

Linear and Quadratic FE for Advection

D F Griffiths

This draft: 27 Aug 1999

We solve $u_t + u_x = 0$ on the interval $0 \leq x \leq 1$ with periodic boundary conditions and initial data

$$u(x, 0) = \exp(\sin(2\pi x)),$$

chosen to be a smooth function.

Methods

1. Quadratic FE using N elements ($h = 1/N$ denotes the element length),
2. Quadratic spline with N knots,
3. Quadratic FE using N elements and the Hedstrom filter, that is if $u_{0,j} = u(jh/2, 0)$, $j = 0, 1, \dots, 2N$ are the nodal values of the quadratic finite interpolant of the initial data, the Hedstrom filtered method begins with the data

$$u_{0,j}^H = \begin{cases} u(jh/2, 0), & j = 0, 2, 4, \dots, 2N \\ \frac{1}{12}(u((j-1)h/2, 0) + 10u(jh/2, 0) + u((j+1)h/2, 0)), & j = 1, 3, \dots, 2N-1. \end{cases}$$

That is, the mid-element values are “averaged”.

4. Linear finite elements using $2N$ elements.

The maximum errors were measured at the $2N + 1$ nodes with grid sizes $h = 1/2^j$, $j = 2, 3, \dots, 11$ and the results are shown in Figure 1

Method	Convergence rate
1e: Quad FE, even nodes	2
1o: Quad FE, odd nodes	2
2: Quad spline	6
3e: Filtered Quad FE, even nodes	4
3o: Filtered Quad FE, odd nodes	2
4: Linear FE	4

Table 1: Convergence rates of maximum nodal errors.

Figures 5–7: These show the solutions at $x = 0$ together with the nodal errors at $x = 0$ and $x = h/2$ (the first mid-node) as functions of time. We treat the cases $N = 8, 16, 32$.

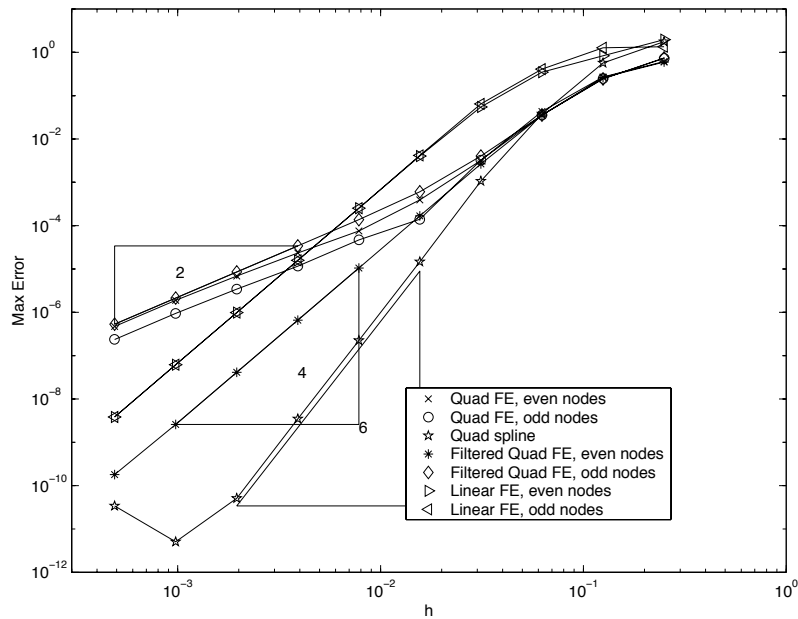


Figure 1: Convergence rates of maximum nodal errors. There is no significant distinction between the accuracy of linear FE at odd and even nodes. Although linear FEs converge at 4th order, it is only for $N > 256$ that one sees the benefit (Figure 1). The errors for quad splines at $N = 1024, 2048$ are dominated by rounding errors.

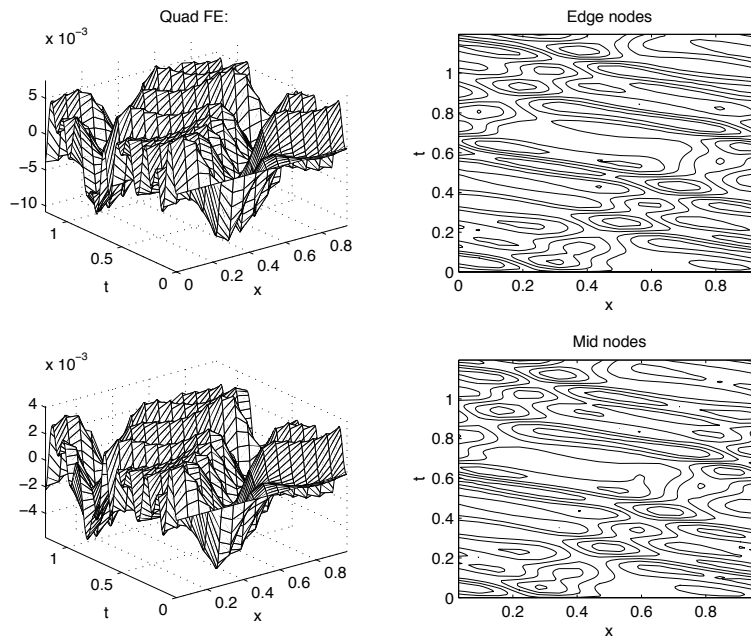


Figure 2: Quadratic FE, $N = 16$. Top: Surface and contour plots of error at edge nodes. Notice the tendency for errors to propagate at speeds of approximately 1 and -5 . The bottom plots show the **negative** of the errors at mid-nodes. The surface has the same structure as the upper surface except that it has **half** the amplitude, in agreement with the structure of eigenvectors.

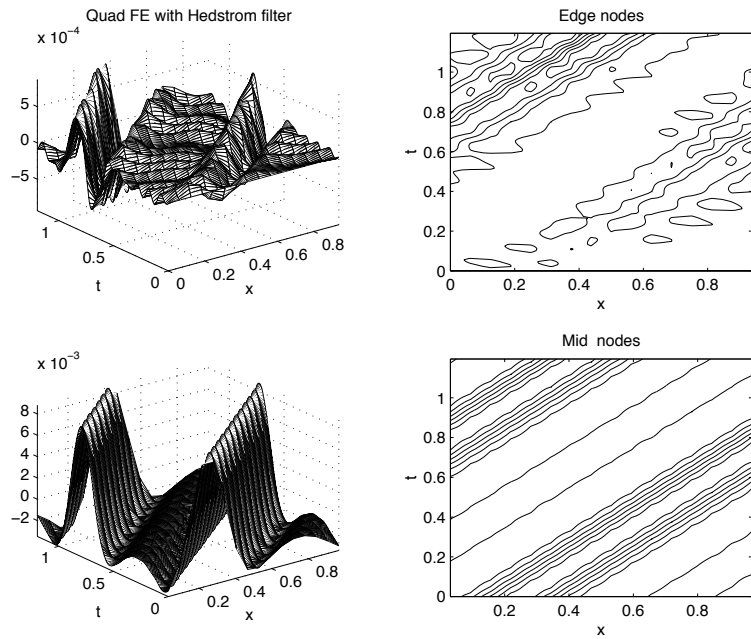


Figure 3: As for Fig. 2 except that the Hedstrom filter is used. The main part of the error propagates with characteristic speed (1) at edge nodes but there is a small amplitude contribution moving with a negative speed. Contrastingly, the error at mid-nodes travels at the characteristic speed.

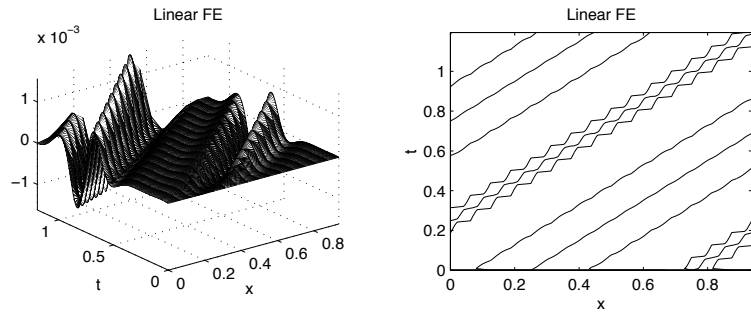


Figure 4: Linear FE, $N = 16$ (32 elements). The error appears to propagate at the characteristic speed.

In each case the solutions at $x = 0$ are indistinguishable to graphical accuracy. It is clear that the Hedstrom filter has a significant effect on the accuracy at edge nodes. Notable features are: the error with standard quads is relatively high frequency (hence the tendency to propagate at the appropriate group velocity) while the mid-node error in the filtered solution has a much longer wavelength and the corresponding group speed is approximately the characteristic speed.

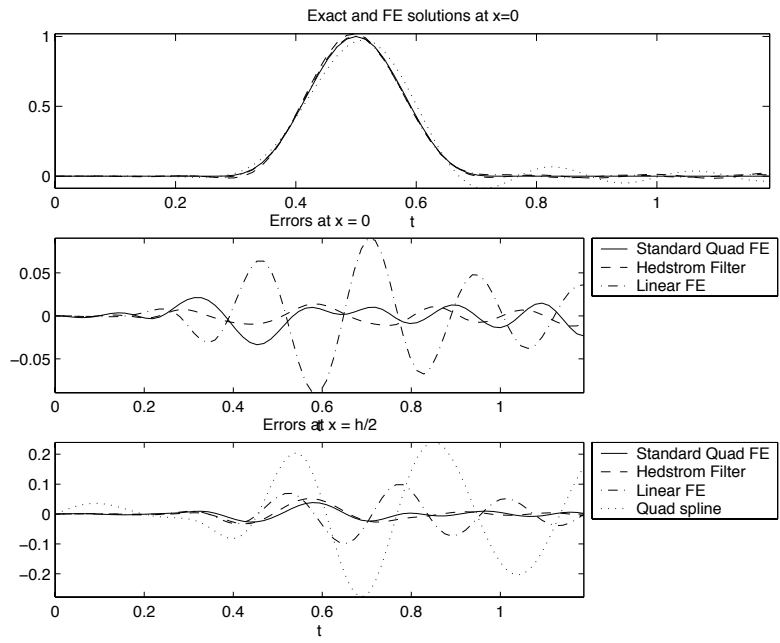


Figure 5: Errors at $x = 0$ and $x = h/2$ ($N = 8$) as functions of time.

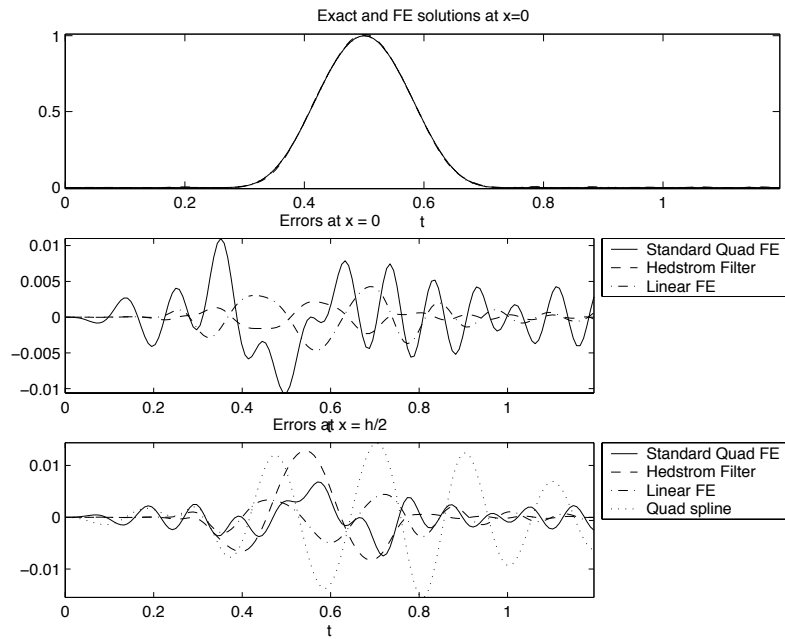


Figure 6: Errors at $x = 0$ and $x = h/2$ ($N = 16$) as functions of time.

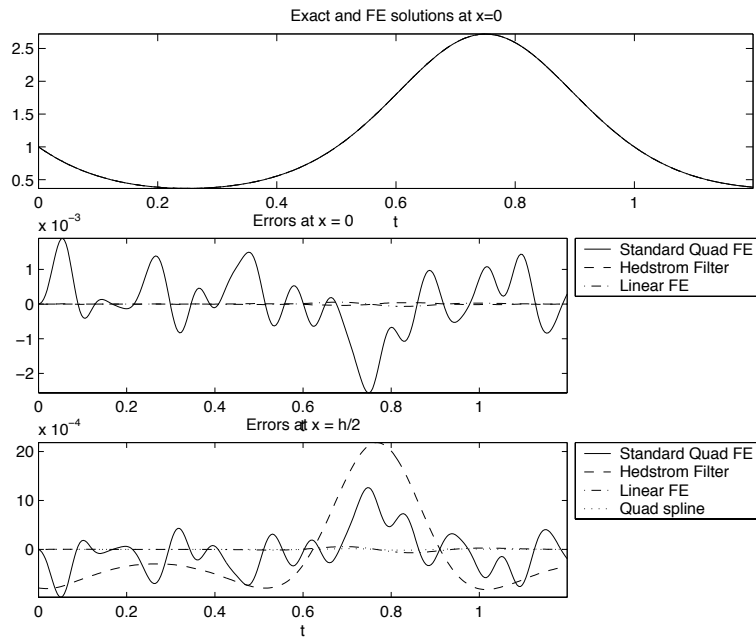


Figure 7: Errors at $x = 0$ and $x = h/2$ ($N = 32$) as functions of time.

Pulse initial data

We change the initial data to

$$u(x, 0) = \begin{cases} \frac{1}{2^m}(\cos 4\pi(x - 1/2) + 1)^m, & \text{for } |x - 1/2| < 1/4 \\ 0, & \text{otherwise} \end{cases}$$

which lies in C^{2m-1} (in the first example $u(x, 0) \in C^\infty$). The convergence rates are shown in Figure 8 for $m = 2$. Most notably the reduction in smoothness of the initial data has reduced the convergence rate of quadratic splines from 6 to 3.5, the filtered edge nodes from 4 to 3 and linear FE from 4 to 3.5.

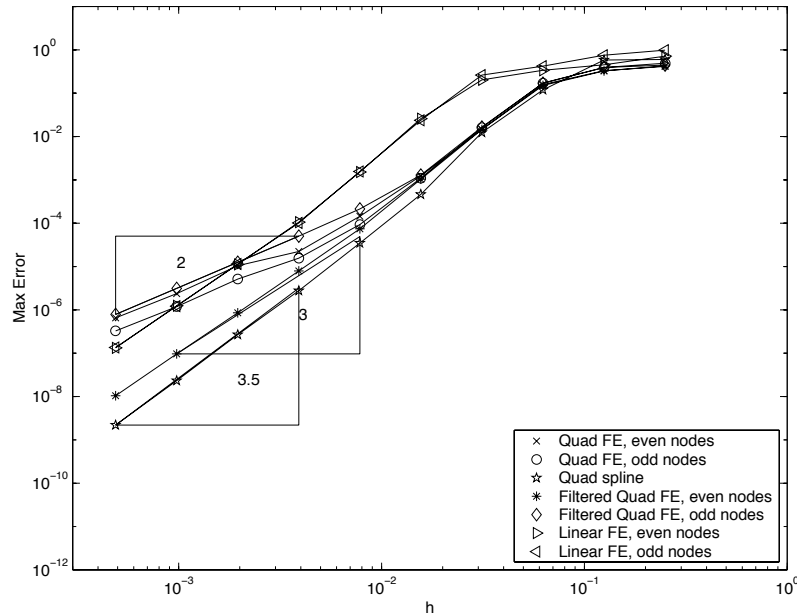


Figure 8: Convergence rates of maximum nodal errors for the pulse with $m = 2$. Despite the higher convergence rate, it requires $N > 1024$ before linear FE outperforms standard quads.

The analogues of Figures 2 and 3 for quadratic FE are essentially unchanged except for vertical scale—the amplitudes of the error are increased by about a factor of 1.8. However, there is a greater difference when using linear FE as shown in Figure 9. The amplitude has increased by a factor of approximately 10 and there is evidence that some components of the error are propagated at the speed -1 .

In the next figures we show the solutions at times $T = 10, 10 + h/2, 10 + h$.

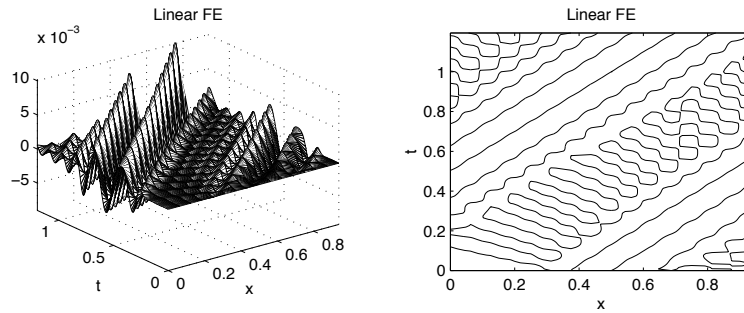


Figure 9: Linear FE, $N = 16$ (32 elements). The error appears to propagate at speeds ± 1 (approx.).

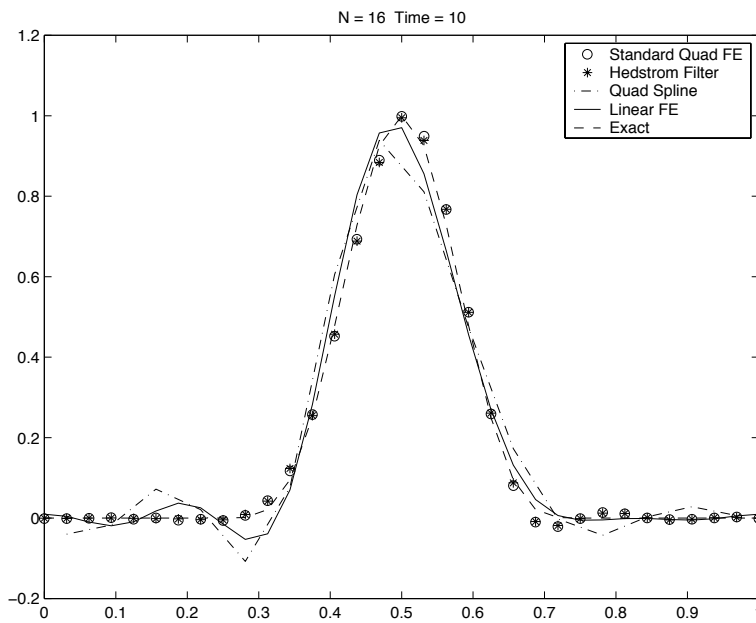


Figure 10: Solutions at time $T = 10$.

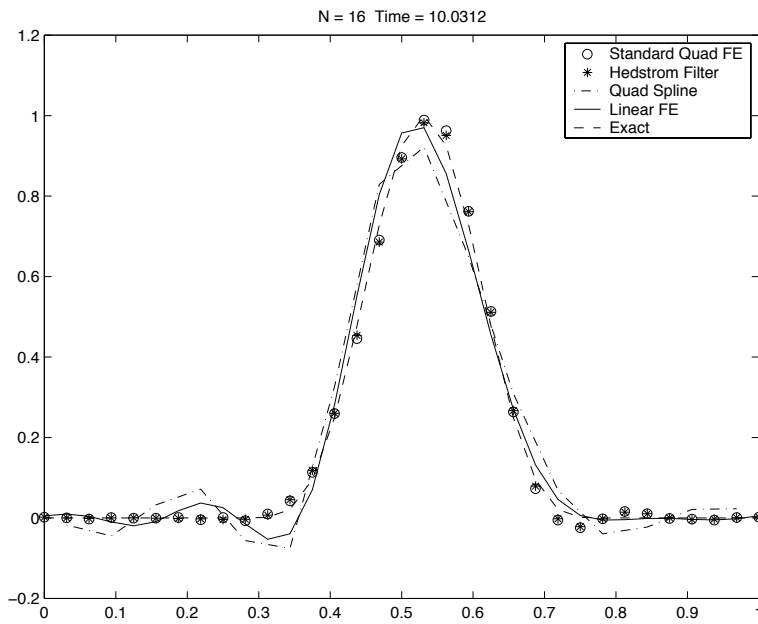


Figure 11: Solutions at time $T = 10 + h/2$.

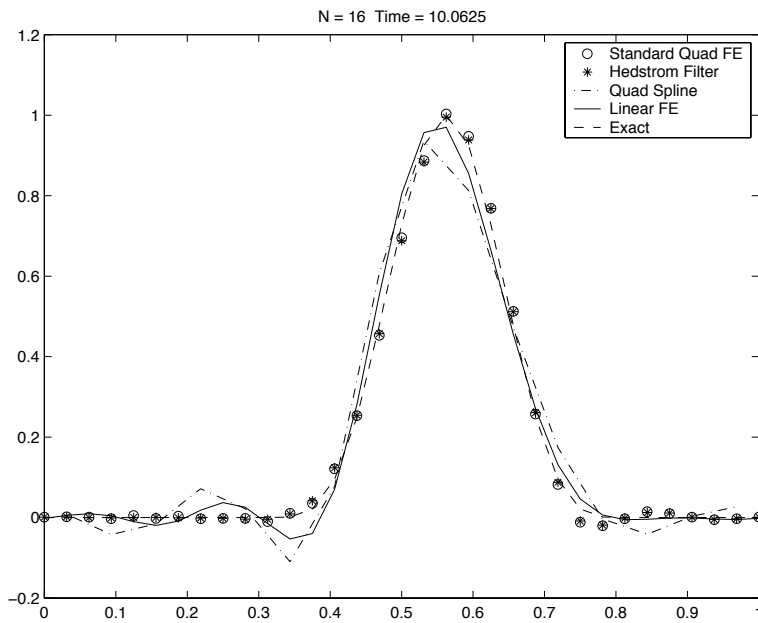


Figure 12: Solutions at time $T = 10 + h$.

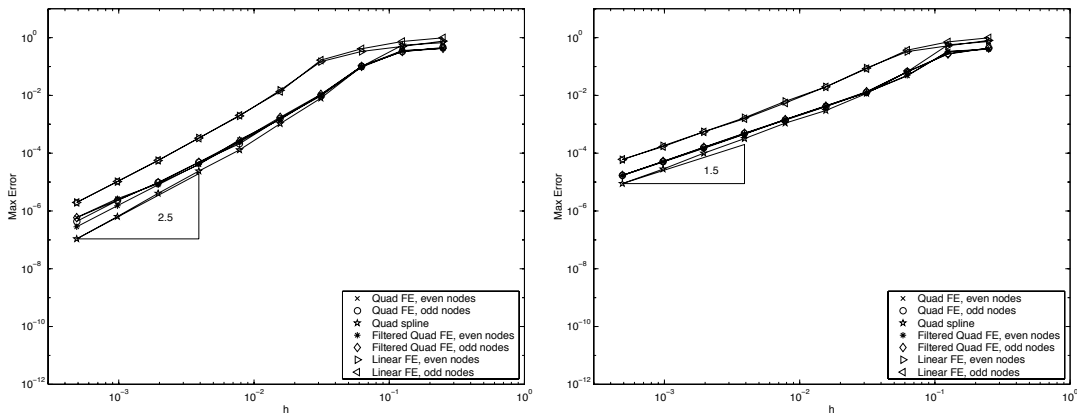


Figure 13: Convergence rates. Left: $m = 3/2$. The highest convergence rate is now only 2.5 (quad spline). Right: $m = 1$. There is no instance when linear FEs are more accurate than quads.

Rougher Pulses The relative accuracies among the quad FE methods remains the same over all three problems.

Conclusions

1. Standard quads are very robust. The nodal accuracy is roughly the same regardless of the initial data chosen except in the roughest case ($m = 1$), when approximately one digit of accuracy is lost.
2. In all cases the filtered quads (at edge nodes) and quad splines are more accurate than standard quads but this superiority is diminished as the data gets rougher (their convergence rates are more sensitive to smoothness of the initial data).
3. Although linear FEs have a superior convergence rate to standard quads for smooth initial data, there needs to be a very fine grid before this effect is apparent. Any superiority of linears over standard quads diminishes significantly as the data becomes less smooth. These results suggest that the error for quad FE has a much smaller error constant (i.e. the C in Ch^p) than linears.