,m.$$

By similar arguments as in (2.19) and (2.23) we can derive

$$|u_i| < \max\left\{\left|\ln\frac{\overline{r}_i}{\overline{a}_{ii}}\right|,\ |\ln\rho_i|\right\},$$

$$|v_j| < \max\left\{\left|\ln\left\{\frac{1}{\overline{d}_{jj}}\sum_{k=1}^n \frac{\overline{r}_k\overline{c}_{jk}}{\overline{a}_{kk}}\right\}\right|,\ |\ln\zeta_j|\right\},$$

where ρ_i and ζ_j are defined by (2.26). Thus, it follows from (2.25) that

$$\sum_{i=1}^{n} |u_i| + \sum_{j=1}^{m} |v_j| \ < \ M \,,$$

which yields a contradiction. Using the property of topological degree and taking

$$J \ = \ I \ : \ \operatorname{Im} Q \ \longrightarrow \ \operatorname{Ker} L \,, \quad (u_1,\dots,u_n,v_1,\dots,v_m)^T \ \longrightarrow \ (u_1,\dots,u_n,v_1,\dots,v_m)^T \,,$$

we have

$$\begin{aligned}
&\deg\bigl(JQN(u_1,\dots,u_n,v_1,\dots,v_m)^T,\ \Omega\cap\operatorname{Ker} L,\ (0,\dots,0,0,\dots,0)^T\bigr)\\
&\quad = \ \deg\bigl(\phi(u_1,\dots,u_n,v_1,\dots,v_m,1),\ \Omega\cap\operatorname{Ker} L,\ (0,\dots,0,0,\dots,0)^T\bigr)\\
&\quad = \ \deg(\phi(u_1,\dots,u_n,v_1,\dots,v_m,0),\ \Omega\cap\operatorname{Ker} L,\ (0,\dots,0,0,\dots,0)^T)\\
&\quad = \ \deg\Biggl(\Biggl(\overline{r}_1-\overline{a}_{11}e^{u_1},\dots,\overline{r}_n-\overline{a}_{nn}e^{u_n},\sum_{k=1}^{n}\overline{c}_{1k}e^{u_k}-\overline{d}_{11}e^{v_1},\dots,\sum_{k=1}^{n}\overline{c}_{mk}e^{u_k}-\overline{d}_{mm}e^{v_m}\Biggr)^T,\\
&\qquad\qquad \Omega\cap\operatorname{Ker} L,\ (0,\dots,0,0,\dots,0)^T\Biggr).
\end{aligned}$$

Thus, a direct calculation shows that

$$\deg\bigl(JQN(u_1,\dots,u_n,v_1,\dots,v_m)^T,\ \Omega\cap\operatorname{Ker} L,\ (0,\dots,0,0,\dots,0)^T\bigr) \ = \ (-1)^{m+n}\,.$$

Finally, it is easy to show that the set $\bigl\{K_P(I-Q)Nu \mid u\in\overline{\Omega}\bigr\}$ is equicontinuous and uniformly bounded. By using the Arzelà–Ascoli Theorem, we see that $K_P(I-Q)N:\overline{\Omega}\to X$ is compact. Moreover, $QN(\overline{\Omega})$ is bounded. Consequently, N is L-compact.

By now we have proved that Ω satisfies all the requirements in Lemma 2.1. Hence, (2.2) has at least one ω-periodic solution. Accordingly, system (1.1) has at least one positive ω-periodic solution. □

3 Uniqueness and global stability

In this section, we are concerned with the uniqueness and global stability of positive periodic solutions of system (1.1). It suffices to verify that each positive periodic solution of system (1.1) is globally asymptotically stable. We first derive certain estimates for the upper bounds for solutions of system (1.1) with initial conditions (1.2).

Lemma 3.1 *Let $z(t)=(x_1(t),\dots,x_n(t),y_1(t),\dots,y_m(t))$ denote any positive solution of system* (1.1) *with initial conditions* (1.2)*. Then there exists a $T>0$ such that if $t>T$,*

$$0 \ < \ x_i(t) \ \le \ M_i\,, \quad 0 \ < \ y_j(t) \ \le \ N_j\,, \quad i \ = \ 1,2,\dots,n\,,\ j \ = \ 1,2,\dots,m\,, \tag{3.1}$$

where

$$\begin{gathered}
M_i \ = \ \frac{r_i^M}{a_{ii}^L}\,e^{r_i^M\tau_{ii}}\,, \quad N_j \ = \ \frac{\sum_{k=1}^{n} c_{jk}^M M_k}{d_{jj}^L}\,\exp\left\{\sum_{k=1}^{n} c_{jk}^M M_k\mu_{jj}\right\},\\
i \ = \ 1,2,\dots,n\,, \quad j \ = \ 1,2,\dots,m\,.
\end{gathered} \tag{3.2}$$

The proof of Lemma 3.1 is similar to that of Lemma 2.1 in Wang and Ma [21], we therefore omit it here.

We are now able to state and prove our result on the uniqueness and global asymptotic stability of positive ω-periodic solutions of system (1.1).

Theorem 3.2 *In addition to* (H1)–(H3)*, assume further that*

(H4) $\liminf\limits_{t\to\infty} A_i(t) \ > \ 0\,, \quad i \ = \ 1,2,\dots,n\,,$

(H5) $\liminf\limits_{t\to\infty} B_j(t) \ > \ 0\,, \quad j \ = \ 1,2,\dots,m\,,$

where

$$
\begin{aligned}
A_i(t) \;&=\; a_{ii}(t) - \sum_{k=1,k\neq i}^{n} a_{ki}(t+\tau_{ki}) - \sum_{j=1}^{m} c_{ji}(t+\sigma_j)\\
&\quad - \left(r_i(t) + \sum_{k=1}^{n} a_{ik}(t)M_k + \sum_{l=1}^{m} b_{il}(t)N_l\right)\int_t^{t+\tau_{ii}} a_{ii}(s)\,ds\\
&\quad - \sum_{k=1}^{n} a_{ki}(t+\tau_{ki})M_k \int_{t+\tau_{ki}}^{t+\tau_{kk}+\tau_{ki}} a_{kk}(s)\,ds\\
&\quad - \sum_{j=1}^{m} c_{ji}(t+\sigma_j)N_j \int_{t+\sigma_j}^{t+\sigma_j+\mu_{jj}} d_{jj}(s)\,ds\,,\\
B_j(t) \;&=\; d_{jj}(t) - \sum_{l=1,\; l\neq j}^{m} d_{lj}(t+\mu_{lj}) - \sum_{k=1}^{n} b_{kj}(t)\\
&\quad - \left(e_j(t) + \sum_{k=1}^{n} c_{jk}(t)M_k + \sum_{l=1}^{m} d_{jl}(t)N_l\right)\int_t^{t+\mu_{jj}} d_{jj}(s)\,ds\\
&\quad - \sum_{l=1}^{m} d_{lj}(t+\mu_{lj})N_l \int_{t+\mu_{lj}}^{t+\mu_{lj}+\mu_{ll}} d_{ll}(s)\,ds\\
&\quad - \sum_{i=1}^{n} b_{ij}(t)M_i \int_t^{t+\tau_{ii}} a_{ii}(s)\,ds\,.
\end{aligned}
\tag{3.3}
$$

Then system (1.1) *has a unique positive* ω*-periodic solution* $(x_1^*(t),\dots,x_n^*(t),y_1^*(t),\dots,y_m^*(t))^T$ *which is globally stable.*

P r o o f. Thanks to the conclusion of Theorem 2.2, we need only to verify the global stability of the positive periodic solutions of (1.1). Let $(x_1^*(t),\dots,x_n^*(t),y_1^*(t),\dots,y_m^*(t))^T$ be a positive ω-periodic solution of system (1.1)–(1.2), and $(x_1(t),\dots,x_n(t),y_1(t),\dots,y_m(t))^T$ denote any positive solution of system (1.1).

By Lemma 3.1 we see that there exist positive constants T and M_i, N_j such that if $t \geq T$,

$$
\begin{aligned}
&0 \;<\; x_i^*(t) \;\leq\; M_i\,, \qquad 0 \;<\; y_j^*(t) \;\leq\; N_j\,,\\
&0 \;<\; x_i(t) \;\leq\; M_i\,, \qquad 0 \;<\; y_j(t) \;\leq\; N_j\,,
\end{aligned}
\qquad i \;=\; 1,2,\dots,n\,,\;\; j \;=\; 1,2,\dots,m\,. \tag{3.4}
$$

Let

$$
V_{i1}(t) \;=\; |\ln x_i(t) - \ln x_i^*(t)|\,. \tag{3.5}
$$

Using a similar argument as in [19], we calculate the upper right derivative of $V_{i1}(t)$ along solutions of system (1.1). It follows from (1.1) and (3.5) that

$$
\begin{aligned}
D^+V_{i1}(t) \;&=\; \left(\frac{\dot{x}_i(t)}{x_i(t)} - \frac{\dot{x}_i^*(t)}{x_i^*(t)}\right)\operatorname{sgn}(x_i(t)-x_i^*(t)) \qquad (3.6)\\
&=\; \operatorname{sgn}(x_i(t)-x_i^*(t))\\
&\qquad \times \left[-\sum_{k=1}^{n} a_{ik}(t)(x_k(t-\tau_{ik}) - x_k^*(t-\tau_{ik})) - \sum_{l=1}^{m} b_{il}(t)(y_l(t)-y_l^*(t))\right]\\
&=\; \operatorname{sgn}(x_i(t)-x_i^*(t))\times
\end{aligned}
$$

$$\begin{aligned} &\times \Bigg[- a_{ii}(t)(x_i(t) - x_i^*(t)) - \sum_{k=1,\ k\neq i}^{n} a_{ik}(t)(x_k(t-\tau_{ik}) - x_k^*(t-\tau_{ik})) \\ &\quad - \sum_{l=1}^{m} b_{il}(t)(y_l(t) - y_l^*(t)) + a_{ii}(t)\int_{t-\tau_{ii}}^{t} (\dot{x}_i(u) - \dot{x}_i^*(u))\, du \Bigg] \\ \leq\ & - a_{ii}(t)\,|x_i(t) - x_i^*(t)| + \sum_{l=1}^{m} b_{il}(t)\,|y_l(t) - y_l^*(t)| \\ &+ \sum_{k=1,\ k\neq i}^{n} a_{ik}(t)\,|x_k(t-\tau_{ik}) - x_k^*(t-\tau_{ik})| + a_{ii}(t)\left|\int_{t-\tau_{ii}}^{t} (\dot{x}_i(u) - \dot{x}_i^*(u))\, du\right| . \end{aligned}$$

On substituting (1.1) into (3.6), it follows that

$$\begin{aligned} D^+V_{i1}(t) \ \leq\ & - a_{ii}(t)\,|x_i(t) - x_i^*(t)| + \sum_{l=1}^{m} b_{il}(t)\,|y_l(t) - y_l^*(t)| \\ &+ \sum_{k=1,\ k\neq i}^{n} a_{ik}(t)\,|x_k(t-\tau_{ik}) - x_k^*(t-\tau_{ik})| \\ &+ a_{ii}(t)\Bigg|\int_{t-\tau_{ii}}^{t} \Bigg\{ (x_i(u) - x_i^*(u)) \Bigg[r_i(u) - \sum_{k=1}^{n} a_{ik}(u) x_k(u-\tau_{ik}) - \sum_{l=1}^{m} b_{il}(u) y_l(u) \Bigg] \\ &\quad - x_i^*(u)\Bigg[\sum_{k=1}^{n} a_{ik}(u)(x_k(u-\tau_{ik}) - x_k^*(u-\tau_{ik})) + \sum_{l=1}^{m} b_{il}(u)(y_l(u) - y_l^*(u))\Bigg]\Bigg\} du\Bigg| . \end{aligned} \tag{3.7}$$

We derive from (3.4) and (3.7) that for $t \geq T + 2\tau$,

$$\begin{aligned} D^+V_{i1}(t) \ \leq\ & - a_{ii}(t)\,|x_i(t) - x_i^*(t)| + \sum_{l=1}^{m} b_{il}(t)\,|y_l(t) - y_l^*(t)| \\ &+ \sum_{k=1,\ k\neq i}^{n} a_{ik}(t)\,|x_k(t-\tau_{ik}) - x_k^*(t-\tau_{ik})| \\ &+ a_{ii}(t)\int_{t-\tau_{ii}}^{t} \Bigg\{ |x_i(u) - x_i^*(u)| \Bigg[r_i(u) + \sum_{k=1}^{n} a_{ik}(u) M_k + \sum_{l=1}^{m} b_{il}(u) N_l \Bigg] \\ &\quad + M_i\Bigg[\sum_{k=1}^{n} a_{ik}(u)\,|x_k(u-\tau_{ik}) - x_k^*(u-\tau_{ik})| + \sum_{l=1}^{m} b_{il}(u)\,|y_l(u) - y_l^*(u)|\Bigg]\Bigg\} du . \end{aligned} \tag{3.8}$$

Define

$$\begin{aligned} V_{i2}(t) \ =\ & \sum_{k=1,\ k\neq i}^{n} \int_{t-\tau_{ik}}^{t} a_{ik}(s+\tau_{ik})\,|x_k(s) - x_k^*(s)|\, ds \\ &+ \int_{t}^{t+\tau_{ii}} \int_{s-\tau_{ii}}^{t} a_{ii}(s) \Bigg\{ |x_i(u) - x_i^*(u)| \Bigg[r_i(u) + \sum_{k=1}^{n} a_{ik}(u) M_k + \sum_{l=1}^{m} b_{il}(u) N_l \Bigg] \\ &\quad + M_i\Bigg[\sum_{k=1}^{n} a_{ik}(u)\,|x_k(u-\tau_{ik}) - x_k^*(u-\tau_{ik})| + \sum_{l=1}^{m} b_{il}(u)\,|y_l(u) - y_l^*(u)|\Bigg]\Bigg\} du\, ds . \end{aligned} \tag{3.9}$$

It then follows from (3.8) and (3.9) that for $t \geq T + 2\tau$,

$$\begin{aligned} D^+V_{i1}(t) + \dot{V}_{i2}(t) \;\leq\; & -a_{ii}(t)\,|x_i(t) - x_i^*(t)| + \sum_{l=1}^{m} b_{il}(t)\,|y_l(t) - y_l^*(t)| \\ & + \sum_{k=1,\ k\neq i}^{n} a_{ik}(t+\tau_{ik})\,|x_k(t) - x_k^*(t)| \\ & + \int_t^{t+\tau_{ii}} a_{ii}(s)\,ds \Bigg\{ |x_i(t) - x_i^*(t)| \left[r_i(t) + \sum_{k=1}^{n} a_{ik}(t) M_k + \sum_{l=1}^{m} b_{il}(t) N_l \right] \\ & \quad + M_i \left[\sum_{k=1}^{n} a_{ik}(t)\,|x_k(t-\tau_{ik}) - x_k^*(t-\tau_{ik})| + \sum_{l=1}^{m} b_{il}(t)\,|y_l(t) - y_l^*(t)| \right] \Bigg\} . \end{aligned} \tag{3.10}$$

We now define

$$V_i(t) \;=\; V_{i1}(t) + V_{i2}(t) + V_{i3}(t)\,, \tag{3.11}$$

where

$$V_{i3}(t) \;=\; M_i \sum_{k=1}^{n} \int_{t-\tau_{ik}}^{t} \int_{l+\tau_{ik}}^{l+\tau_{ii}+\tau_{ik}} a_{ii}(s) a_{ik}(l+\tau_{ik})\,|x_k(l) - x_k^*(l)|\,ds\,dl\,. \tag{3.12}$$

Then it follows from (3.10)–(3.12) that for $t \geq T + 2\tau$

$$\begin{aligned} D^+V_i(t) \;\leq\; & -a_{ii}(t)\,|x_i(t) - x_i^*(t)| + \sum_{l=1}^{m} b_{il}(t)\,|y_l(t) - y_l^*(t)| \\ & + \sum_{k=1,\ k\neq i}^{n} a_{ik}(t+\tau_{ik})\,|x_k(t) - x_k^*(t)| \\ & + \left(r_i(t) + \sum_{k=1}^{n} a_{ik}(t) M_k + \sum_{l=1}^{m} b_{il}(t) N_l \right) \int_t^{t+\tau_{ii}} a_{ii}(s)\,ds\,|x_i(t) - x_i^*(t)| \\ & + M_i \sum_{l=1}^{m} b_{il}(t) \int_t^{t+\tau_{ii}} a_{ii}(s)\,ds\,|y_l(t) - y_l^*(t)| \\ & + M_i \sum_{k=1}^{n} a_{ik}(t+\tau_{ik}) \int_{t+\tau_{ik}}^{t+\tau_{ii}+\tau_{ik}} a_{ii}(s)\,ds\,|x_k(t) - x_k^*(t)|\,. \end{aligned} \tag{3.13}$$

Next, let

$$U_j(t) \;=\; U_{j1}(t) + U_{j2}(t) + U_{j3}(t)\,, \tag{3.14}$$

where

$$U_{j1}(t) = |\ln y_j(t) - \ln y_j^*(t)|\,,$$

$$\begin{aligned} U_{j2}(t) = & \sum_{k=1}^{n} \int_{t-\sigma_j}^{t} c_{jk}(t+\sigma_j)\,|x_k(s) - x_k^*(s)|\,ds \\ & + \sum_{l=1,\ l\neq j}^{m} \int_{t-\mu_{jl}}^{t} d_{jl}(t+\mu_{jl})\,|y_l(s) - y_l^*(s)|\,ds \\ & + \int_t^{t+\mu_{jj}} \int_{s-\mu_{jj}}^{t} d_{jj}(s) \Bigg\{ |y_j(u) - y_j^*(u)| \left[e_j(u) + \sum_{k=1}^{n} c_{jk}(u) M_k + \sum_{l=1}^{m} d_{jl}(u) N_l \right] \\ & + N_j \sum_{k=1}^{n} c_{jk}(u) |x_k(u-\sigma_j) - x_k^*(u-\sigma)| + N_j \sum_{l=1}^{m} d_{jl}(u) |y_l(u-\mu_{jl}) - y_l^*(u-\mu_{jl})| \Bigg\}\,du\,ds\,, \end{aligned} \tag{3.15}$$

$$U_{j3}(t) \;=\; N_j \sum_{k=1}^{n} \int_{t-\sigma_j}^{t} \int_{u+\sigma_j}^{u+\sigma_j+\mu_{jj}} d_{jj}(s) c_{jk}(u+\sigma_j)\, |x_k(u) - x_k^*(u)|\, ds\, du$$
$$+ N_j \sum_{l=1}^{m} \int_{t-\mu_{jl}}^{t} \int_{u+\mu_{jl}}^{u+\mu_{jl}+\mu_{jj}} d_{jj}(s) d_{jl}(u+\mu_{jl})\, |y_l(u) - y_l^*(u)|\, ds\, du\,.$$

Calculating the upper right derivative of $U_j(t)$ along solutions of system (1.1), it follows for $t \geq T + 2\tau$ that

$$\begin{aligned} D^+U_j(t) \;\leq\; & - d_{jj}(t)\, |y_j(t) - y_j^*(t)| + \sum_{k=1}^{n} c_{jk}(t+\sigma_j)\, |x_k(t) - x_k^*(t)| \\ & + \sum_{l=1,\, l\neq j}^{m} d_{jl}(t+\mu_{jl})\, |y_l(t) - y_l^*(t)| \\ & + \left(e_j(t) + \sum_{k=1}^{n} c_{jk}(t) M_k + \sum_{l=1}^{m} d_{jl}(t) N_l \right) \int_t^{t+\mu_{jj}} d_{jj}(s)\, ds\, |y_j(t) - y_j^*(t)| \\ & + N_j \sum_{k=1}^{n} c_{jk}(t+\sigma_j) \int_{t+\sigma_j}^{t+\sigma_j+\mu_{jj}} d_{jj}(s)\, ds\, |x_k(t) - x_k^*(t)| \\ & + N_j \sum_{l=1}^{m} d_{jl}(t+\mu_{jl}) \int_{t+\mu_{jl}}^{t+\mu_{jl}+\mu_{jj}} d_{jj}(s)\, ds\, |y_l(t) - y_l^*(t)|\,. \end{aligned} \tag{3.16}$$

We now define

$$V(t) \;=\; \sum_{i=1}^{n} V_i(t) + \sum_{j=1}^{m} U_j(t)\,. \tag{3.17}$$

Then it follows from (3.13), (3.16) and (3.17) that for $t \geq T + 2\tau$

$$D^+V(t) \;\leq\; -\sum_{i=1}^{n} A_i(t)\, |x_i(t) - x_i^*(t)| - \sum_{j=1}^{m} B_j(t)\, |y_j(t) - y_j^*(t)|\,, \tag{3.18}$$

where $A_i(t)$ and $B_j(t)$ are defined in (3.3).

By assumptions (H4) and (H5), there exist constants $\alpha_i > 0$ $(i = 1, 2, \ldots, n)$, $\beta_j > 0$ $(j = 1, 2, \ldots, m)$ and $T^* \geq T + 2\tau$ such that if $t \geq T^*$,

$$A_i(t) \;\geq\; \alpha_i \;>\; 0\,, \quad B_j(t) \;\geq\; \beta_j \;>\; 0\,, \quad i \;=\; 1, 2, \ldots, n\,, \quad j \;=\; 1, 2, \ldots, m\,. \tag{3.19}$$

Integrating both sides of (3.18) on interval $[T^*, t]$, we obtain

$$V(t) + \sum_{i=1}^{n} \int_{T^*}^{t} A_i(s)\, |x_i(s) - x_i^*(s)|\, ds + \sum_{j=1}^{m} \int_{T^*}^{t} B_j(s)\, |y_j(s) - y_j^*(s)|\, ds \;\leq\; V(T^*)\,. \tag{3.20}$$

It follows from (3.19) and (3.20) that

$$V(t) + \sum_{i=1}^{n} \alpha_i \int_{T^*}^{t} |x_i(s) - x_i^*(s)|\, ds + \sum_{j=1}^{m} \beta_j \int_{T^*}^{t} |y_j(s) - y_j^*(s)|\, ds \;\leq\; V(T^*)\,, \quad t \;\geq\; T^*\,.$$

Therefore, $V(t)$ is bounded on $[T^*, \infty)$ and also

$$\int_{T^*}^{\infty} |x_i(s) - x_i^*(s)|\, ds \;<\; \infty\,, \quad i \;=\; 1, 2, \ldots, n\,,$$

and

$$\int_{T^*}^{\infty} |y_j(s) - y_j^*(s)|\, ds \;<\; \infty\,, \quad j \;=\; 1\,,2\,,\dots\,,m\,.$$

By Lemma 3.1, $|x_i(t) - x_i^*(t)|$ $(i = 1, 2, \dots, n)$ and $|y_j(t) - y_j^*(t)|$ $(j = 1, 2, \dots, m)$ are bounded on $[T^*, \infty)$. On the other hand, it is easy to see that $\dot{x}_i(t), \dot{y}_j(t), \dot{x}_i^*(t)$ and $\dot{y}_j^*(t)$ are bounded for $t \geq T^*$. Therefore, $|x_i(t) - x_i^*(t)|$ $(i = 1, 2, \dots, n)$ and $|y_j(t) - y_j^*(t)|$ $(j = 1, 2, \dots, m)$ are uniformly continuous on $[T^*, \infty)$. By Barbalat's lemma (Lemmas 1.2.2 and 1.2.3, Gopalsamy [11]), we conclude that

$$\lim_{t\to\infty} |x_i(t) - x_i^*(t)| \;=\; 0\,, \quad \lim_{t\to\infty} |y_j(t) - y_j^*(t)| \;=\; 0\,.$$

This shows that the positive periodic solution $(x_1^*(t), \dots, x_n^*(t), y_1^*(t), \dots, y_m^*(t))^T$ attracts all other positive solutions to system (1.1). □

Finally, we give an example to illustrate our main results in Theorems 2.2 and 3.2.

Example 3.3 As an example, we consider the following system

$$\begin{cases} \dot{x}_1(t) \;=\; x_1(t)\left(4 + \sin t - 4x_1\left(t - \frac{1}{10^2}\right) - 0.1x_2(t - \tau_{12}) - y_1(t) - y_2(t)\right), \\ \dot{x}_2(t) \;=\; x_2(t)\left(3 + \cos t - 0.1x_1(t - \tau_{21}) - 4x_2\left(t - \frac{1}{10^2}\right) - y_1(t) - y_2(t)\right), \\ \dot{y}_1(t) \;=\; y_1(t)\big(- 0.01 + 0.001\sin t + x_1(t - \sigma_1) + x_2(t - \sigma_1) \\ \qquad\qquad -4y_1\left(t - \frac{1}{10^2}\right) - 0.1y_2(t - \mu_{12})\big)\,, \\ \dot{y}_2(t) \;=\; y_2(t)\big(- 0.015 + 0.001\sin t + x_1(t - \sigma_2) + x_2(t - \sigma_2) \\ \qquad\qquad -0.1y_1(t - \mu_{21}) - 4y_2\left(t - \frac{1}{10^2}\right)\big)\,, \end{cases} \tag{3.21}$$

where $\tau_{12}, \tau_{21}, \sigma_1, \sigma_2, \mu_{12}$ and μ_{21} are nonnegative constants.

It is easy to verify that (H1)–(H3) hold for system (3.21). By direct calculation, we obtain $M_1 = 1.3141$, $M_2 = 1.0408$, $N_1 = N_2 = 0.6028$, $A_1(t) \geq 1.1747$, $A_2(t) \geq 1.3000$, $B_1(t) \geq 1.5134$ and $B_2(t) \geq 1.5132$. Thus, the coefficients of system (3.21) satisfy (H4) and (H5). By Theorem 3.2, system (3.21) admits a unique positive 2π-periodic solution. Taking $\tau_{12} = \tau_{21} = \sigma_1 = \sigma_2 = \mu_{12} = \mu_{21} = 1$, numerical integration of system (3.21) can now be carried out using standard algorithms in MATLAB. As shown in Fig. 1, numerical simulation also confirms our observation above.

4 Discussion

In this paper, we have discussed the combined effects of periodicity of ecological and environmental parameters and time delays due to gestations and negative feedbacks on the dynamics of a Lotka–Volterra type predation-competition model. By using Gaines and Mawhin's continuation theorem of coincidence degree theory and by constructing suitable Lyapunov functionals, we have established sufficient conditions for the existence, uniqueness and global stability of positive periodic solutions of system (1.1) in which n different prey species at the same trophic level compete common resource and m different predator species at the same trophic level compete and feed on n common preys. By Theorem 3.2, we see that system (1.1) with initial conditions (1.2) will have a unique positive periodic solution if the coefficients of (1.1) satisfy (H1)–(H5). We note that under the assumptions of Theorem 2.2, time delays due to gestations and negative feedbacks have no influence on the existence of positive periodic solutions of system (1.1). By Theorem 3.2, we know that small delays are negligible for the uniqueness and global stability of positive periodic solutions of system (1.1) provided that the intra-specific competitions dominate other interspecific interaction effects with or without time delays.

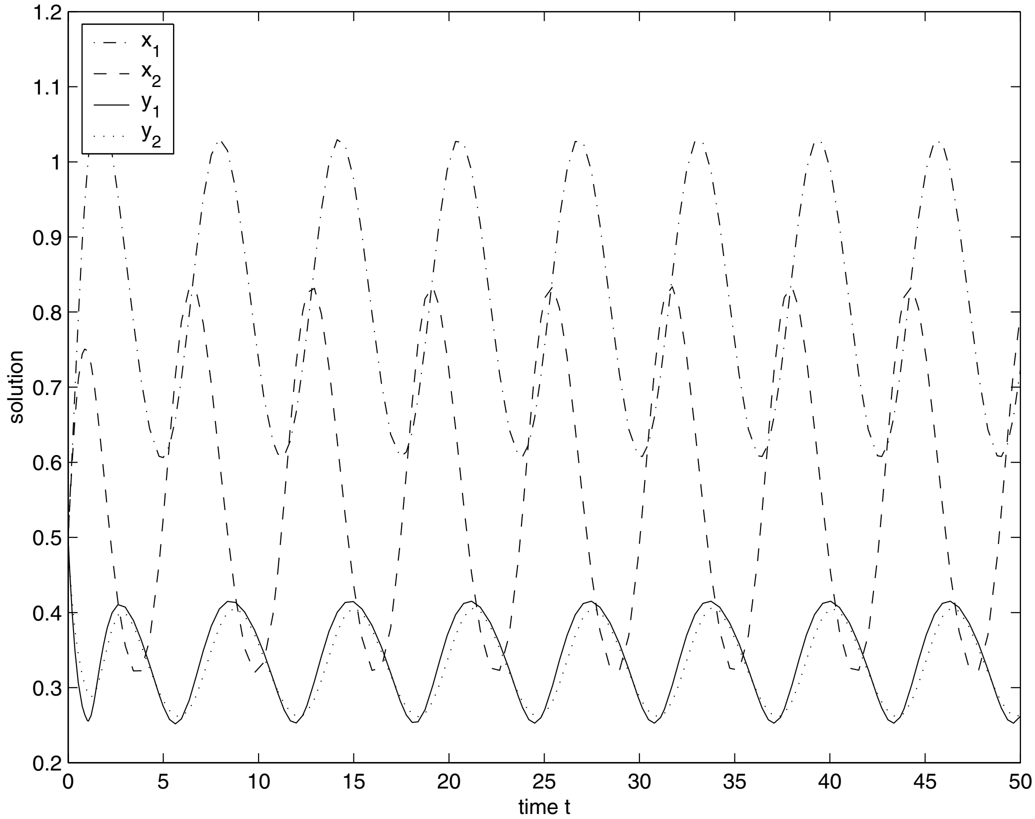

Fig. 1 The positive periodic solution found by numerical integration of system (3.21) with $\tau_{12} = \tau_{21} = \sigma_1 = \sigma_2 = \mu_{12} = \mu_{21} = 1$, and $(\phi_1(\theta), \phi_2(\theta), \psi_1(\theta), \psi_2(\theta)) = (0.5, 0.5, 0.5, 0.5)$

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