Discretised Eigenvalue Problems, LBB Constants and Stabilization

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Abstract

Standard finite element methods applied to certain classes of differential equations give rise to unstable approximations. The instability is attributable to the fact that the associated LBB (Ladyzhenskaya–Babuska–Brezzi) constant is either small relative to the grid size h or tends to zero with h. This can be investigated by analysing the eigenvalues of related discrete eigenvalue problems.

We consider three instances of such instabilities, convection–diffusion, a curl–curl system and the Q_1–P_0 mixed finite element approximation of the Stokes equations. We identify the LBB constants and describe ways in which the problem may be stabilized.

1 Introduction

Standard Galerkin finite element methods applied to certain classes of differential equations are prone to instabilities in the form of $2-\Delta x$ modes, that is, unphysical oscillations in solutions that have a wavelength equal to twice the element length. The analysis of such problems has been well–known for a number of years (see, for instance, [7, 4]) and the aspect we shall focus on is the failure to satisfy the LBB (Ladyzhenskaya–Babuska–Brezzi) condition. Some of the underlying ideas are described in the next section.

Stability is often associated with problems involving first derivatives and a simple example is provided by the use of linear elements to approximate a derivative such as $u'(x)$ on a periodic domain of length 1. With $h = 1/n$, say, this produces a discrete operator $D_{1,h}$, where

$$D_{1,h}U_j = \tfrac{1}{2}(U_{j+1} - U_{j-1})$$

and U is a typical grid function with $U_j \approx u(jh)$ (with subscript evaluated modulo n). The oscillations are introduced because the grid function $U_j = (-1)^j$ lies in the null space of $D_{1,h}$—this is usually termed a "pure mode", an eigenfunction of the operator with zero eigenvalue. This can sometimes be offset by the use of a filter to remove such components from the solution[24]. This may be only partly successful in that the operator will have small eigenvalues ($\mathcal{O}(h)$) with eigenfunctions of the form $(-1)^j \bar{u}(jh)$, where $\bar{u}$ is a smooth function (in this case, trigonometric). The LBB condition quantifies the smallness in terms of eigenvalues of a related problem. In this example it might take the form

$$D_{1,h}^T D_{2,h}^{-1} D_{1,h} U = \sigma I_h U$$

where $D_{2,h} U_j = (-U_{j+1} + 2U_j - U_{j-1})/h$ (and whose domain is resticted to the orthogonal complement of constant functions to ensure an inverse) is associated with the second derivative and $I_h U_j = h(U_{j+1} + 4U_j + U_{j-1})/6$ with the mass matrix. This, in essence, compares the two approximations $D_{1,h}^T D_{1,h}$ and $D_{2,h}$ for approximating second derivatives. This example is treated in more depth in Section 4.

An outline of the paper is as follows. The LBB condition and its relation to discrete eigenvalue problems is discussed in Section 2. The remaining sections then deal with three sample problems, convection–diffusion, a curl–curl system and the Q_1–P_0 mixed finite element approximation of the Stokes equations. We identify the LBB constants and show that all three may be stabilized using the same basic idea which may be interpreted as either preconditioning of the given systems or by using a Petrov–Galerkin finite element method.

2 The LBB Condition

The Petrov–Galerkin weak formulation of an elliptic boundary problem, written in the form

$$Au = f, \tag{1}$$

involves selecting test and trial spaces $\mathcal{S}$ and $\mathcal{T}$, respectively, a bilinear form $a(\cdot,\cdot)$ on $\mathcal{S} \times \mathcal{T}$ associated with A, and a linear functional $\ell(\cdot)$ on $\mathcal{S}$ associated with f. The weak form then consists of finding $u \in \mathcal{T}$ such that

$$a(v,u) = \ell(v), \qquad \forall v \in \mathcal{S}. \tag{2}$$

It follows from a generalization of the Lax–Milgram theorem due to Babuska and Aziz[1, Theorem 5.2.1] that (2) has a unique solution provided that

$$|a(v,u)| \le C \|v\|_{\mathcal{S}} \|u\|_{\mathcal{T}}, \qquad (C < \infty), \tag{3}$$

$$\sup_{u \in \mathcal{T}} |a(v,u)| > 0, \qquad (\forall v \ne 0) \tag{4}$$

$$\inf_{u \in \mathcal{T}} \sup_{v \in \mathcal{S}} \frac{a(v,u)}{\|v\|_{\mathcal{S}} \|u\|_{\mathcal{T}}} \ge k_0 > 0, \tag{5}$$

and that $\ell(\cdot)$ is a bounded linear functional on $\mathcal{S}$. We then have the stability estimate

$$\|u\|_{\mathcal{T}} \leq \frac{1}{k_0} \sup_{v \in \mathcal{S}} \frac{|\ell(v)|}{\|v\|_{\mathcal{S}}}.$$

For the finite element approximation of (2) we select finite dimensional subspaces $\mathcal{S}_h \subset \mathcal{S}, \mathcal{T}_h \subset \mathcal{T}$, with $\mathcal{S}_h$ and $\mathcal{T}_h$ having equal dimension proportional to h^{-d} (h being a notional element diameter) in d space dimensions, and seek $u^h \in \mathcal{T}_h$ such that

$$a(v, u^h) = \ell(v), \qquad \forall v \in \mathcal{S}_h. \tag{6}$$

This problem, in turn, has a unique solution [1, Chapter 6] if

$$\sup_{u \in \mathcal{T}_h} |a(v,u)| > 0, \qquad (\forall v \neq 0) \tag{7}$$

$$\inf_{u \in \mathcal{T}_h} \sup_{v \in \mathcal{S}_h} \frac{a(v,u)}{\|v\|_{\mathcal{S}} \|u\|_{\mathcal{T}}} \geq k_h > 0. \tag{8}$$

Moreover,

$$\|u - u^h\|_{\mathcal{T}} \leq (1 + C/k_h) \min_{w \in \mathcal{T}_h} \|u - w\|_{\mathcal{T}}, \tag{9}$$

where C is the continuity constant defined by (3) and k_h is the LBB, or "inf–sup" constant defined by (8). We say that the method satisfies the (discrete) LBB condition if k_h is uniformly bounded away from zero as $h \to 0$. This being the case, (9) gives a quasi–optimal error bound. Otherwise there will be a deterioration in the rate of convergence or, in extreme cases, non-convergence. In situations that lead to $k_h = 0$ (which may, or may not coincide with the conditions for $k_0 = 0$) it is often appropriate to replace $\mathcal{S}_h$ with $\mathcal{S}_h / \ker A^{hT}$ and $\mathcal{T}_h$ with $\mathcal{T}_h / \ker A^h$, where A^h is the discrete version of A (see below). A discussion of the LBB condition from the point of view of linear algebra is given by Brezzi and Bathe[3].

It is often quite difficult to establish the satisfaction (or otherwise) of the LBB condition and one way of proceeding is to express the condition as a discrete eigenvalue problem[21, 4]. We choose bases $\{\phi_j\}_{j=1}^N$ and $\{\psi_j\}_{j=1}^N$, respectively, for $\mathcal{T}_h$ and $\mathcal{S}_h$ so that

$$v^h = \sum_{j=1}^N v_j \psi_j \in \mathcal{S}_h, \qquad u^h = \sum_{j=1}^N u_j \phi_j \in \mathcal{T}_h.$$

Defining $\mathbf{u} = (u_1, u_2, \ldots, u_N)^T$ with a similar definition for $\mathbf{v}$, we assume that $N \times N$ matrices A^h, S^h and T^h are defined by

$$a(v^h, u^h) = \mathbf{v}^T A^h \mathbf{u}, \qquad \|v^h\|_{\mathcal{S}}^2 = \mathbf{v}^T S^h \mathbf{u}, \qquad \|u^h\|_{\mathcal{S}}^2 = \mathbf{v}^T T^h \mathbf{u}.$$

Condition (8) then leads to the generalised eigenvalue problem

$$\begin{bmatrix} 0 & A^{hT} \\ A^h & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \mu \begin{bmatrix} T^h & 0 \\ 0 & S^h \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}. \tag{10}$$

The eigenvalues of this system occur in pairs $\pm\mu$ and it may be shown that[4]

$$k_h = \min_{\mu\neq 0} |\mu|.$$

In the case $S^h = T^h = I$, (10) corresponds to a singular value decomposition of A^h.

3 A Convection–Diffusion Problem

We consider the boundary value problem

$$\begin{aligned} -\varepsilon\nabla^2 u + \mathbf{p}\cdot\nabla u &= f, \qquad \text{in } \Omega, \\ u &= 0, \qquad \text{on } \partial\Omega, \end{aligned} \tag{11}$$

where Ω is a closed bounded domain in $\mathbb{R}^d (d \geq 1)$, $\varepsilon > 0$ and $\mathbf{p} \in \mathbb{R}^d$ is a constant vector. The bilinear form associated with (11) is given by

$$a(v, u) = \varepsilon(\nabla v, \nabla u) + (v, \mathbf{p}\cdot\nabla u),$$

where $(\cdot,\cdot)$ is the usual L_2 inner product and $\mathcal{S} = \mathcal{T} = H_0^1(\Omega)$.

The one–dimensional case $d = 1$ with $\Omega = (0, 1)$ has been studied by Griffiths & Lorenz[8]. It is shown there that the continuity constant C is given by

$$C = (\varepsilon^2 + p^2/4\pi^2)^{1/2}$$

and that, for both the continuous case and the Galerkin finite element approximation where $\mathcal{T}_h = \mathcal{S}_h$ consist of continuous piecewise linear functions,

$$k_0 = k_h = \varepsilon.$$

Thus, in this case the stability constant $C/k_h = \mathcal{O}(\varepsilon^{-1}) \gg 1$ is very large and this property is clearly inherited from the continuous problem rather than being induced by the approximation.

A popular means of stabilising the finite element method is to use the Streamline Diffusion method[11, 14, 17]. This is a Petrov–Galerkin method in which the test functions are related to the trial functions through

$$\psi_j = (1 + \alpha h\mathbf{p}\cdot\nabla)\,\phi_j, \quad j = 1, 2, \ldots, N,$$

where α is an adjustable parameter ($\alpha = 1$ leads to the "upwind" scheme and $\alpha = \coth(h/2\varepsilon) - 2\varepsilon/h$ to the Il'in scheme[23], for example). The functions ψ_j are, in general, discontinuous at element boundaries and therefore lead to nonconforming approximations. One way of proceeding is to define a new bilinear form according to

$$a^h(v, u) = \sum_{e\in\Omega} \varepsilon(\nabla v, \nabla u)_e + (v, \mathbf{p}\cdot\nabla u)_e,$$

where e denotes an element of the discretised domain Ω and $(\cdot,\cdot)_e$ the L_2 inner product on e. Alternatively, one may adopt equivalent, conforming, test functions[5, 8, 9]. The precise form does not concern us here since both approaches lead to essentially similar results.

Restricting attention once more to the one–dimensional case, it is shown in [8] that the LBB constant now becomes

$$k_h = \frac{\varepsilon + \alpha p h/2}{(1+3\alpha^2)^{1/2}}$$

so that, in cases where $h \gg \varepsilon$ and $\alpha p > 0$ we have $k_h \gg k_0$ thus leading to a much more stable situation.

Stabilization of the problem has been achieved by choosing an operator $P = \mathbf{p}\cdot\nabla$ and replacing the original problem $Au = f$ by the "preconditioned" system

$$(1 + \alpha h P^*)Au = (1 + \alpha h P^*)f, \tag{12}$$

P^* being the adjoint of P, and this equation clearly has the same solutions as $Au = f$ modulo boundary conditions. With A defined by (11), equation (12) takes the form

$$-(\nabla\cdot(\varepsilon + \alpha h \mathbf{p}\mathbf{p}^T\nabla u) + \mathbf{p}\cdot\nabla u + \varepsilon\alpha h(\mathbf{p}\cdot\nabla)\nabla^2 u = (1 - \alpha h\mathbf{p}\cdot\nabla)f \tag{13}$$

from which we observe that the diffusion coefficient ε has been enhanced by a tensor diffusivity $\alpha h\mathbf{p}\mathbf{p}^T$ directed in the flow direction $\mathbf{p}$. This leads to the small "cross–wind" diffusion of the method in multidimensions. The streamline diffusion method is equivalent to using a Galerkin finite element method on the equation (13). Then, because of the local support of the test functions and their low degree, the method effectively ignores the term $\varepsilon\alpha h(\mathbf{p}\cdot\nabla)\nabla^2 u$.

Were we to adopt a finite difference strategy, for instance, approximate each term in (13) to a sufficiently high degree and choose a suitable value for α, we would obtain the QUICK scheme of Leonard[20].

The regularization may thus be interpreted as "preconditioning" of the differential equation (in such a way as to leave the solutions invariant) prior to discretization.

4 A curl–curl System

Certain electromagnetic field problems associated with Maxwell's equations lead to an eigenvalue problem involving the curl–curl operator

$$A = \begin{bmatrix} 0 & -\mathrm{curl} \\ \mathrm{curl} & 0 \end{bmatrix}. \tag{14}$$

The discretization of such problems by Galerkin finite element methods (and many other schemes) gives rise to spurious modes, that is, eigenvalues of the discrete system that have no counterpart in the continuous problem. Schroeder and Wolff[25] give a comprehensive review of this phenomenon and offer the opinion that "The spurious mode problem is severe and often renders the numerical output useless".

Our aim is to show that the spuriosity is a consequence of a violation of the LBB condition and to propose a stabilization of the problem in the spirit of the preconditioning described in the previous section.

We shall illustrate our ideas by considering the problem in one dimension on the interval $[0, 2\pi]$ with periodic boundary conditions. The operator A then takes the form

$$A = \begin{bmatrix} 0 & -\mathrm{d}/\mathrm{dx} \\ \mathrm{d}/\mathrm{dx} & 0 \end{bmatrix}. \tag{15}$$

We choose the test and trial spaces

$$\mathcal{S} = [L_2[0, 2\pi]]^2, \qquad \mathcal{T} = \left[H_p^1[0, 2\pi]\right]^2,$$

respectively, where $H_p^1[0, 2\pi]$ denotes the space of periodic functions on $[0, 2\pi]$ with first derivatives in $L_2[0, 2\pi]$. The LBB constant for this problem is $k_0 = 1$ and the continuous eigenproblem $Au = \lambda u$ has eigenvalues $\lambda = 0, \pm 1, \pm 2, \ldots$, each with multiplicity 2.

Following Schroeder and Wolff[25], we begin by discretizing the eigenvalue problem by a Galerkin method in which both test and trial spaces $\mathcal{S}_h$, $\mathcal{T}_h$ are periodic, continuous, piecewise linear functions on a uniform grid of size $h = 2\pi/N$. This leads to the discrete operator

$$A^h = \begin{bmatrix} 0 & C^T \\ C & 0 \end{bmatrix}, \tag{16}$$

where C denotes the $N \times N$ cyclic tridiagonal matrix $\mathrm{cyctri}(-\frac{1}{2}, 0, \frac{1}{2})$, and

$$\mathrm{cyctri}(a, b, c) = \begin{bmatrix} b & c & & & a \\ a & b & c & & \\ & \ddots & \ddots & \ddots & \\ & & a & b & c \\ c & & & a & b \end{bmatrix}.$$

The discrete form of $Au = \lambda u$ is the matrix eigenproblem

$$\begin{bmatrix} 0 & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} u^h \\ v^h \end{bmatrix} = \lambda^h \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} u^h \\ v^h \end{bmatrix}, \tag{17}$$

where $M = \frac{1}{6} h\, \mathrm{cyctri}(1, 4, 1)$ is the mass matrix for linear elements. The eigenvalues are given by

$$\lambda_j^h = \pm \frac{\sin jh}{h} \frac{1}{1 - \frac{2}{3}\sin^2 \frac{1}{2} jh}, \qquad j = 0, 1, 2, \ldots, (N-1) \tag{18}$$

from which we see that $\lambda_j^h = \lambda_j + \mathcal{O}(h^2)$ for $jh \ll 1$. Also $\lambda_{N-j}^h = \lambda_j^h$ so that each eigenvalue has multiplicity 2, mimicking the continuous problem. The difficulties arise for values of $jh \approx \frac{1}{2}N$. For instance, if N is even,

$$\lambda_{N/2-k}^h = \pm \frac{\sin kh}{h} \frac{1}{1 - \frac{2}{3}\cos^2 \frac{1}{2}kh}$$

so that $\lambda_{N/2-k}^h = 3k + \mathcal{O}(h^2)$—the eigenvalues near the centre of the spectrum are aliased to certain physical eigenvalues (multiples of 3) at the extreme ends; spurious eigenvalues "intermingle" with the true ones. A typical situation is shown in Table 1 for $N = 20$. It is only by looking at the corresponding eigenvectors that one may identify with any certainty the spurious modes as they oscillate with a wavelength $2h$.

λ_j	λ_j^h			
	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 1/\sqrt{15}$	$\alpha = 0.5$
1	0.9999	1.0000	1.0000	1.0001
2	1.9982	1.9985	2.0000	2.0046
3	2.8132	2.9877	3.0003	3.0336
4	[2.8132]	[3.8197]	4.0022	4.1360
5	2.9854	3.9445	5.0110	5.4019
6	3.9332	[4.5138]	6.0408	6.9840
7	[4.7129]	4.8153	7.1194	9.1391
8	4.7746	[5.4744]	8.2661	12.2578
9	5.3708	5.4929	9.3535	16.4489
10	5.4705	5.7919	9.8625	19.0986

Table 1: The positive eigenvalues $\lambda_j = j$ and their discrete counterparts λ_j^h for $N = 20$ and different values of the stabilizing parameter α. The case $\alpha = 0$ corresponds to the unstabilized case given by (18)and the boxed entries identify the spurious modes.

To compute the discrete LBB constant in this case we follow the development in Section 2 in which case the generalised eigenvalue problem (10) leads to a block 4×4 system where A^h is given by (16), $T^h = K \oplus K$, $S^h = M \oplus M$ and $K = (1/h)\text{cyctri}(-1, 2, -1)$ is the "stiffness" matrix for linear elements. Each of the submatrices in this system are cyclic so that they may be simultaneously diagonalised. It is then a straightforward exercise to show that the problem has eigenvalues

$$\mu_j = \pm \cos \tfrac{1}{2}jh \frac{1}{\sqrt{1 - \frac{2}{3}\sin^2 \frac{1}{2}jh}}, \qquad j = 0, 1, 2, \ldots, (N-1).$$

Thus, $|\mu_j| = 1 + \mathcal{O}(h^2)$ for $jh \ll 1$ but, for N even,

$$\mu_{N/2-k} = \pm \sin \tfrac{1}{2}kh \frac{1}{\sqrt{1 - \frac{2}{3}\cos^2 \frac{1}{2}kh}}.$$

Thus $k_h = \mu_{N/2-1} = \sqrt{3}h/2 + \mathcal{O}(h^2)$ and the LBB condition is violated since $k_h \to 0$ with h. The continuous problem is easily shown to have an LBB constant $k_0 = 1$.

Schroeder and Wolff[25] propose a stabilization which involves a Petrov–Galerkin formulation in which the trial space is left unchanged but the test space consists of the same piecewise linear functions on a grid of size $h/2$ so that its dimension is doubled. The eigenvalue problem then takes the form of a generalised Rayleigh quotient and their numerical results indicate that the spurious modes are eliminated.

We also adopt a Petrov–Galerkin formulation in which only the test space $\mathcal{S}_h$ is modified, but we choose to define

$$\mathcal{S}_h = \{v \,|\, v = (I + \alpha h P)u, u \in \mathcal{T}_h\}$$

where

$$P = \begin{bmatrix} 0 & \mathrm{d/dx} \\ \mathrm{d/dx} & 0 \end{bmatrix}.$$

The discrete form of the eigenvalue problem $Au = \lambda u$ is equivalent to the Galerkin finite element approximation of (compare with (12))

$$(1 + \alpha h P^*)Au = \lambda(1 + \alpha h P^*)u$$

and leads to the algebraic eigenvalue problem

$$\begin{bmatrix} \alpha h K & C^T \\ C & -\alpha h K \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \lambda^h \begin{bmatrix} I & \alpha h C^T \\ -\alpha h C & I \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}. \qquad (19)$$

The LBB condition associated with the operator on the left of this equation is given by

$$\begin{bmatrix} 0 & 0 & -\alpha h K & C \\ 0 & 0 & -C & \alpha h K \\ -\alpha h K & C & 0 & 0 \\ -C & \alpha h K & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \mu \begin{bmatrix} K & 0 & 0 & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & M \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix}.$$

from which we obtain

$$\mu_j^2 = \frac{\cos^2 \frac{1}{2}jh + 4\alpha^2 \sin^2 \frac{1}{2}jh}{1 + (4\alpha^2 - \frac{2}{3})\sin^2 \frac{1}{2}jh}.$$

The LBB constant is deduced from the minimum of this expression over $\theta \in [0, 2\pi]$ and we find that

$$k_h = \frac{2\alpha\sqrt{3}}{\sqrt{1 + 12\alpha^2}} + \mathcal{O}(h^2).$$

Thus the Petrov–Galerkin formulation is LBB stable provided that α is bounded away from zero independent of h.

The eigenvalues of the stabilized problem (19) are given by

$$\lambda_j^h = \pm \left(\frac{\sin^2 jh + 16\alpha^2 \sin^4 \frac{1}{2}jh}{h^2 \left((1 - \frac{2}{3}\sin^2 \frac{1}{2}jh)^2 + \alpha^2 \sin^2 jh \right)} \right)^{1/2}.$$

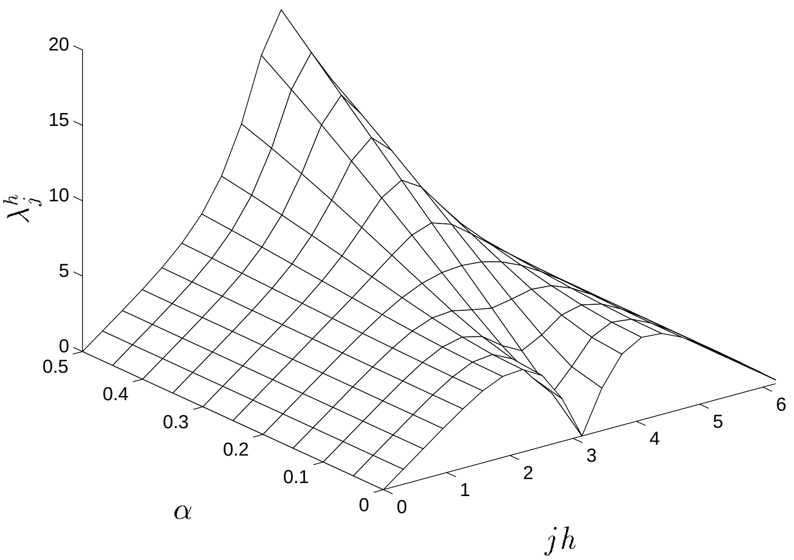

Figure 1: The eigenvalues λ_j^h shown as a function of $jh \in [0, 2\pi]$ and $a \in [0, 1/2]$.

Hence,

$$\lambda_j^h = \pm j \pm \frac{1}{180}(15\alpha^2 - 1)j^5 h^4 + \mathcal{O}(h^6), \qquad jh \ll 1,$$

which suggests the value $\alpha = 1/\sqrt{15}$ since this gives eigenvalues of order $\mathcal{O}(h^6)$. Values of λ_j^h for $\alpha = 0.1, 1/\sqrt{15}, 0.5$ are given in Table 1. It is seen that spurious eigenvalues are not precluded unless α is sufficiently large; a value $\alpha > 0$ will ensure that there are no spurious zero eigenvalues (The fact that there appear to be more spurious eigenvalues for $\alpha = 0.1$ than for $\alpha = 0$ is because the latter case also has an additional eigenvalue $\lambda^h = 0$ which is not shown). The effect of the stabilizing parameter on the eigenvalues is shown in Figure 1. It is readily shown, for

$$0 \leq \alpha < \alpha_{crit} \equiv \tfrac{1}{6}\sqrt{3\sqrt{2} - 3} \approx 0.185\ldots,$$

that λ_j^h has two local maxima for $j \in [0, N]$. Additionally, for α in this range, λ_j^h has a local minimum $\lambda_{N/2}^h = 6\alpha N/\pi$ at $j = N/2$. Consequently, spurious eigenvalues can only occur when $\alpha < \alpha_{crit}$ in which case the spurious eigenvalues must satisfy $\lambda^h \geq 6\alpha N/\pi$.

For $\alpha \geq \alpha_{crit}$, λ_j^h has a local maximum $\lambda_{N/2}^h = 6\alpha N/\pi$ at $j = N/2$ which will agree with the corresponding exact eigenvalue $\lambda_{N/2} = N/2$ if

$$\alpha = \pi/12 \approx 0.2618.$$

This value is close to the optimum $\alpha = 1/\sqrt{15} \approx 0.2582$ suggested earlier.

5 Mixed Approximations of the Stokes Equations

The Stokes equations which govern the slow flow of a viscous fluid are given by

$$\begin{aligned} -\nabla^2 \mathbf{u} + \nabla p &= \mathbf{f} \quad &\text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad &\text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} \quad &\text{on } \Gamma. \end{aligned} \tag{20}$$

In order for this problem to have a solution it is necessary that

$$\int_\Gamma \mathbf{g} \cdot \mathbf{n} \, ds = 0, \tag{21}$$

where $\mathbf{n}$ denotes the outward normal to Γ.

We shall take Ω to be the unit square in $\mathbb{R}^2$ and impose a finite element grid of $n \times n$ square elements of size $h = 1/n$. The Galerkin finite element formulation of this problem takes the form: Find $\mathbf{u}^h \in \mathcal{V}^h, p^h \in \mathcal{P}^h$ such that

$$\begin{aligned} (\nabla \mathbf{v}, \nabla \mathbf{u}^h) + (\mathbf{v}, \nabla p^h) &= (\mathbf{v}, \mathbf{f}) \quad \forall \mathbf{v} \in \mathcal{V}_0^h, \\ (q, -\nabla \cdot \mathbf{u}^h) &= 0 \qquad \forall q \in \mathcal{P}^h, \end{aligned} \tag{22}$$

where $\mathcal{V}_0^h \subset H^1(\Omega)$, $\mathcal{V}^h \cap H_0^1(\Omega)$ and $\mathcal{P}^h \subset L_2(\Omega)$. We shall consider the Q_1–P_0approximation where $\mathcal{V}^h$ is the space (usually called Q_1) of vector–valued continuous piecewise bilinear functions on the grid which interpolate the velocity data on the boundary Γ, and $\mathcal{P}^h$ (usually called P_0) is the space of piecewise constant functions. It is well documented in the literature (see, for instance, [24, 16, 2, 7, 4, 10]) that this combination of finite element spaces is LBB unstable. In particular, it is shown in [10] that the LBB constant is given by

$$k_h = \sqrt{\tfrac{3}{8}} \pi h + \mathcal{O}(h^2).$$

This result was established by working with the algebraic form of the discrete equations (22), which may be written as

$$\begin{bmatrix} L & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}, \tag{23}$$

in which $L \in \mathbb{R}^{2(n-1)^2 \times 2(n-1)^2}$ denotes the approximation of the vector–valued Laplacian (and is positive definite), $B \in \mathbb{R}^{n^2 \times 2(n-1)^2}$ the approximation of the divergence operator, and $\mathbf{U} \in \mathbb{R}^{2(n-1)^2}$, $\mathbf{P} \in \mathbb{R}^{n^2}$ contain, respectively, the nodal values of $\mathbf{u}^h$ and p^h. We associate with (23) the generalised eigenvalue problem

$$\begin{bmatrix} L & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \lambda^h \begin{bmatrix} L & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix}, \tag{24}$$

$M \in I\!R^{n^2 \times n^2}$ being the mass matrix on $\mathcal{P}^h$ ($M = h^2 I$), which has real eigenvalues λ^h and corresponding eigenvectors that are orthogonal with respect to the inner product

$$(\mathbf{W}_1, \mathbf{W}_2) = \mathbf{U_1}^T L \mathbf{U_2} + \mathbf{P_1} M \mathbf{P_2}, \qquad \mathbf{W}_k = \begin{bmatrix} \mathbf{U}_k \\ \mathbf{P}_k \end{bmatrix}, \quad k = 1, 2. \qquad (25)$$

The LBB constant is given by[21]

$$k_h = \min_{\lambda^h \neq 0,1} \sqrt{\lambda^h(\lambda^h - 1)}.$$

The eigenvalues λ_j^h for (24) are shown in Figure 2 for $h = 1/10$. They may be classified

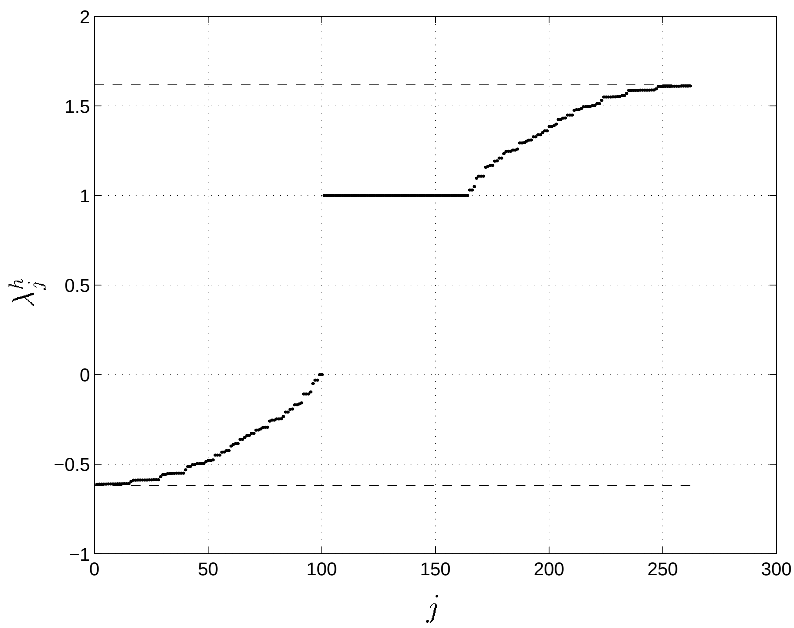

Figure 2: The eigenvalues λ_j^h of (24) for $h = 1/10$. The horizontal broken lines are drawn at $(1 \pm \sqrt{5})/2$.

into four groups with corresponding subspaces which are orthogonal with respect to (25).

1. Pure pressure modes. There are two vectors, $\mathbf{P}_{0,1}$ and $\mathbf{P}_{0,2}$, lying in the nullspace of B^T. These correspond to two eigenvalues $\lambda^h = 0$ and generate the space

$$\mathcal{W}_p^h = \{ \begin{bmatrix} \mathbf{0} \\ \mathbf{P} \end{bmatrix}; B^T \mathbf{P} = 0 \}, \qquad \dim \mathcal{W}_p^h = 2.$$

   The vectors $\mathbf{P}_{0,1}$ and $\mathbf{P}_{0,2}$ may be identified with the hydrostatic mode $p^h = 1$ and the "pure chequerboard" mode p_{CB}^h which takes on the values $+1$ and -1 on contiguous elements.

2. Discretely divergence–free vectors. These correspond to $\lambda^h = 1$ and generate the space

$$\mathcal{W}^h_{div} = \{ \begin{bmatrix} \mathbf{U} \\ \mathbf{0} \end{bmatrix} ; B\mathbf{U} = 0 \}, \qquad \dim \mathcal{W}^h_{div} = 2n^2 - (n+1)^2 + 2.$$

We shall suppose that the vectors $\{\mathbf{U}_{0,j}, j = 1, 2, \ldots, (n-1)^2\}$ form a basis for this space which is orthogonal with respect to (25) (with corresponding zero pressure vectors).

3. Discretely irrotational vectors. Eliminating $\mathbf{U}$ from (24) we obtain an eigenvalue problem for $\mathbf{P}$ based on the Schur complement of L:

$$BL^{-1}B^T\mathbf{P} = \sigma^h M\mathbf{P}, \qquad \sigma^h = \lambda^h(\lambda^h - 1) > 0. \tag{26}$$

The case $\sigma^h = 0$ leads to elements of $\mathcal{W}^h_{div}$ and $\mathcal{W}^h_p$.
Problem (26) generates $(n+1)^2 - 2$ eigenvalues σ^h_j (counting multiplicities), ordered such that $0 \le \sigma^h_1 \le \sigma^h_2 < \cdots$, with corresponding eigenvectors denoted by $\mathbf{P}_{\sigma_j}$. We let $\lambda^h_{+j}(> 1), \lambda^h_{-j}(< 0)$ denote the two roots of

$$\lambda^h(\lambda^h - 1) = \sigma^h_j$$

and, to each eigenvector of (26), we compute a velocity

$$\mathbf{U}_{\sigma_j} = L^{-1}B^T\mathbf{P}_{\sigma_j}.$$

Two spaces $\mathcal{W}^h_+$ and $\mathcal{W}^h_-$ may then be defined by

$$\mathcal{W}^h_\pm = \{ \begin{bmatrix} \mathbf{U}_{\sigma_j} \\ (\lambda^h_{\pm j} - 1)\mathbf{P}_{\sigma_j} \end{bmatrix}, \qquad j = 1, 2, \ldots, (n+1)^2 - 2 \}$$

so that $\dim \mathcal{W}^h_+ = \dim \mathcal{W}^h_- = (n+1)^2 - 2$. These are termed "discretely irrotational vectors" in that they are orthogonal to $\mathcal{W}^h_{div}$; no explicit constraint to this effect is imposed on either space.

We return now to the system of equations (23) and observe that, since the coefficient matrix is singular, it will have a solution if, and only if, the vector $\mathbf{G}$ is orthogonal to both $\mathbf{P}_{0,1}$ and $\mathbf{P}_{0,2}$. When these conditions are satisfied, a short calculation reveals that the solution of may be written in terms of an eigenvector expansion as

$$\begin{aligned} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} &= \sum_{j=1,2} a_j \begin{bmatrix} \mathbf{0} \\ \mathbf{P}_{0,j} \end{bmatrix} + \sum_{j=1}^{(n-1)^2} \frac{\mathbf{U}^T_{0,j}\mathbf{F}}{\mathbf{U}^T_{0,j} L \mathbf{U}_{0,j}} \begin{bmatrix} \mathbf{U}_{0,j} \\ \mathbf{0} \end{bmatrix} + \\ & \sum_{j=1}^{(n+1)^2-2} \frac{1}{\sigma^h_j \mathbf{P}^T_{\sigma_j} M \mathbf{P}_{\sigma_j}} \left(\mathbf{P}^T_{\sigma_j}\mathbf{G} \begin{bmatrix} \mathbf{U}_{\sigma_j} \\ -\mathbf{P}_{\sigma_j} \end{bmatrix} + \mathbf{U}^T_{\sigma_j}\mathbf{F} \begin{bmatrix} \mathbf{0} \\ \mathbf{P}_{\sigma_j} \end{bmatrix} \right), \end{aligned} \tag{27}$$

where the coefficients a_1, a_2 are arbitrary, reflecting the non–uniqueness of solutions. The terms involving these coefficients may be filtered out of computed solutions[24] by requiring that $\mathbf{P}$ be orthogonal to the pure pressure modes. An expansion of this form holds for all mixed approximations which lead to systems of the form (23), except that the dimensions of the various spaces will vary from method to method.

It follows from (27) that the "weakly unstable modes" (or "impure modes")—the contributions associated with the small LBB constant $k_h = (\sigma_1^h)^{1/2} = \mathcal{O}(h)$—are contained in the third sum on the right of (27). In cases where $\mathbf{G} = 0$ (the "leaky" lid driven cavity, for instance, where the velocity is zero on the boundary Γ except on the upper lid, where the horizontal component of velocity takes on the same value at all nodes) the impure modes are seen to affect only the pressure part of the solution. It may be shown, more generally, that the the effect of the impure modes is substantially greater on the pressure components than on the velocity.

It may be shown[10] for Q_1–P_0 that $\sigma_1^h = \sigma_2^h = 3/8\pi^2 h^2 + \mathcal{O}(h^3)$ and the corresponding eigenfunctions have nodal values given to leading order terms by

$$\left.\begin{array}{rcl} \mathbf{u}_{\ell,m}^h & = & (-1)^{\ell+m+1}\begin{bmatrix} \frac{3}{8}\pi^2 h^2 \sin m\pi h \\ 0 \end{bmatrix} \\ p_{\ell+1/2,m+1/2}^h & = & (-1)^{\ell+m+1}\pi\cos\pi(m+1/2)h \end{array}\right\} \qquad (28)$$

and

$$\left.\begin{array}{rcl} \mathbf{u}_{\ell,m}^h & = & (-1)^{\ell+m+1}\begin{bmatrix} 0 \\ \frac{3}{8}\pi^2 h^2 \sin \ell\pi h \end{bmatrix} \\ p_{\ell+1/2,m+1/2}^h & = & (-1)^{\ell+m+1}\pi\cos\pi(\ell+1/2)h, \end{array}\right\} \qquad (29)$$

where $\mathbf{u}_{\ell,m}^h$ corresponds to the velocity node $(\ell h, mh)$ (a vertex) and $p_{\ell+1/2,m+1/2}^h$ with a pressure node $((\ell+1/2)h, (m+1/2)h)$ (a centroid).

We now extend these results to certain stabilizations of the equations; for reviews of this area see [4, 26]. The stabilizations are obtained by replacing the second of equations (22) by

$$(q, -\nabla\cdot\mathbf{u}^h) - \beta c_h(q, p^h) = 0 \qquad \forall q \in \mathcal{P}^h, \qquad (30)$$

where β is the so–called stabilization parameter and $c_h(\cdot,\cdot)$ a suitable bilinear form defined on $\mathcal{P}^h \times \mathcal{P}^h$. Some formulations also modify the right hand side in order to maintain consistency at a higher level[6, 18].

Using (30), the system (23) takes the form

$$\begin{bmatrix} L & B^T \\ B & -\beta C \end{bmatrix}\begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}, \qquad (31)$$

and an essential requirement is that the $n^2 \times n^2$ matrix C should be positive semi-definite. We shall investigate stability by studying the behaviour of the smallest (non-zero) and largest eigenvalues σ_β^h of the system

$$\begin{bmatrix} L & B^T \\ B & -\beta C \end{bmatrix}\begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \sigma_\beta^h \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} \qquad (32)$$

which is equivalent to

$$(BL^{-1}B^T + \beta C)\mathbf{P} = \sigma_\beta^h M\mathbf{P}, \tag{33}$$

so that σ_0^h is consistent with the earlier use of σ^h.

Stabilization 1: (Global)[12, 18] We take the bilinear form defined by

$$c_h(q, p^h) = \sum_{e\in\Gamma_h} h_e \int_e [\![\, q \,]\!]_e [\![\, p^h \,]\!]_e ds, \tag{34}$$

where Γ_h is the set of all interior edges of the mesh, h_e the length of the element edge e and $[\![\, \cdot \,]\!]_e$ denotes the jump across e (the direction in which the jump is measured is immaterial). This approach generates a matrix $C \equiv C_1$ such that $C_1\mathbf{P}$ is equivalent to the use of the standard 5–point finite difference approximation of $-h^2\nabla^2 p$ (with nodes at element centroids) and satisfying homogeneous Neumann boundary conditions. This form of stabilization may be interpreted as discretizing a preconditioned system of the form (12) where A denotes the operator defined by the left side of (20), the preconditioner P takes the form

$$P^* = \begin{pmatrix} 0 & 0 \\ -\beta h\nabla\cdot & 0 \end{pmatrix}$$

and we explicitly impose the constraint $\nabla \cdot \mathbf{u} = 0$ on the result. This idea is developed in greater depth by Idelsohn and colleagues[15, 27].

Stabilization 2: (Global) This is a minor modification of the preceding case where we replace the mass matrix M on the right of (32) by a diagonal matrix D obtained by dividing the diagonal entries of C_1 by 4. Thus D has diagonal entries $h^2/2$ corresponding to corner elements, $3h^2/4$ corresponding to other boundary elements and h^2 corresponding to interior elements.

Stabilization 3: (Local)[19, 28] We assume, for simplicity, that the domain is decomposed into an even number of elements in each direction and then group them into blocks of 2×2 macro–elements $\mathcal{M}_h$. We then amend the definition (34) to read

$$c_h(q, p^h) = 2 \sum_{m\in\mathcal{M}_h} \sum_{e\in\Gamma_m} h_e \int_e [\![\, q \,]\!]_e [\![\, p^h \,]\!]_e ds, \tag{35}$$

where Γ_m is now the set of four interior edges of the macro m. There is an additional factor 2 on the right which is included so as to harmonize the dependence of eigenvalues on β with the other methods. When the pressure variables are suitably ordered (counterclockwise in each macro–element), this leads to a stabilization matrix $C = C_3$ that is block diagonal and whose diagonal blocks are the 4×4 matrix

$$C_3^m = h^2 \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & -2 \end{bmatrix}.$$

Stabilization 4: (Local)[22] This uses the bilinear form based on macro–elements

$$c_h(q, p^h) = 2 \sum_{m \in \mathcal{M}_h} \int_m \Pi_m q \Pi_m p^h dm, \tag{36}$$

where Π_m denotes the L_2–projection onto the local chequerboard mode on each macro. The corresponding stabilization matrix $C = C_4$ has a similar structure to the preceding case but the diagonal blocks are the rank one matrix

$$C_4^m = h^2 \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \end{bmatrix}.$$

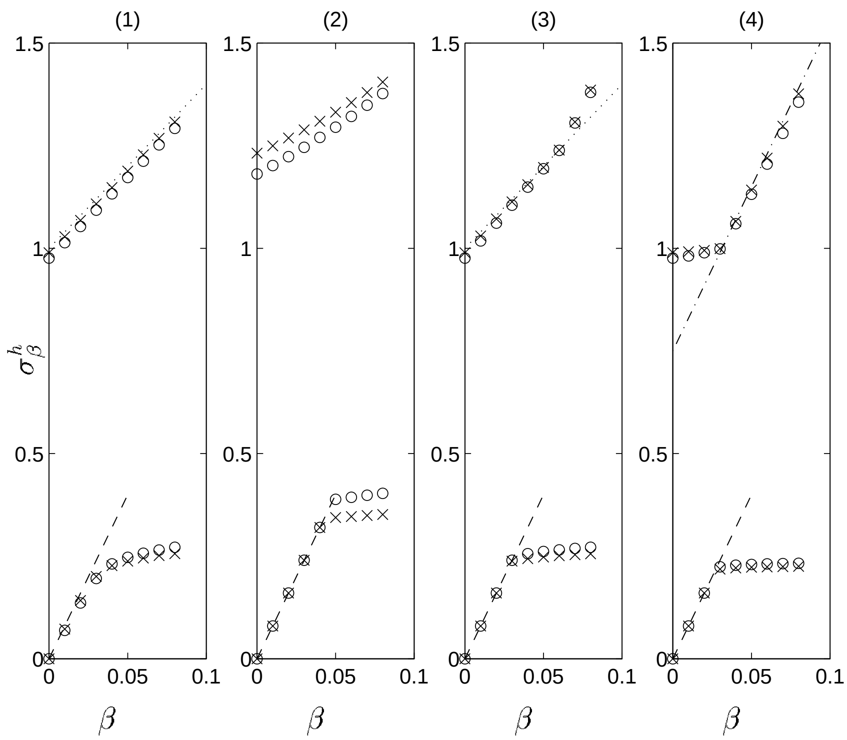

Figure 3: The smallest and largest eigenvalues σ_β^h of (33) shown as a function of β for $h = 1/8$ ($\circ$) and $h = 1/12$ ($\times$). Also shown are he lines $\sigma = 8\beta$ (broken), $\sigma = 1 + \beta$ (dotted) and $\sigma = 3/4 + 8\beta$ (dot–dash).

Broadly speaking, the dependence of the eigenvalues of all four methods on β is qualitatively similar: there are eigenvalues in the interval $[1/4, 1]$ ($[1/3, 4/3]$ in case 2) that are relatively insensitive to β. The corresponding eigenvectors do not display the typical chequerboard pattern associated with weakly unstable modes. The remaining eigenvalues depend strongly on β and those closest to zero can be identified with weakly unstable modes.

More specifically, it is easily observed from the structure of the matrices C for stabilizations 2–4, that there is an eigenvector $\mathbf{P}$ of (33) which has entries that correspond to $+1$ and -1 on contiguous elements (the pure pressure mode of the unstabilized problem) with corresponding eigenvalue

$$\sigma_\beta^h = 8\beta$$

and velocity components $\mathbf{U} = \mathbf{0}$. There is no exact representation of this eigenvector for stabilization 1, but we may use the techniques described in [10] to show that $\sigma_\beta^h = 8\beta + \mathcal{O}(h)$ (numerical results suggest that $\sigma_\beta^h = 8\beta(1-8\beta) + \mathcal{O}(h), 0 \leq \beta \leq 1/16$).

The eigenvalues are shown in Figure 3 for $h = 1/8$ ($\circ$) and $h = 1/12$ ($\times$). These results show that the smallest eigenvalue does indeed grow as 8β for small values of β, until this value exceeds $\approx 1/4$ ($\approx 1/3$ in case 2), whereupon the smallest eigenvalue becomes one of the set of eigenvalues relatively insensitive to β. Thus, stabilization of the smallest eigenvalue may be achieved by choosing $\beta \geq 1/32$; larger values having relatively little effect.

Practical considerations, however, often depend more on the condition number ($\max \sigma_\beta^h / \min \sigma_\beta^h$) of the system (33) than the smallest eigenvalue[28] and we therefore require information regarding the largest eigenvalues.

In cases 1, 3 and 4 we assume that there is an eigenvector having the form (compare with (29) and (28))

$$\left.\begin{array}{rcl} \mathbf{u}_{\ell,m}^h & = & (-1)^\ell \begin{bmatrix} \bar{u}_{\ell,m} \\ 0 \end{bmatrix} \\ p_{\ell+1/2,m+1/2}^h & = & (-1)^\ell \bar{p}_{\ell+1/2,m+1/2}, \end{array}\right\} \qquad (37)$$

where $\bar{u}_{\ell,m}$ and $\bar{p}_{\ell+1/2,m+1/2}$ are the grid values of smooth functions (There is also a second set of grid functions with u and v components of velocity interchanged which lead to the same eigenvalues). It may then be shown, following [10], that

$$\sigma_\beta^h = 1 + 4\beta + \mathcal{O}(h),$$

in cases 1 and 3, while, $\sigma_\beta^h = 1 + \mathcal{O}(h)$ for case 4 (the essential difference is that the pressure vector with entries corresponding to the grid function which has the value $(-1)^\ell$ on the element centred at $x = (\ell + 1/2)h, y = (m + 1/2)h$ is an eigenvector of C_3 and C_4 with eigenvalues 4 and 0, respectively). These provide lower bounds for the largest eigenvalues. The case of stabilization 2 and stabilization 4 (for $\beta > 1/32$) may be tackled differently. The associated velocity eigenvectors may be represented by a tensor product of two simple functions, but the details are, as yet, incomplete.

The expressions for the eigenvalues are in agreement with Vincent[28] who proves that the condition number in case 3 is bounded by

$$\frac{a + b\beta}{\min(c, 8\beta)},$$

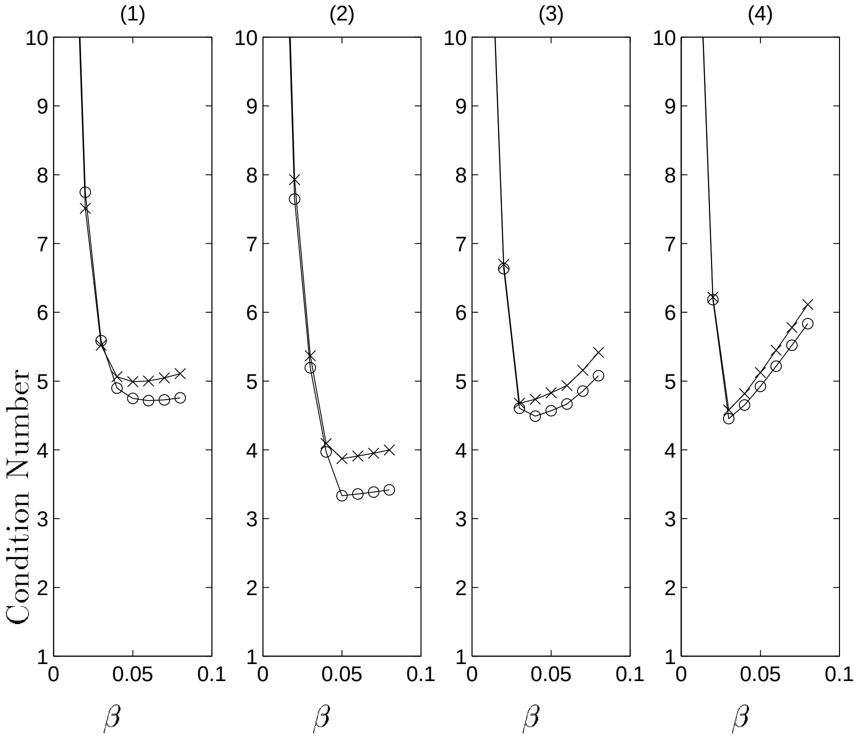

Figure 4: The condition number shown as a function of β.

where a, b and c are constants (undetermined) that are independent of h (we have taken into account the additional factor 2 introduced into the definition f the bilinear form (35).

The condition numbers of the four cases are shown in Figure 4 for $h = 1/8$ and $h = 1/12$. All four methods are seen to have effectively stabilized the problem. The optimum value of $\beta = 1/32$ is suggested both by the estimates of eigenvalues and the computations. The condition number of method 4 appears to grow particularly rapidly if too large a value of β is used.

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