HYBRID SPLINE FRAMES

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ABSTRACT. Using their unitary extension principle, Ron and Shen have constructed a normalized tight frame for $L^2(\mathbb{R})$ consisting of spline functions with uniform knots. This paper constructs a normalized tight frame for $L^2((0,\infty))$ comprising spline functions with knots on a hybrid of uniform and geometric mesh. The construction is motivated by applications in adaptive approximation using spline functions on a hybrid mesh that admits a natural dyadic multiresolution approximation of $L^2((0,\infty))$ based on dilation and translation.

1. INTRODUCTION

Cardinal splines or polynomial splines on $\mathbb{R}$ with integer knots were introduced and studied by Schoenberg in a series of papers (see for instance ([16, 17]). They have been used by various authors (for instance [4, 5, 8, 9, 15]) to construct wavelet bases and frames for $L^2(\mathbb{R})$. In particular Chui and Wang [8] constructed the semi-orthogonal spline wavelets and Ron and Shen [15] constructed normalized tight frames of spline functions for $L^2(\mathbb{R})$. Polynomial splines on geometric mesh, q^i, $i \in \mathbb{Z}$, or *geometric splines* were first studied by Micchelli [14]. They were independently rediscovered by Mallat in the study of image foveation [13]. In this paper we consider spline functions on $(0,\infty)$ with knots on a hybrid of geometric and uniform knots as follows. Take an integer $n \geq 2$. Let $\widetilde{V}_0$ denote the space of all spline functions of order n in $L^2((0,\infty))$ with knots x_i, $i \in \mathbb{Z}$, where

$$x_i = \begin{cases} 2^i, & i \leq -1, \\ i+1, & i \geq 0. \end{cases} \tag{1.1}$$

The mesh x_i, $i \in \mathbb{Z}$, is a hybrid of uniform and geometric knots, and the corresponding piecewise polynomial functions will be referred to as *hybrid uniform-geometric splines* or simply *hybrid splines*. For $j \in \mathbb{Z}$, let $\widetilde{V}_j := \{f(2^j\cdot) : f \in \widetilde{V}_0\}$. For each j, $\widetilde{V}_j$ is the space of polynomial splines of order n on $(0,\infty)$ with knots at $2^{-j}x_i$, $i \in \mathbb{Z}$, where x_i are as in (1.1). We shall construct a normalized tight frame for $L^2((0,\infty))$ from functions in $\widetilde{V}_j$, $j \in \mathbb{Z}$.

This construction is motivated by applications in adaptive approximation, for instance, image foveation ([1, 2, 13]) and feature preserving processes in smoothing and image compression ([10, 11]). Multiresolution approximation of $L^2((0,\infty))$ by hybrid splines and hybrid wavelets associated with other scaling functions has been studied in [10], where the focus is on the construction of filters and Riesz bases, while applications in image foveation and feature preserving smoothing are given in [11]. It turns out that the analysis and construction of

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normalized tight frames of hybrid splines is much simpler and neater than that of the Riesz bases and may have important ramifications. Interestingly the hybrid spline tight frame (Theorem 3.1) comprises the dyadic dilations of a finite set of spline functions with mixed uniform and geometric knots and a single geometric B-spline with knots 2^ℓ, $\ell = 0, 1, ..., n$, together with the dyadic dilations and shifts by positive integers of the uniform spline frame generators of Ron and Shen. In [6, 7] Chui, He and Stöckler developed a general theory on nonstationary tight frames using the kernels of projection operators on multiresolution subspaces. However the construction of nonstationary tight frames for particular cases remains an interesting and challenging problem as shown in the construction of the nonstationary spline frames in [7]. For the case of hybrid splines we use a direct method based on matrix extension in the time domain. The idea is to integrate uniform splines on a half infinite interval with geometric splines near the origin. In Section 2 we construct a normalized tight frame of uniform splines for $L^2([0,\infty))$ from the non-negative integer shifts of the uniform spline frame generators and the uniform B-spline with knots at $0, 1, \ldots, n$. They are then used in Section 3 for the construction of tight frames of hybrid splines for $L^2((0,\infty))$.

2. Spline Frames on the Half Line

Take an integer $n \geq 2$. Let $\mathcal{S}_n$ denote the space of all spline functions of order n, i.e. C^{n-2} piecewise polynomials of degree $n-1$, with knots in $\mathbb{Z}$. Let

$$V_0 := \{f \in \mathcal{S}_n : \operatorname{supp}(f) \subset [0,\infty)\} \cap L^2(\mathbb{R})$$

and

$$V_j := \{f(2^j\cdot\,) : f \in V_0\}, \quad j \in \mathbb{Z}.$$

We shall construct a normalized tight frame for $L^2([0,\infty))$ from functions in V_j, $j \in \mathbb{Z}$, where for $f \in V_j$ we confuse f and $f|_{[0,\infty)}$.

Let $N \in \mathcal{S}_n$ denote the B-spline with support $[0, n]$, normalized by $\int_0^n N = 1$, so that $\sum_{j=-\infty}^{\infty} N(\,\cdot - j) = 1$. Note that

$$N(x) = \sum_{i=0}^{n} a_{0i}^n N(2x - i), \quad x \in \mathbb{R}, \tag{2.1}$$

where

$$P_0^n(z) := 2^{-n+1}(1+z)^n =: \sum_{i=0}^{n} a_{0i}^n z^i. \tag{2.2}$$

For $k = 1, 2, ..., n$, let

$$P_k^n(z) := 2^{-n+1}(1-z)^k(1+z)^{n-k} =: \sum_{i=0}^{n} a_{ki}^n z^i,$$

and

$$\psi_k(x) := \sqrt{\binom{n}{k}} \sum_{i=0}^{n} a_{ki}^n N(2x - i), \quad x \in \mathbb{R}. \tag{2.3}$$

By applying the 'unitary extension principle', introduced by Ron and Shen in [15], it can easily be seen that the functions

$$2^{j/2}\psi_k(2^j\cdot\ -i),\ i,j\in\mathbb{Z},\ k=1,2,...,n, \tag{2.4}$$

form a normalized tight frame for $L^2(\mathbb{R})$. The essential idea of this construction was given in [15]. We shall see that the functions ψ_k, $k=1,2,...,n$, in (2.3) together with the uniform B-spline will lead to a normalized tight frame for $L^2([0,\infty))$.

Theorem 2.1. *Let $S\subset L^2([0,\infty))$ denote the set*

$$\psi_k(\ \cdot-i),\ i=0,1,...,\ k=1,2,...,n,\ c_\ell^{1/2}N(\ \cdot-\ell),\ \ell=0,1,...,n-2,$$

where

$$c_\ell:=2^{-n+1}\sum_{j=\ell+1}^{n-1}\binom{n-1}{j},\quad \ell=0,1,\dots,n-2. \tag{2.5}$$

Then $\{2^{j/2}f(2^j\cdot\ ):f\in S,\ j\in\mathbb{Z}\}$ is a normalized tight frame for $L^2([0,\infty))$.

We shall need the following lemma to prove the theorem.

Lemma 2.2. *For $n\geq 1$ and $i,j\in\{0,1,...,n\}$,*

$$\sum_{k=0}^{n}\binom{n}{k}a_{ki}^n a_{kj}^n=\begin{cases}0, & i\neq j,\\ 2^{-n+2}\binom{n}{i}, & i=j.\end{cases}$$

Proof. We prove by induction on n. The identity is easily seen to be true for $n=1$. Assume it is true for some $n\geq 1$. Now for $k=0,1,...,n$,

$$P_k^{n+1}(z)=2^{-1}(1+z)P_k^n(z),\qquad P_{k+1}^{n+1}(z)=2^{-1}(1-z)P_k^n(z),$$

and hence for $i=0,1,...,n+1$,

$$a_{ki}^{n+1}=\frac{1}{2}(a_{k,i-1}^n+a_{ki}^n),\qquad a_{k+1,i}^{n+1}=\frac{1}{2}(a_{ki}^n-a_{k,i-1}^n),$$

where we have taken $a_{k,-1}^n=a_{k,n+1}^n=0$. Then for $i,j\in\{0,1,...,n+1\}$,

$$\begin{aligned}\sum_{k=0}^{n+1}\binom{n+1}{k}a_{ki}^{n+1}a_{kj}^{n+1} &= \sum_{k=0}^{n+1}\left(\binom{n}{k-1}+\binom{n}{k}\right)a_{ki}^{n+1}a_{kj}^{n+1}\\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{4}(a_{ki}^n-a_{k,i-1}^n)(a_{kj}^n-a_{k,j-1}^n)\\ &\qquad+\sum_{k=0}^{n}\binom{n}{k}\frac{1}{4}(a_{k,i-1}^n+a_{ki}^n)(a_{k,j-1}^n+a_{kj}^n)\\ &= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}(a_{k,i-1}^n a_{k,j-1}^n+a_{ki}^n a_{kj}^n).\end{aligned}$$

So our induction hypothesis gives for $i \neq j$,

$$\sum_{k=0}^{n+1} \binom{n+1}{j} a_{ki}^{n+1} a_{kj}^{n+1} = 0,$$

and for $i = j$,

$$\begin{aligned}\sum_{k=0}^{n+1} \binom{n+1}{k} (a_{ki}^{n+1})^2 &= \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} (a_{k,i-1}^{n})^2 + \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} (a_{ki}^{n})^2 \\ &= 2^{-n+1} \binom{n}{i-1} + 2^{-n+1} \binom{n}{i} = 2^{-n+1} \binom{n+1}{i}.\end{aligned}$$

Thus the result is true for $n+1$, and this concludes the inductive proof. ♠

That (2.4) forms a normalized tight frame depends on the following (see [3], p. 319). For simplicity we write $\langle f, g \rangle := \int_{-\infty}^{\infty} fg$ for $f, g \in L^2(\mathbb{R})$, and for $j \in \mathbb{Z}$, $N_j := N(\,\cdot - j)$, $\widetilde{N}_j := \sqrt{2} N(2 \cdot - j)$, $\psi_{kj} := \psi_k(\,\cdot - j)$, $k = 1, 2, ..., n$. Then for $f \in L^2(\mathbb{R})$,

$$\sum_{j=-\infty}^{\infty} \langle f, \widetilde{N}_j \rangle^2 = \sum_{j=-\infty}^{\infty} \langle f, N_j \rangle^2 + \sum_{j=-\infty}^{\infty} \sum_{k=1}^{n} \langle f, \psi_{kj} \rangle^2. \tag{2.6}$$

(In fact in [3] this is stated for f in a dense subset of $L^2(\mathbb{R})$ but by passing to the limit it easily follows that it holds for all $L^2(\mathbb{R})$.)

In a similar manner Theorem 2.1 is based upon the following lemma, where $\langle f, g \rangle$ now denotes $\int_0^{\infty} fg$ for $f, g \in L^2([0, \infty))$.

Lemma 2.3. *For $f \in L^2([0, \infty))$,*

$$\sum_{j=0}^{\infty} \langle f, \widetilde{N}_j \rangle^2 = \sum_{j=0}^{\infty} \langle f, N_j \rangle^2 + \sum_{j=0}^{\infty} \sum_{k=1}^{n} \langle f, \psi_{kj} \rangle^2 + \sum_{j=0}^{n-2} c_j \langle f, \widetilde{N}_j \rangle^2. \tag{2.7}$$

Proof. Take $f \in L^2([0, \infty))$. By (2.6),

$$\sum_{j=1-n}^{\infty} \langle f, \widetilde{N}_j \rangle^2 = \sum_{j=1-n}^{\infty} \langle f, N_j \rangle^2 + \sum_{j=1-n}^{\infty} \sum_{k=1}^{n} \langle f, \psi_{kj} \rangle^2.$$

Thus (2.7) is equivalent to

$$\sum_{j=1-n}^{-1} \langle f, N_j \rangle^2 + \sum_{j=1-n}^{-1} \sum_{k=1}^{n} \langle f, \psi_{kj} \rangle^2 - \sum_{j=1-n}^{-1} \langle f, \widetilde{N}_j \rangle^2 = \sum_{j=0}^{n-2} c_j \langle f, \widetilde{N}_j \rangle^2,$$

which is

$$\frac{1}{2} \sum_{j=1-n}^{-1} \sum_{k=0}^{n} \binom{n}{k} \langle f, \sum_{i=0}^{n} a_{ki}^{n} \widetilde{N}_{i+2j} \rangle^2 - \sum_{j=1-n}^{-1} \langle f, \widetilde{N}_j \rangle^2 = \sum_{j=0}^{n-2} c_j \langle f, \widetilde{N}_j \rangle^2, \tag{2.8}$$

by (2.1) and (2.3). Letting $\alpha_j := \langle f, \widetilde{N}_j \rangle$, $j \in \mathbb{Z}$, (2.8) becomes

$$\frac{1}{2}\sum_{j=1-n}^{-1}\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{i=0}^{n} a_{ki}^n \alpha_{i+2j}\right)^2 - \sum_{j=1-n}^{-1}\alpha_j^2 = \sum_{j=0}^{n-2} c_j \alpha_j^2. \tag{2.9}$$

Now applying Lemma 2.2,

$$\begin{aligned}\frac{1}{2}\sum_{j=1-n}^{-1}\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{i=0}^{n} a_{ki}^n \alpha_{i+2j}\right)^2 &= \frac{1}{2}\sum_{j=1-n}^{-1}\sum_{i=0}^{n}\sum_{\ell=0}^{n}\alpha_{i+2j}\alpha_{\ell+2j}\sum_{k=0}^{n}\binom{n}{k} a_{ki}^n a_{k\ell}^n \\ &= 2^{-n+1}\sum_{j=1-n}^{-1}\sum_{i=0}^{n}\binom{n}{i}\alpha_{i+2j}^2. \qquad (2.10)\end{aligned}$$

Performing further calculations and recalling $\alpha_j = 0$, $j = 2-2n, \ldots, -n$, we have

$$\begin{aligned}2^{-n+1}\sum_{j=1-n}^{-1}\sum_{i=0}^{n}\binom{n}{i}\alpha_{i+2j}^2 &= 2^{-n+1}\sum_{j=1-n}^{-1}\sum_{i=0}^{n}\left\{\binom{n-1}{i-1}+\binom{n-1}{i}\right\}\alpha_{i+2j}^2 \\ &= 2^{-n+1}\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\sum_{j=\ell+2-2n}^{\ell-1}\alpha_j^2 \\ &= 2^{-n+1}\sum_{j=1-n}^{n-2}\alpha_j^2\sum_{\ell=j+1}^{n-1}\binom{n-1}{\ell} \\ &= \sum_{j=1-n}^{-1}\alpha_j^2 + \sum_{j=0}^{n-2} c_j\alpha_j^2. \qquad (2.11)\end{aligned}$$

Hence by (2.10) and (2.11), we obtain (2.9) which gives (2.7) and completes the proof. ♠

Proof of Theorem 2.1. Take $f \in L^2([0,\infty))$. By scaling (2.7) we see that for any $j \in \mathbb{Z}$,

$$\begin{aligned}\sum_{i=0}^{\infty}\langle f, 2^{(j+1)/2}N(2^{j+1}\cdot -i)\rangle^2 &= \sum_{i=0}^{\infty}\langle f, 2^{j/2}N(2^j\cdot -i)\rangle^2 \\ &+ \sum_{i=0}^{\infty}\sum_{k=1}^{n}\langle f, 2^{j/2}\psi_k(2^j\cdot -i)\rangle^2 + \sum_{i=0}^{n-2} c_i\langle f, 2^{(j+1)/2}N(2^{j+1}\cdot -i)\rangle^2.\end{aligned}$$

So for any integers $r < s$,

$$\begin{aligned}\sum_{i=0}^{\infty}\langle f, 2^{s/2}N(2^s\cdot -i)\rangle^2 &= \sum_{i=0}^{\infty}\langle f, 2^{r/2}N(2^r\cdot -i)\rangle^2 \\ &+ \sum_{j=r}^{s-1}\sum_{i=0}^{\infty}\sum_{k=1}^{n}\langle f, 2^{j/2}\psi_k(2^j\cdot -i)\rangle^2 + \sum_{j=r+1}^{s}\sum_{i=0}^{n-2} c_i\langle f, 2^{j/2}N(2^j\cdot -i)\rangle^2. \qquad (2.12)\end{aligned}$$

It is well known that

$$\lim_{s\to\infty}\sum_{i=-\infty}^{\infty}\langle f, 2^{s/2}N(2^s\cdot -i)\rangle\, 2^{s/2}N(2^s\cdot -i) = f$$

in $L^2([0,\infty))$. (For more general results, see [12].) It follows immediately that

$$\lim_{s\to\infty}\sum_{i=-\infty}^{\infty}\langle f, 2^{s/2}N(2^s\cdot -i)\rangle^2 = \|f\|^2. \tag{2.13}$$

Taking any $\epsilon > 0$ and applying (2.13) to $f|_{[0,\epsilon)}$, we see that

$$\overline{\lim_{s\to\infty}}\sum_{i=1-n}^{-1}\langle f, 2^{s/2}N(2^s\cdot -i)\rangle^2 \le \int_0^\epsilon f^2,$$

and letting $\epsilon \to 0$ shows that

$$\lim_{s\to\infty}\sum_{i=1-n}^{-1}\langle f, 2^{s/2}N(2^s\cdot -i)\rangle^2 = 0. \tag{2.14}$$

Since $\langle f, N(2^s\cdot -i)\rangle = 0$ for $i \le -n$, (2.13) and (2.14) give

$$\lim_{s\to\infty}\sum_{i=0}^{\infty}\langle f, 2^{s/2}N(2^s\cdot -i)\rangle^2 = \|f\|^2. \tag{2.15}$$

It is shown in ([3], p. 320) that

$$\lim_{r\to-\infty}\sum_{i=-\infty}^{\infty}\langle f, 2^{r/2}N(2^r\cdot -i)\rangle^2 = 0,$$

and hence

$$\lim_{r\to-\infty}\sum_{i=0}^{\infty}\langle f, 2^{r/2}N(2^r\cdot -i)\rangle^2 = 0. \tag{2.16}$$

From (2.12), (2.15) and (2.16) we then have

$$\|f\|^2 = \sum_{j=-\infty}^{\infty}\sum_{i=0}^{\infty}\sum_{k=1}^{n}\langle f, 2^{j/2}\psi_k(2^j\cdot -i)\rangle^2 + \sum_{j=-\infty}^{\infty}\sum_{i=0}^{n-2}c_i\langle f, 2^{j/2}N(2^j\cdot -i)\rangle^2,$$

which proves the result. ♠

3. Spline Frames on a Hybrid Mesh

Again take an integer $n \ge 2$, the knot sequence x_i, $i \in \mathbb{Z}$, as in (1.1) and as before let $\widetilde{V}_0$ denote the space of all spline functions of order n in $L^2((0,\infty))$ with knots x_i, $i \in \mathbb{Z}$, and $\widetilde{V}_j := \{f(2^j\cdot) : f \in \widetilde{V}_0\}$ for $j \in \mathbb{Z}$. We now construct a normalized tight frame for $L^2((0,\infty))$ from functions in $\widetilde{V}_j$, $j \in \mathbb{Z}$.

For $j \in \mathbb{Z}$, let N_j denote the B-spline with knots $x_j, ..., x_{j+n}$, normalized so that

$$\sum_{j=-\infty}^{\infty} N_j(x) = 1, \quad x > 0. \tag{3.1}$$

We write

$$I_j := \int_0^\infty N_j = \frac{x_{j+n} - x_j}{n}, \qquad j \in \mathbb{Z}. \tag{3.2}$$

Since $\widetilde{V}_0 \subset \widetilde{V}_1$, the functions (N_j) satisfy a refinement equation of form

$$N_j = \sum_{k=-\infty}^{\infty} b_{jk}^n N_k(2\cdot), \quad j \in \mathbb{Z}. \tag{3.3}$$

For $j \geq 0$, $N_j = N(\cdot - j - 1)$, $I_j = 1$, and from (2.1) and (2.2),

$$N_j = 2^{-n+1} \sum_{k=0}^{n} \binom{n}{k} N_{k+1+2j}(2\cdot).$$

Thus

$$b_{jk}^n = 2^{-n+1} \binom{n}{k-2j-1}, \quad j \geq 0, \ k \in \mathbb{Z}. \tag{3.4}$$

Note that for $j \leq 1-n$, N_j has knots $2^j, ..., 2^{j+n}$, and so

$$N_j = M(2^{-j}\cdot), \quad j \leq 1-n, \tag{3.5}$$

where M is the 'geometric' B-spline with knots 2^ℓ, $\ell = 0, 1, ..., n$, with $\int_0^\infty M = \frac{2^n-1}{n}$. Thus

$$N_j = N_{j+1}(2\cdot), \quad j \leq -n, \tag{3.6}$$

and

$$b_{jk}^n = \delta_{j,k-1}, \quad j \leq -n.$$

It can be shown that $b_{jk}^n \geq 0$, $j, k \in \mathbb{Z}$, and $b_{jk}^n > 0$ if and only if

$$\left.\begin{array}{ll} k = j+1, & j \leq -n, \\ j+1 \leq k \leq 2j+n+1, & -n < j \leq 0, \\ 2j+1 \leq k \leq 2j+n+1, & j > 0. \end{array}\right\} \tag{3.7}$$

From (3.1) and (3.3),

$$1 = \sum_{j=-\infty}^{\infty} N_j = \sum_{k=-\infty}^{\infty} N_k(2\cdot) \sum_{j=-\infty}^{\infty} b_{jk}^n,$$

and so

$$\sum_{j=-\infty}^{\infty} b_{jk}^n = 1, \quad k \in \mathbb{Z}. \tag{3.8}$$

In order to state our main result, we define for integers k, ℓ,

$$c_{k\ell} := \frac{1}{2} I_k I_\ell \sum_{j=1-n}^{-1} \frac{1}{I_j} b_{jk}^n b_{j\ell}^n. \tag{3.9}$$

Clearly $c_{k\ell} = c_{\ell k} \geq 0$ for any integers $k \neq \ell$. We can see from (3.7) that for $k < \ell$, $c_{k\ell} > 0$ if and only if

$$2-n \leq k < \ell \leq \min\{n-1+2k, n-1\}.$$

Note that (3.3) gives

$$I_j = \frac{1}{2} \sum_{k=-\infty}^{\infty} b_{jk}^n I_k. \tag{3.10}$$

Theorem 3.1. *Let $T \subset L^2((0,\infty))$ denote the set of functions $\left(\frac{n}{2^n-1}\right)^{\frac{1}{2}} M,\ \psi_k(\cdot - i),\ i = 1, 2, ...,\ k = 1, 2, ..., n,$ and*

$$c_{\ell m}^{1/2}\left(\frac{N_\ell}{I_\ell} - \frac{N_m}{I_m}\right), \quad 2-n \le \ell < m \le \min\{n-1+2\ell, n-1\}. \tag{3.11}$$

Then $\{2^{j/2} f(2^j\cdot) : f \in T, j \in \mathbb{Z}\}$ is a normalized tight frame for $L^2((0,\infty))$.

We shall see in due course that the $(n-1)^2$ functions in (3.11) can in fact be replaced by $2n-3$ functions, but not of such simple form. The form of (3.11) comes from the following.

Lemma 3.2. *Let $A = (A_{ij})_{i,j=1}^m$ be a symmetric matrix satisfying $\sum_{j=1}^m A_{ij} = 0,\ i = 1, 2, ..., m$. Then for any $\beta_1, ..., \beta_m$,*

$$\sum_{i=1}^m \sum_{j=1}^m A_{ij}\beta_i\beta_j = -\sum_{1\le i<j\le m} A_{ij}(\beta_i - \beta_j)^2.$$

Proof. For any $\beta_1, ..., \beta_m$,

$$\begin{aligned}
\sum_{1\le i<j\le m} A_{ij}(\beta_i-\beta_j)^2 &= \sum_{i=1}^m \beta_i^2 \sum_{\substack{j=1\\ j\ne i}}^m A_{ij} - 2\sum_{1\le i<j\le m} A_{ij}\beta_i\beta_j \\
&= -\sum_{i=1}^m \beta_i^2 A_{ii} - 2\sum_{1\le i<j\le m} A_{ij}\beta_i\beta_j \\
&= -\sum_{i=1}^m\sum_{j=1}^m A_{ij}\beta_i\beta_j.
\end{aligned}$$

♠

Now, in a similar manner to Lemma 2.3, the proof of Theorem 3.1 is based upon the following.

Lemma 3.3. *For $f \in L^2((0,\infty))$,*

$$\begin{aligned}
2\sum_{j=-\infty}^{\infty} I_j^{-1}\langle f, N_j(2\cdot)\rangle^2 &= \sum_{j=-\infty}^{\infty} I_j^{-1}\langle f, N_j\rangle^2 + \sum_{j=1}^{\infty}\sum_{k=1}^{n}\langle f, \psi_{kj}\rangle^2 \\
&\quad +2\sum_{2-n\le\ell<m\le n-1} c_{\ell m}\left\langle f, \frac{N_\ell(2\cdot)}{I_\ell} - \frac{N_m(2\cdot)}{I_m}\right\rangle^2.
\end{aligned} \tag{3.12}$$

Proof. By Lemma 2.3 applied on $[1,\infty)$,

$$2\sum_{j=1}^{\infty}\langle f, N_j(2\cdot)\rangle^2 = \sum_{j=0}^{\infty}\langle f, N_j\rangle^2 + \sum_{j=1}^{\infty}\sum_{k=1}^{n}\langle f,\psi_{kj}\rangle^2 + 2\sum_{j=1}^{n-1} c_{j-1}\langle f, N_j(2\cdot)\rangle^2. \tag{3.13}$$

Now let

$$Y := 2\sum_{j=-\infty}^{\infty} I_j^{-1}\langle f, N_j(2\cdot)\rangle^2 - \sum_{j=-\infty}^{\infty} I_j^{-1}\langle f, N_j\rangle^2 - \sum_{j=1}^{\infty}\sum_{k=1}^{n}\langle f,\psi_{kj}\rangle^2. \tag{3.14}$$

By (3.6), $I_j^{-1}N_j = 2I_{j+1}^{-1}N_{j+1}(2\cdot)$, $j \le -n$, and so

$$Y = 2\sum_{j=2-n}^{\infty} I_j^{-1}\langle f, N_j(2\cdot)\rangle^2 - \sum_{j=1-n}^{\infty} I_j^{-1}\langle f, N_j\rangle^2 - \sum_{j=1}^{\infty}\sum_{k=1}^{n}\langle f, \psi_{kj}\rangle^2. \tag{3.15}$$

Adding (3.13) and (3.15) and recalling, $I_j = 1$, $j \ge 0$, gives

$$Y = 2\sum_{j=2-n}^{0} I_j^{-1}\langle f, N_j(2\cdot)\rangle^2 + 2\sum_{j=1}^{n-1} c_{j-1}\langle f, N_j(2\cdot)\rangle^2 - \sum_{j=1-n}^{-1} I_j^{-1}\langle f, N_j\rangle^2.$$

Putting $\beta_j := \sqrt{2}I_j^{-1}\langle f, N_j(2\cdot)\rangle$, $j \in \mathbb{Z}$, and applying (3.3), this becomes

$$Y = \sum_{j=2-n}^{0} I_j\beta_j^2 + \sum_{j=1}^{n-1} c_{j-1}\beta_j^2 - \sum_{j=1-n}^{-1}\frac{1}{2I_j}\left\{\sum_{k=-\infty}^{\infty} b_{jk}^n I_k\beta_k\right\}^2,$$

which simplifies to

$$Y = \sum_{j=2-n}^{0} I_j\beta_j^2 + \sum_{j=1}^{n-1} c_{j-1}\beta_j^2 - \sum_{k=2-n}^{n-1}\sum_{\ell=2-n}^{n-1} c_{k\ell}\beta_k\beta_\ell,$$

using (3.7) and (3.9). We may write this as

$$Y = \sum_{k=2-n}^{n-1}\sum_{\ell=2-n}^{n-1} A_{k\ell}\beta_k\beta_\ell, \tag{3.16}$$

where $A = (A_{k\ell})_{k,\ell=2-n}^{n-1}$ is a symmetric matrix given by

$$\left.\begin{aligned} A_{kk} &= I_k - c_{kk}, && k = 2-n, ..., 0,\\ A_{kk} &= c_{k-1} - c_{kk}, && k = 1, ..., n-1,\\ A_{k\ell} &= -c_{k\ell}, && 2-n \le k < \ell \le n-1. \end{aligned}\right\} \tag{3.17}$$

We shall show that for $k = 2-n, ..., n-1$,

$$\sum_{\ell=2-n}^{n-1} A_{k\ell} = 0. \tag{3.18}$$

First take $2-n \le k \le 0$. Then using (3.17) and (3.9),

$$\begin{aligned} \sum_{\ell=2-n}^{n-1} A_{k\ell} &= I_k - \frac{1}{2}I_k\sum_{j=1-n}^{-1}\frac{1}{I_j}b_{jk}^n\sum_{\ell=2-n}^{n-1} b_{j\ell}^n I_\ell \\ &= I_k - I_k\sum_{j=1-n}^{-1} b_{jk}^n, \end{aligned} \tag{3.19}$$

by (3.10), on noting from (3.7) that $b_{j\ell}^n = 0$ except for $2-n \le j+1 \le \ell \le 2j+n+1 \le n-1$. Also from (3.7) we see that $b_{jk}^n = 0$ except for $1-n \le \frac{k-n-1}{2} \le j \le k-1 \le -1$. Thus

$$\sum_{j=1-n}^{-1} b_{jk}^n = \sum_{j=-\infty}^{\infty} b_{jk}^n = 1,$$

by (3.8), and so (3.19) gives (3.18) .

Now take $1 \le k \le n-1$. As in the derivation of (3.19), and noting that $I_k = 1$,

$$\sum_{\ell=2-n}^{n-1} A_{k\ell} = c_{k-1} - \sum_{j=1-n}^{-1} b_{jk}^n. \tag{3.20}$$

From (3.7) we see that $b_{jk}^n \neq 0$ implies $j \ge \frac{k-n-1}{2}$, and since $1-n \le \frac{k-n-1}{2}$,

$$\sum_{j=1-n}^{-1} b_{jk}^n = \sum_{j=-\infty}^{-1} b_{jk}^n = 1 - \sum_{j=0}^{\infty} b_{jk}^n,$$

by (3.8). Thus by (3.20), (2.5) and (3.4),

$$\begin{aligned}
\sum_{\ell=2-n}^{n-1} A_{k\ell} &= 2^{-n+1}\sum_{j=k}^{n-1}\binom{n-1}{j} + 2^{-n+1}\sum_{j=0}^{\infty}\binom{n}{k-2j-1} - 1 \\
&= 2^{-n+1}\sum_{j=0}^{\infty}\binom{n-1}{k+j} + 2^{-n+1}\sum_{j=0}^{\infty}\left\{\binom{n-1}{k-2j-2} + \binom{n-1}{k-2j-1}\right\} - 1 \\
&= 2^{-n+1}\sum_{j=-\infty}^{\infty}\binom{n-1}{k+j} - 1 \\
&= 2^{-n+1}2^{n-1} - 1 = 0.
\end{aligned}$$

Hence (3.18) holds for $k = 2-n, ..., n-1$.

By Lemma 3.2, (3.16) and (3.17),

$$\begin{aligned}
Y &= -\sum_{2-n\le i<j\le n-1} A_{ij}(\beta_i - \beta_j)^2 \\
&= 2\sum_{2-n\le i<j\le n-1} c_{ij}\left\langle f, \frac{N_i(2\cdot)}{I_i} - \frac{N_j(2\cdot)}{I_j}\right\rangle^2.
\end{aligned}$$

From (3.14), this gives (3.12). ♠

Proof of Theorem 3.1. Take $f \in L^2((0,\infty))$. By scaling (3.12) we see that for any $j \in \mathbb{Z}$,

$$\begin{aligned}
\sum_{i=-\infty}^{\infty} 2^{j+1} I_i^{-1}\langle f, N_i(2^{j+1}\cdot)\rangle^2 &= \sum_{i=-\infty}^{\infty} 2^j I_i^{-1}\langle f, N_i(2^j\cdot)\rangle^2 \\
&+ \sum_{i=1}^{\infty}\sum_{k=1}^{n} 2^j\langle f, \psi_{ki}(2^j\cdot)\rangle^2 + \sum_{2-n\le\ell<m\le n-1} 2^{j+1} c_{\ell m}\left\langle f, \frac{N_\ell(2^{j+1}\cdot)}{I_\ell} - \frac{N_m(2^{j+1}\cdot)}{I_m}\right\rangle^2.
\end{aligned}$$

So for any integers $r < s$,

$$\begin{aligned}
\sum_{i=-\infty}^{\infty} I_i^{-1}\langle f, 2^{s/2}N_i(2^s\cdot)\rangle^2 &= \sum_{i=-\infty}^{\infty} I_i^{-1}\langle f, 2^{r/2}N_i(2^r\cdot)\rangle^2 + \sum_{j=r}^{s-1}\sum_{i=1}^{\infty}\sum_{k=1}^{n}\langle f, 2^{j/2}\psi_{ki}(2^j\cdot)\rangle^2 \\
&+ \sum_{j=r+1}^{s}\sum_{2-n\le\ell<m\le n-1} c_{\ell m}\left\langle f, 2^{j/2}\left(\frac{N_\ell(2^j\cdot)}{I_\ell} - \frac{N_m(2^j\cdot)}{I_m}\right)\right\rangle^2.
\end{aligned} \tag{3.21}$$

Now for any integers i, s, recalling from (3.1) that $0 \le N_i(x) \le 1$, $x > 0$, we have

$$\begin{aligned} I_i^{-1}\langle f, 2^{\frac{s}{2}} N_i(2^s\cdot)\rangle^2 &\le 2^s I_i^{-1} \int_{2^{-s}x_i}^{2^{-s}x_{i+n}} f^2 \int_{2^{-s}x_i}^{2^{-s}x_{i+n}} N_i(2^s\cdot)^2 \\ &\le I_i^{-1}(x_{i+n} - x_i) \int_{2^{-s}x_i}^{2^{-s}x_{i+n}} f^2 = n \int_{2^{-s}x_i}^{2^{-s}x_{i+n}} f^2, \end{aligned}$$

by (3.2). Thus

$$\begin{aligned} \sum_{i=-\infty}^{-1} I_i^{-1}\langle f, 2^{s/2} N_i(2^s\cdot)\rangle^2 &\le n \sum_{i=-\infty}^{-1} \int_{2^{-s}x_i}^{2^{-s}x_{i+n}} f^2 \\ &\le n^2 \int_0^{2^{-s}x_{n-1}} f^2 \to 0 \text{ as } s \to \infty. \end{aligned} \tag{3.22}$$

Also

$$\begin{aligned} \sum_{i=0}^{\infty} I_i^{-1}\langle f, 2^{s/2} N_i(2^s\cdot)\rangle^2 &= \sum_{i=1}^{\infty} \langle f, 2^{s/2} N(2^s \cdot - i)\rangle^2 \\ &\to \|f\|^2 \text{ as } s \to \infty, \end{aligned} \tag{3.23}$$

as in the proof of Theorem 2.1. As above,

$$\begin{aligned} \sum_{i=2-n}^{\infty} I_i^{-1}\langle f, 2^{r/2} N_i(2^r\cdot)\rangle^2 &\le n \sum_{i=2-n}^{\infty} \int_{2^{-r}x_i}^{2^{-r}x_{i+n}} f^2 \\ &\le n^2 \int_{2^{-r}x_{2-n}}^{\infty} f^2 \to 0 \text{ as } r \to -\infty. \end{aligned} \tag{3.24}$$

Also by (3.5),

$$\begin{aligned} \sum_{i=-\infty}^{1-n} I_i^{-1}\langle f, 2^{r/2} N_i(2^r\cdot)\rangle^2 &= \sum_{i=-\infty}^{1-n} \frac{2^{-i} n}{2^n - 1} \langle f, 2^{r/2} M(2^{r-i}\cdot)\rangle^2 \\ &= \sum_{j=-\infty}^{1-n-r} \frac{n}{2^n - 1} \langle f, 2^{-j/2} M(2^{-j}\cdot)\rangle^2. \end{aligned} \tag{3.25}$$

From (3.21) - (3.25),

$$\begin{aligned} \|f\|^2 = & \sum_{j=-\infty}^{\infty} \frac{n}{2^n - 1} \langle f, 2^{-j/2} M(2^{-j}\cdot)\rangle^2 + \sum_{j=-\infty}^{\infty} \sum_{i=1}^{\infty} \sum_{k=1}^{r} \langle f, 2^{j/2} \psi_{ki}(2^j\cdot)\rangle^2 \\ & + \sum_{j=-\infty}^{\infty} \sum_{2-r \le \ell < m \le n-1} c_{\ell m} \left\langle f, 2^{j/2} \left(\frac{N_\ell(2^j\cdot)}{I_\ell} - \frac{N_m(2^j\cdot)}{I_m} \right) \right\rangle^2, \end{aligned}$$

which proves the result. ♠

Now recall the matrix A, as in (3.17), and consider the Cholesky type factorization $A = LL^T$. Here $L = (L_{k\ell})_{k,\ell=2-n}^{n-1}$, where $L_{k\ell} = 0$ unless

$$2 - n \le \ell \le k \le \min\{n - 1 + 2\ell, n - 1\},$$

and

$$\sum_{k=\ell}^{n-1} L_{k\ell} = 0, \quad 2-n \leq \ell \leq n-1.$$

Then $L_{n-1,n-1} = 0$ and from (3.16),

$$\begin{aligned} Y &= \sum_{\ell=2-n}^{n-2} \left(\sum_{k=\ell}^{n-1} L_{k\ell}\beta_k \right)^2 \\ &= 2 \sum_{\ell=2-n}^{n-2} \langle f, \sum_{k=\ell}^{n-1} L_{k\ell} I_k^{-1} N_k(2\cdot)\rangle^2. \end{aligned}$$

So in Theorem 3.1 we may replace the $(n-1)^2$ functions in (3.11) by the $2n-3$ functions

$$\sum_{k=\ell}^{n-1} L_{k\ell}\frac{N_k}{I_k}, \quad \ell = 2-n, ..., n-2.$$

Example Take $n = 2$. From (3.9), $c_{01} = \frac{1}{2I_{-1}} b^2_{-1,0} b^2_{-1,1}$. Since $b^2_{-1,0} = 1$, $b^2_{-1,1} = \frac{1}{2}$, $I_{-1} = \frac{x_1 - x_{-1}}{2} = \frac{3}{4}$, we have $c_{01} = \frac{1}{3}$, and so (3.11) comprises the function $\frac{1}{\sqrt{3}}(N_0 - N_1)$.

Example Take $n = 3$. Here $b^3_{-1,0} = \frac{2}{3}$ and the other values follow from (3.4) or (3.8). From (3.9), the matrix A in (3.17) is given by

$$A = \begin{bmatrix} \frac{5}{21} & -\frac{5}{21} & 0 & 0 \\ -\frac{5}{21} & \frac{67}{105} & -\frac{3}{10} & -\frac{1}{10} \\ 0 & -\frac{3}{10} & \frac{33}{80} & -\frac{9}{80} \\ 0 & -\frac{1}{10} & -\frac{9}{80} & \frac{17}{80} \end{bmatrix}.$$

Thus (3.11) comprises

$$\frac{\sqrt{105}}{105}\left(6N_{-1} - 5N_0\right), \ \frac{\sqrt{30}}{10}\left(N_0 - N_1\right), \ \frac{\sqrt{10}}{10}\left(N_0 - N_2\right), \ \frac{3\sqrt{5}}{20}\left(N_1 - N_2\right).$$

A simple calculation shows $A = LL^T$, where

$$L = \begin{bmatrix} \frac{\sqrt{105}}{21} & 0 & 0 & 0 \\ -\frac{\sqrt{105}}{21} & \frac{\sqrt{10}}{5} & 0 & 0 \\ 0 & -\frac{3\sqrt{10}}{20} & \frac{\sqrt{3}}{4} & 0 \\ 0 & -\frac{\sqrt{10}}{20} & -\frac{\sqrt{3}}{4} & 0 \end{bmatrix}.$$

So we may replace (3.11) by the three functions:

$$\frac{\sqrt{105}}{105}\left(6N_{-1} - 5N_0\right), \ \frac{\sqrt{10}}{20}\left(4N_0 - 3N_1 - N_2\right), \ \frac{\sqrt{3}}{4}\left(N_1 - N_2\right).$$

Example Take $n = 4$. Here $(b^4_{j,k})^{1,5}_{j=-4,k=-4}$ is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{2}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{7} & \frac{3}{5} & \frac{3}{20} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{5} & \frac{29}{40} & \frac{1}{2} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{4} & \frac{1}{2} & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}.$$

From (3.9), the functions in (3.11) are

$$\frac{4\sqrt{165}}{1155}(14N_{-2} - 11N_{-1}), \ \frac{\sqrt{33}}{77}(8N_{-1} - 7N_0), \ \frac{\sqrt{33}}{154}(8N_{-1} - 7N_1),$$
$$\frac{\sqrt{34265}}{385}(N_0 - N_1), \ \frac{2\sqrt{35}}{35}(N_0 - N_2), \ \frac{\sqrt{1015}}{70}(N_1 - N_2),$$
$$\frac{\sqrt{35}}{35}(N_0 - N_3), \ \frac{\sqrt{1015}}{140}(N_1 - N_3), \ \frac{\sqrt{7}}{14}(N_2 - N_3).$$

Also from (3.9), the matrix A in (3.17) is given by

$$A = \begin{bmatrix} \frac{11}{60} & -\frac{11}{60} & 0 & 0 & 0 & 0 \\ -\frac{11}{60} & \frac{173}{330} & -\frac{3}{11} & -\frac{3}{44} & 0 & 0 \\ 0 & -\frac{3}{11} & \frac{249}{385} & -\frac{89}{385} & -\frac{4}{35} & -\frac{1}{35} \\ 0 & -\frac{3}{44} & -\frac{89}{385} & \frac{3439}{6160} & -\frac{29}{140} & -\frac{29}{560} \\ 0 & 0 & -\frac{4}{35} & -\frac{29}{140} & \frac{5}{14} & -\frac{1}{28} \\ 0 & 0 & -\frac{1}{35} & -\frac{29}{560} & -\frac{1}{28} & \frac{13}{112} \end{bmatrix}.$$

A simple calculation shows $A = LL^T$, where

$$L = \begin{bmatrix} \frac{\sqrt{165}}{30} & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{165}}{30} & \frac{\sqrt{165}}{22} & 0 & 0 & 0 & 0 \\ 0 & -\frac{2\sqrt{165}}{55} & \frac{\sqrt{21}}{7} & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{165}}{110} & -\frac{2\sqrt{21}}{21} & \frac{\sqrt{51}}{12} & 0 & 0 \\ 0 & 0 & -\frac{4\sqrt{21}}{105} & -\frac{\sqrt{51}}{15} & \frac{\sqrt{10}}{10} & 0 \\ 0 & 0 & -\frac{\sqrt{21}}{105} & -\frac{\sqrt{51}}{60} & -\frac{\sqrt{10}}{10} & 0 \end{bmatrix}.$$

So we may replace (3.11) by the five functions:

$$\frac{\sqrt{165}}{1155}(14N_{-2} - 11N_{-1}), \ \frac{\sqrt{165}}{110}(40N_{-1} - 28N_0 - 7N_1),$$
$$\frac{\sqrt{21}}{105}(15N_0 - 10N_1 - 4N_2 - N_3), \ \frac{\sqrt{51}}{60}(5N_1 - 4N_2 - N_3), \ \frac{\sqrt{10}}{10}(N_2 - N_3).$$

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