CONSTRUCTING TIGHT FRAMES OF MULTIVARIATE FUNCTIONS

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In honored memory of George Lorentz

Abstract

The paper presents a method of construction of tight frames for $L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$. The construction is based on local orthogonal matrix extension of vectors associated with the transition matrices across consecutive resolution levels. Two explicit constructions are given, one for linear splines on triangular polygonal surfaces with arbitrary topology and the other for quadratic splines associated with Powell-Sabin elements on a six-direction mesh.

Keywords. Tight frames, triangular polygonal surfaces, Powell-Sabin elements.

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1. Introduction

In the past two decades, wavelets and frames have proved to be a useful tool in image and signal processing. Recent developments in geometric modeling and numerical approximations have motivated the construction of wavelets and frames for other multiscale and multiresolution information, such as discrete geometry information generated by subdivision processes or 3-D scanners and discrete functional data generated by numerical solution of partial differential equations. For instance, wavelets based on surface subdivision schemes on polyhedral meshes of arbitrary topology have been studied in [15], [14], [2] for the purpose of geometry compression, while frames from surface subdivision schemes have been constructed recently by Charina and Stöckler ([3], [4]).

In many situations, frames provide more flexibility than wavelets and their constructions from multiresolution analyses (MRA) are also simpler. A catalyst for this development is the Unitary Extension Principle introduced by Ron and Shen [18], which provides a general construction of tight wavelet frames for $L^2(\mathbb{R}^n)$ in the shift-invariant setting, which we shall refer to as the stationary case. Some historical pointers on the development of the theory and construction of frames can be found in [9]. The paper [9] and a parallel and independent study in [6] provide further development of the subject. The Unitary Extension Principle of Ron and Shen is formulated in the frequency domain and allows for the construction of affine frames that are shifts and dilates of a finite set of functions. With the aim of constructing tight frames for square-integrable functions on compact sets and to extend the frame theory beyond the shift-invariant setting, Chui, He and Stöckler [7] have developed a parallel theory, formulated in the time domain, together with a general construction of tight frames for $L^2(I)$, where I is a finite interval in $\mathbb{R}$. The multiresolution subspaces $(V_j)_{j=0}^{\infty}$ of $L^2(I)$ are generated by finite sequences of functions, Φ_j, that satisfy the non-stationary refinement relations, $\Phi_j = \Phi_{j+1}A_j$, $j = 0, 1, \dots$, where A_j is a non-square transition matrix. They provide necessary and sufficient conditions for sequences of functions $\Phi_{j+1}B_j$, $j = 0, 1, \dots$, to form a tight frame, in terms of the existence of symmetric positive semi-definite matrices, Γ_j, that define the kernels, $\Phi_j\Gamma_j\Phi_j^T$, $j = 0, 1, \dots$, for a sequence of approximation operators that converges in norm to f for any $f \in L^2(\mathbb{R})$. The construction of frame elements is then carried out by factoring the matrices $\Gamma_{j+1} - A_j\Gamma_j A_j^T$ to obtain B_j. The method was demonstrated in the construction of non-uniform spline tight frames in [7]. The theory and the general method of construction have been extended in [8] to spaces of square-integrable functions on non-compact subsets of $\mathbb{R}$, and are also partially extended to higher dimensions in [3]. On the other hand, for the space $L^2([0,2\pi)^n)$ of periodic functions, in which the wavelet frames are generally non-stationary, Goh and Teo [12] have developed the corresponding Unitary Extension Principle in the frequency domain and introduced a new method of construction based on local matrix extension to matrices with orthonormal columns. The method employs the singular value decomposition, which simplifies considerably the problem of matrix extension.

While general theories and methods for the construction of wavelets and framelets in the stationary and non-stationary settings have been established, explicit construction of wavelets and framelets in most cases remains a challenging task. The object of this paper is to develop a new method for constructing tight frames for the space $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$, and provide two explicit constructions. The construction, which is largely motivated by that in [12], is based on the local matrix extension of vectors associated with transition matrices of the refinement relations across consecutive resolution levels. It is applicable in both the stationary as well as the non-stationary setting. The matrix extension can be carried out by the Householder transformation in the scalar case and by an adaptation of the method based on singular value decomposition developed in [12] in the scalar as well as the vector case. These are illustrated in Sections 4 and 5. A general theory is developed in Sections 2 and 3. Although the construction of frame elements in Section 2 is developed for $L^2(\Omega)$ on a n-dimensional region $\Omega \subset \mathbb{R}^n$, it is applicable to locally parametrizable piecewise smooth manifolds in $\mathbb{R}^n$, such as a sphere or polyhedral surfaces. In Section 4 we construct tight frames for linear splines on triangular polyhedral surfaces in $\mathbb{R}^3$ of arbitrary topology. Piecewise linear functions defined on polyhedral surfaces are useful in applications in computer graphics (see for instance, [1], [11]). Section 5 deals with the construction of piecewise quadratic tight frames from Powell-Sabin elements on a six-direction mesh [16].

After completing this work we discovered that similar, but different, constructions of tight frames appear in preprints by Charina and Stöckler [3], [4] and Reimers [17]. Explicit constructions of tight frames from MRA generated by surface subdivision schemes are given in [3], [4], while some univariate tight spline frames and bivariate linear spline frames are constructed in [17].

2. Construction of tight frames

Let ϕ_i, $i \in S$, and $\widetilde{\phi}_j$, $j \in T$, be real-valued functions in $L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, satisfying

$$\phi_i = \sum_{j \in T} a_{ij} \widetilde{\phi}_j, \quad i \in S, \tag{2.1}$$

where $a_{ij} \geq 0,\ i \in S,\ j \in T$,

$$\sum_{i \in S} a_{ij} = 1, \quad j \in T, \tag{2.2}$$

and each row of the matrix $(a_{ij})_{i\in S, j\in T}$ has finitely many non-zero entries. We assume that $\{\phi_i : i \in S\}$ and $\{\widetilde{\phi}_j : j \in T\}$ are locally finite, i.e. for any $x \in \Omega$, there is a neighborhood of x in which all but a finite number of them vanish. Note that

$$\sum_{i\in S} \phi_i = \sum_{i\in S}\sum_{j\in T} a_{ij}\widetilde{\phi}_j = \sum_{j\in T}\widetilde{\phi}_j \sum_{i\in S} a_{ij} = \sum_{j\in T}\widetilde{\phi}_j.$$

Suppose $I_i := \int \phi_i > 0,\ i \in S$, and $\widetilde{I}_j := \int \widetilde{\phi}_j > 0,\ j \in T$. Then integrating (2.1) gives

$$I_i = \sum_{j\in T} a_{ij}\widetilde{I}_j, \quad i \in S. \tag{2.3}$$

Let $\Phi_i := I_i^{-1/2}\phi_i,\ i \in S$, and $\widetilde{\Phi}_j := \widetilde{I}_j^{-1/2}\widetilde{\phi}_j,\ j \in T$. Then (2.1) gives

$$\Phi_i = \sum_{j\in T}\left(\widetilde{I}_j/I_i\right)^{1/2} a_{ij}\widetilde{\Phi}_j, \quad i \in S. \tag{2.4}$$

Take $i \in S$. Let $\{j : a_{ij} \neq 0\} = \{j_1, \ldots, j_{r(i)}\}$. For simplicity put $r(i) = r$, and define

$$v_\ell \equiv v_\ell(i) := \left((\widetilde{I}_{j_\ell}/I_i)a_{ij_\ell}\right)^{1/2}, \quad \ell = 1, \ldots, r.$$

Then

$$\sum_{\ell=1}^{r} v_\ell^2 = \sum_{\ell=1}^{r}\left(\widetilde{I}_{j_\ell}/I_i\right) a_{ij_\ell} = 1,$$

by (2.3). We can therefore extend the row vector $(v_1, v_2, \ldots, v_r)$ to an orthogonal matrix $V = (v_{k\ell})_{k,\ell=1}^r$ with $v_{1\ell} = v_\ell,\ \ell = 1, \ldots, r$.

For $k = 2, \ldots, r$, define

$$\psi_k \equiv \psi_k^i := \sum_{\ell=1}^{r} v_{k\ell} a_{ij_\ell}^{1/2}\widetilde{\Phi}_{j_\ell}. \tag{2.5}$$

Proposition 2.1. *For $f \in L^2(\Omega)$,*

$$\sum_{i\in S}\langle f, \Phi_i\rangle^2 + \sum_{i\in S}\sum_{k=2}^{r(i)}\langle f, \psi_k^i\rangle^2 = \sum_{j\in T}\langle f, \widetilde{\Phi}_j\rangle^2.$$

Proof. Take $f \in L^2(\Omega)$ and let $\alpha_j := \langle f, \widetilde{\Phi}_j\rangle$, $j \in T$. Then for a fixed $i \in S$, by (2.4) and (2.5),

$$\begin{aligned}\langle f, \Phi_i\rangle^2 + \sum_{k=2}^{r}\langle f, \psi_k\rangle^2 &= \left(\sum_{\ell=1}^{r}\left(\widetilde{I}_{j_\ell}/I_i\right)^{1/2} a_{ij_\ell}\alpha_{j_\ell}\right)^2 + \sum_{k=2}^{r}\left(\sum_{\ell=1}^{r} v_{k\ell}a_{ij_\ell}^{1/2}\alpha_{j_\ell}\right)^2 \\ &= \sum_{k=1}^{r}\left(\sum_{\ell=1}^{r} a_{ij_\ell}^{1/2} v_{k\ell}\alpha_{j_\ell}\right)^2 \\ &= \left\|VD[\alpha_{j_1},\dots,\alpha_{j_r}]^T\right\|^2,\end{aligned}$$

where $D := \operatorname{diag}(a_{ij_1}^{1/2},\dots,a_{ij_r}^{1/2})$. Thus

$$\begin{aligned}\langle f, \Phi_i\rangle^2 + \sum_{k=2}^{r}\langle f, \psi_k\rangle^2 &= [\alpha_{j_1},\dots,\alpha_{j_r}]D^TV^TVD[\alpha_{j_1},\dots,\alpha_{j_r}]^T \\ &= [\alpha_{j_1},\dots,\alpha_{j_r}]\operatorname{diag}(a_{ij_1},\dots,a_{ij_r})[\alpha_{j_1},\dots,\alpha_{j_r}]^T \\ &= \sum_{k=1}^{r} a_{ij_k}\alpha_{j_k}^2 = \sum_{j\in T} a_{ij}\langle f, \widetilde{\Phi}_j\rangle^2.\end{aligned}$$

Hence it follows that

$$\sum_{i\in S}\langle f, \Phi_i\rangle^2 + \sum_{i\in S}\sum_{k=2}^{r(i)}\langle f, \psi_k^i\rangle^2 = \sum_{i\in S}\sum_{j\in T} a_{ij}\langle f, \widetilde{\Phi}_j\rangle^2 = \sum_{j\in T}\langle f, \widetilde{\Phi}_j\rangle^2,$$

by (2.2). ♠

Next, we consider the vector case. Here, we consider sets of functions rather than a single set. Let ϕ_i, $i \in S$, and $\widetilde{\phi}_j$, $j \in T$, be functions in $L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, as before and we still assume the situation of (2.1)–(2.4), except that we now allow $a_{ij} < 0$. Now suppose S is the union of finite disjoint subsets S_ν, $\nu \in J$. Take any $\nu \in J$ and suppose $S_\nu = \{i_1,\dots,i_m\}$, where $m = m(\nu) \geq 1$. Let $\{j \in T : a_{ij} \neq 0 \text{ for some } i \in S_\nu\} \equiv \{j_1,\dots,j_r\}$, where we suppose $r \geq m$. For simplicity we write $b_{pq} = a_{i_p,j_q}$, $1 \leq p \leq m,\ 1 \leq q \leq r$, and assume

$$b_{1\ell} + \cdots + b_{m\ell} > 0, \quad \ell = 1,\dots,r.$$

Now let $U := (v_{pq})_{p=1,q=1}^{m,\ \ r}$, where

$$v_{pq} := \left(\widetilde{I}_{j_q}/I_{i_p}\right)^{1/2}(b_{1q} + \cdots + b_{mq})^{-1/2}b_{pq}.$$

Then for $1 \leq p, q \leq m$,

$$(UU^T)_{pq} = \sum_{\ell=1}^{r} v_{p\ell}v_{q\ell} = (I_{i_p}I_{i_q})^{-1/2}\sum_{\ell=1}^{r}\frac{b_{p\ell}b_{q\ell}\widetilde{I}_{j_\ell}}{b_{1\ell} + \cdots + b_{m\ell}}.$$

Note that

$$\begin{aligned}\sum_{q=1}^{m}(UU^T)_{pq}I_{i_q}^{1/2} &= I_{i_p}^{-1/2}\sum_{\ell=1}^{r}\sum_{q=1}^{m}\frac{b_{p\ell}b_{q\ell}\widetilde{I}_{j_\ell}}{b_{1\ell}+\cdots+b_{m\ell}} \\ &= I_{i_p}^{-1/2}\sum_{\ell=1}^{r}b_{p\ell}\widetilde{I}_{j_\ell} = I_{i_p}^{-1/2}\sum_{\ell=1}^{r}a_{i_p,j_\ell}\widetilde{I}_{j_\ell} \\ &= I_{i_p}^{-1/2}\sum_{j\in T}a_{i_p,j}\widetilde{I}_j = I_{i_p}^{1/2},\end{aligned}$$

by (2.3). Thus $(I_{i_1}^{1/2},\ldots,I_{i_m}^{1/2})^T$ is an eigenvector of UU^T with eigenvalue 1. We shall assume that all eigenvalues of UU^T are less than or equal to 1. We note that this will be the case if all entries of UU^T are non-negative. For let D be the diagonal matrix with entries $I_{i_1}^{1/2},\ldots,I_{i_m}^{1/2}$. Then $D^{-1}UU^TD$ has eigenvector $(1,1,\ldots,1)^T$ with eigenvalue 1. If all entries of UU^T are non-negative, then $D^{-1}UU^TD$ is a stochastic matrix and so all its eigenvalues are less than or equal to 1. Under this assumption on the eigenvalues, we may extend U to a real matrix $V=(v_{pq})_{p,q=1}^{s,r}$, $s=s(\nu)\geq r$, which has orthonormal columns; this is shown in Theorem 4.1 of [12] for complex-valued matrices and the result for real matrices follows as a special case. For $k=m+1,\ldots,s$, define

$$\psi_k^\nu := \sum_{\ell=1}^{r}v_{k\ell}(b_{1\ell}+\cdots+b_{m\ell})^{1/2}\widetilde{\Phi}_{j_\ell}. \tag{2.6}$$

Proposition 2.2. *For $f\in L^2(\Omega)$,*

$$\sum_{i\in S}\langle f,\Phi_i\rangle^2+\sum_{\nu\in J}\sum_{k=m(\nu)+1}^{s(\nu)}\langle f,\psi_k^\nu\rangle^2=\sum_{j\in T}\langle f,\widetilde{\Phi}_j\rangle^2.$$

Proof. Take $f\in L^2(\Omega)$ and for a fixed $\nu\in J$, let $\alpha_q:=\langle f,\widetilde{\Phi}_{j_q}\rangle$, $q=1,\ldots,r$. Then by (2.4) and (2.6),

$$\begin{aligned}&\sum_{p=1}^{m}\langle f,\Phi_{i_p}\rangle^2+\sum_{k=m+1}^{s}\langle f,\psi_k^\nu\rangle^2 \\ =\;&\sum_{p=1}^{m}\left(\sum_{\ell=1}^{r}\left(\widetilde{I}_{j_\ell}/I_{i_p}\right)^{1/2}b_{p\ell}\alpha_\ell\right)^2+\sum_{k=m+1}^{s}\left(\sum_{\ell=1}^{r}v_{k\ell}(b_{1\ell}+\cdots+b_{m\ell})^{1/2}\alpha_\ell\right)^2 \\ =\;&\sum_{k=1}^{s}\left(\sum_{\ell=1}^{r}v_{k\ell}(b_{1\ell}+\cdots+b_{m\ell})^{1/2}\alpha_\ell\right)^2=\left\|VD[\alpha_1,\ldots,\alpha_r]^T\right\|^2,\end{aligned}$$

where $D := \text{diag}((b_{1\ell} + \cdots + b_{m\ell})^{1/2})_{\ell=1}^r$. Thus

$$\begin{aligned}
\sum_{p=1}^{m}\langle f, \Phi_{i_p}\rangle^2 + \sum_{k=m+1}^{s}\langle f, \psi_k^\nu\rangle^2 &= [\alpha_1, \ldots, \alpha_r]D^T V^T V D[\alpha_1, \ldots, \alpha_r]^T \\
&= [\alpha_1, \ldots, \alpha_r]\text{diag}((b_{1\ell} + \cdots + b_{m\ell})^{1/2})_{\ell=1}^r[\alpha_1, \ldots, \alpha_r]^T \\
&= \sum_{k=1}^{r}(b_{1k} + \cdots + b_{mk})\alpha_k^2 \\
&= \sum_{j\in T}(a_{i_1,j} + \ldots + a_{i_m,j})\langle f, \widetilde{\Phi}_j\rangle^2.
\end{aligned}$$

Summing over ν in J gives

$$\sum_{i\in S}\langle f, \Phi_i\rangle^2 + \sum_{\nu\in J}\sum_{k=m(\nu)+1}^{s(\nu)}\langle f, \psi_k^\nu\rangle^2 = \sum_{i\in S}\sum_{j\in T} a_{ij}\langle f, \widetilde{\Phi}_j\rangle^2 = \sum_{j\in T}\langle f, \widetilde{\Phi}_j\rangle^2,$$

by (2.2). ♠

To construct tight frames for $L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, we consider scaling sequences ϕ_i^ℓ, $i \in S_\ell$, $\ell = 0, 1, \ldots$, of real-valued functions in $L^2(\Omega)$ satisfying

$$\phi_i^\ell = \sum_{j\in S_{\ell+1}} a_{ij}^\ell \phi_j^{\ell+1}, \quad i \in S_\ell,\ \ell = 0, 1, \ldots, \tag{2.7}$$

for $a_{ij}^\ell \in \mathbb{R}$ with

$$\sum_{i\in S_\ell} a_{ij}^\ell = 1, \quad j \in S_{\ell+1},\ \ell = 0, 1, \ldots. \tag{2.8}$$

We assume that for $\ell = 0, 1, \ldots$, the collection $\{\phi_i^\ell : i \in S_\ell\}$ is locally finite and that for each $i \in S_\ell$, $\ell = 0, 1, \ldots$, only a finite number of coefficients in (2.7) are non-zero. Note that from (2.7), for $\ell = 0, 1, \ldots$,

$$\sum_{i\in S_\ell}\phi_i^\ell = \sum_{i\in S_\ell}\sum_{j\in S_{\ell+1}} a_{ij}^\ell\phi_j^{\ell+1} = \sum_{j\in S_{\ell+1}}\phi_j^{\ell+1}\sum_{i\in S_\ell} a_{ij}^\ell = \sum_{j\in S_{\ell+1}}\phi_j^{\ell+1},$$

by (2.8). We shall assume that $\sum_{i\in S_0}\phi_i^0 = 1$, so that

$$\sum_{i\in S_\ell}\phi_i^\ell = 1, \quad \ell = 0, 1, \ldots\ . \tag{2.9}$$

We shall also assume that for $i \in S_\ell$, $\ell = 0, 1, \ldots$,

$$I_i^\ell := \int_\Omega \phi_i^\ell > 0,$$

and define

$$\Phi_i^\ell := (I_i^\ell)^{-1/2}\phi_i^\ell.$$

For $i \in S_\ell$, $\ell = 0, 1, \ldots$, we have provided methods to construct, under quite general conditions, $r = r(i, \ell)$ functions ψ_{ik}^ℓ, $k = 1, \ldots, r$, defined by

$$\psi_{ik}^\ell = \sum_{j \in S_{\ell+1}} b(k)_{ij}^\ell \Phi_j^{\ell+1}, \quad i \in S_\ell, \ \ell = 0, 1, \ldots, \tag{2.10}$$

and satisfying for any $f \in L^2(\Omega)$, $\ell \geq 0$,

$$\sum_{j \in S_{\ell+1}} \langle f, \Phi_j^{\ell+1} \rangle^2 = \sum_{i \in S_\ell} \langle f, \Phi_i^\ell \rangle^2 + \sum_{i \in S_\ell} \sum_{k=1}^{r} \langle f, \psi_{ik}^\ell \rangle^2. \tag{2.11}$$

We suppose that for each $i \in S_\ell$, $\ell = 0, 1, \ldots$, only a finite number of coefficients in (2.10) are non-zero.

We say that the collection $\{\Phi_i^0 : i \in S_0\} \cup \{\psi_{ik}^\ell : k = 1, \ldots, r, \ i \in S_\ell, \ \ell = 0, 1, \ldots\}$ forms a *normalized tight frame* for $L^2(\Omega)$ if for every $f \in L^2(\Omega)$,

$$||f||^2 = \sum_{i \in S_0} \langle f, \Phi_i^0 \rangle^2 + \sum_{\ell=0}^{\infty} \sum_{i \in S_\ell} \sum_{k=1}^{r} \langle f, \psi_{ik}^\ell \rangle^2. \tag{2.12}$$

By standard frame theory (see for instance [5], p. 115), this is equivalent to

$$f = \sum_{i \in S_0} \langle f, \Phi_i^0 \rangle \Phi_i^0 + \sum_{\ell=0}^{\infty} \sum_{i \in S_\ell} \sum_{k=1}^{r} \langle f, \psi_{ik}^\ell \rangle \psi_{ik}^\ell \tag{2.13}$$

for all $f \in L^2(\Omega)$. In general, for ψ_{ik}^ℓ, $k = 1, \ldots, r$, $i \in S_\ell$, $\ell = 0, 1, \ldots$, to contribute to a normalized tight frame requires some restrictions on the scaling sequences ϕ_i^ℓ. For simplicity we shall make the following assumptions, which cover most practical cases including those in Sections 4 and 5.

(A1) $\phi_i^\ell(x) \geq 0$, $x \in \Omega$, $i \in S_\ell$, $\ell = 0, 1, \ldots$.

(A2) For some integer N,

$$|\{i : \phi_i^\ell(x) \neq 0\}| \leq N, \quad x \in \Omega, \ i \in S_\ell, \ \ell = 0, 1, \ldots.$$

(A3) For $\ell = 0, 1, \ldots$, $i \in S_\ell$, there is a subset $\Omega_i^\ell \subset \Omega$ such that $\mathrm{supp}(\phi_i^\ell) \subset \Omega_i^\ell$ and $\mathrm{diam}(\Omega_i^\ell) \leq h_\ell$, where $h_\ell \to 0$ as $\ell \to \infty$.

Theorem 2.3. *Suppose that* (A1)–(A3) *are satisfied. Then the collection* $\{\Phi_i^0 : i \in S_0\} \cup \{\psi_{ik}^\ell : k = 1, \ldots, r, \ i \in S_\ell, \ \ell = 0, 1, \ldots\}$ *forms a normalized tight frame for* $L^2(\Omega)$.

We shall give a proof of Theorem 2.3 in the next section.

3. Approximation order and proof of Theorem 2.3

We consider ϕ_i^ℓ, Φ_i^ℓ and ψ_{ik}^ℓ, $k = 1, \ldots, r$, $i \in S_\ell$, $\ell = 0, 1, \ldots$, in $L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, with all the assumptions as in the previous section. To determine the approximation order of the frame system, we take $\nu \geq 0$ and consider a sequence of operators $Q_\ell : C^{\nu+1}(\Omega) \longrightarrow V_\ell$ of the form

$$Q_\ell f := \sum_{i \in S_\ell} \lambda_i^\ell(f)\phi_i^\ell,$$

where in addition to conditions (A1)–(A3) on ϕ_i^ℓ we assume that the linear functionals λ_i^ℓ satisfy

(A4) $|\lambda_i^\ell(f)| \leq A||f|_{\Omega_i^\ell}||_\infty,\ i \in S_\ell,\ \ell = 0, 1, \ldots,$ for some constant $A > 0$.

Theorem 3.1. *Take $\nu \geq 0$ and suppose that for $\ell = 0, 1, \ldots$, $Q_\ell p = p$ for any polynomial p on Ω of degree ν. If $f \in C^{\nu+1}(\Omega)$, then*

$$|Q_\ell f(x) - f(x)| = O(h_\ell^{\nu+1}),$$

uniformly on compact subsets of Ω, where h_ℓ are as in condition (A3).

Proof. Take $x \in \Omega$. For $\ell = 0, 1, \ldots$, let $W_\ell = \cup\{\Omega_i^\ell : i \in S_\ell,\ \phi_i^\ell(x) \neq 0\}$. Letting W be a compact set containing $\{y : d(y, x) \leq h_\ell\}$, $\ell = 0, 1, \ldots$, condition (A3) ensures that $W_\ell \subset W$, $\ell = 0, 1, \ldots$. Since W is compact and $d(x, y) \leq h_\ell$ for any $y \in W_\ell$, there is a constant B depending only on the $(\nu+1)^{th}$ derivative of f on W, such that

$$||(f-p)|_W||_\infty \leq Bh_\ell^{\nu+1}, \quad \ell = 0, 1, \ldots.$$

Then since $f(x) = p(x)$, recalling (A1) and (A4), for $\ell = 0, 1, \ldots$,

$$\begin{aligned} |Q_\ell f(x) - f(x)| &= |Q_\ell(f-p)(x)| \\ &\leq \sum_{i \in S_\ell} |\lambda_i^\ell(f-p)|\phi_i^\ell(x) \\ &\leq A||(f-p)|_{W_\ell}||_\infty \sum_{i \in S_\ell} \phi_i^\ell(x) \\ &\leq ABh_\ell^{\nu+1}. \end{aligned}$$

The result follows. ♠

Remark 1. *In most practical situations, Ω is compact. In this case Theorem* 3.1 *gives*

$$||Q_\ell f - f||_2 = O(h_\ell^{\nu+1}).$$

We now define a sequence of operators $T_\ell : L^2(\Omega) \to V_\ell$ by

$$T_\ell f = \sum_{i \in S_\ell} \lambda_i^\ell(f) \phi_i^\ell,$$

where $\lambda_i^\ell(f) := \frac{1}{I_i^\ell}\langle f, \phi_i^\ell \rangle$. This is a sequence of integral operators with kernels

$$K_\ell(x,y) := \sum_{i \in S_\ell} (I_i^\ell)^{-1} \phi_i^\ell(x) \phi_i^\ell(y), \quad (x,y) \in \Omega \times \Omega.$$

In the univariate case, the kernel K_ℓ corresponds to that of [7] with the matrix $\Gamma_\ell = \operatorname{diag}(\{(I_i^\ell)^{-1/2}\}_{i \in S_\ell})$. We are interested in the norm convergence, $T_\ell f \to f$, for the proof of Theorem 2.3. The conditions for the convergence of such an integral operator have been studied in approximation theory. Since the results are scattered in the literature we shall give a proof of a result (Theorem 3.3 below) that meets our need.

By condition (A3), $|\lambda_i^\ell(f)| \le ||f|_{\Omega_i^\ell}||_\infty$. The following lemma shows that T_ℓ is uniformly bounded.

Lemma 3.2. *For* $f \in L^2(\Omega)$,

$$||T_\ell f||_2 \le N^{1/2} ||f||_2,$$

where N is defined by condition (A2).

Proof. For $x \in \Omega$,

$$\left(\sum_{i \in S_\ell} \lambda_i^\ell(f) \phi_i^\ell(x) \right)^2 \le N \sum_{i \in S_\ell} (\lambda_i^\ell(f))^2 (\phi_i^\ell(x))^2,$$

since condition (A2) implies that there are at most N non-zero terms in the summation. Thus

$$\begin{aligned}
||T_\ell f||_2^2 &\le N \sum_{i \in S_\ell} (I_i^\ell)^{-2} \langle f, \phi_i^\ell \rangle^2 ||\phi_i^\ell||_2^2 \\
&\le N \sum_{i \in S_\ell} (I_i^\ell)^{-2} \int_\Omega f^2 \phi_i^\ell \int_\Omega \phi_i^\ell \int_\Omega (\phi_i^\ell)^2 \\
&\le N \sum_{i \in S_\ell} \int_\Omega f^2 \phi_i^\ell \\
&= N \int_\Omega f^2 \sum_{i \in S_\ell} \phi_i^\ell = N ||f||_2^2,
\end{aligned}$$

where we have used (2.9) and the inequality $\int_\Omega (\phi_i^\ell)^2 \le \int_\Omega \phi_i^\ell = I_i^\ell$, since $0 \le \phi_i^\ell \le 1$. ♠

Theorem 3.3. *For* $f \in L^2(\Omega)$,

$$\lim_{\ell \to \infty} T_\ell f = f \quad \text{in} \quad L^2(\Omega).$$

Proof. Note that if $g(x) = c$ for all $x \in \Omega$ and for a constant c, then for $x \in \Omega$,

$$T_\ell g(x) = \sum_{i \in S_\ell} (I_i^\ell)^{-1} \left(\int_\Omega c\phi_i^\ell \right) \phi_i^\ell(x) = c \sum_{i \in S_\ell} \phi_i^\ell(x) = c.$$

It follows from Theorem 3.1 that if $g \in C^1(\Omega)$, then $|T_\ell g(x) - g(x)| = O(h_\ell)$ uniformly on compact subsets of Ω.

Now take $f \in L^2(\Omega)$ and $\epsilon > 0$. Choose a compact subset $K \subset \Omega$ with $||f - f|_K||_2 < \epsilon$. Let $L = \{x \in \Omega : d(x, K) \leq 1\}$. As in the proof of Lemma 3.2, if $h_\ell \leq 1$, then

$$||T_\ell f - T_\ell f|_L||_2 < N^{1/2} ||f - f|_K||_2 < N^{1/2}\epsilon.$$

Choose $g \in C^1(\Omega)$ such that $||(g - f)|_L||_2 < \epsilon$. Choose M such that for all $\ell \geq M$, $h_\ell < 1$ and $||(T_\ell g - g)|_L||_2 < \epsilon$. Then for $\ell \geq M$,

$$\begin{aligned} ||f - T_\ell f||_2 &\leq ||f - f|_L||_2 + ||f|_L - g|_L||_2 + ||(g - T_\ell g)|_L||_2 \\ &\quad + ||T_\ell(g - f)|_L||_2 + ||T_\ell f|_L - T_\ell f||_2 \\ &< 3\epsilon + 2N^{1/2}\epsilon. \end{aligned}$$

Thus $\lim_{\ell \to \infty} T_\ell f = f$ in $L^2(\Omega)$. ♠

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. It follows from (2.11) (see [5], p. 409) that for $f \in L^2(\Omega)$ and $\ell = 1, 2, \ldots,$

$$\sum_{j \in S_\ell} \langle f, \Phi_j^\ell \rangle \Phi_j^\ell = \sum_{i \in S_{\ell-1}} \langle f, \Phi_i^{\ell-1} \rangle \Phi_i^{\ell-1} + \sum_{i \in S_{\ell-1}} \sum_{k=1}^{r} \langle f, \psi_{ik}^{\ell-1} \rangle \psi_{ik}^{\ell-1}. \tag{3.1}$$

Iterating (3.1), we have for any $f \in L^2(\Omega)$ and $\ell = 1, 2, \ldots,$

$$\sum_{i \in S_\ell} \langle f, \Phi_i^\ell \rangle \Phi_i^\ell = \sum_{i \in S_0} \langle f, \Phi_i^0 \rangle \Phi_i^0 + \sum_{m=0}^{\ell-1} \sum_{i \in S_m} \sum_{k=1}^{r} \langle f, \psi_{ik}^m \rangle \psi_{ik}^m,$$

which gives

$$T_\ell f = \sum_{i \in S_0} \langle f, \Phi_i^0 \rangle \Phi_i^0 + \sum_{m=0}^{\ell-1} \sum_{i \in S_m} \sum_{k=1}^{r} \langle f, \psi_{ik}^m \rangle \psi_{ik}^m.$$

Letting $\ell \to \infty$ and using Theorem 3.3 gives (2.13), which is equivalent to (2.12). ♠

In Sections 4 and 5 we shall construct ψ_{ik}^ℓ for bivariate linear splines and for Powell-Sabin elements respectively.

4. Piecewise linear frames on triangular meshes in $\mathbb{R}^3$

The construction of frame elements in Section 2 is applicable if $\Omega \subset \mathbb{R}^3$ is a polyhedral surface. We shall illustrate the construction of frames on such a manifold by considering linear splines on triangular polyhedral surfaces of arbitrary topology. Let $\mathcal{P}^0$ be a triangular polyhedral mesh in $\mathbb{R}^3$, $P(\mathcal{P}^0)$ be the set of all its vertices and $F(\mathcal{P}^0)$ be the set of all its faces. For any two surface integrable functions $f, g : \mathcal{P}^0 \to \mathbb{R}$, we define their inner product

$$\langle f, g\rangle := \int_{\mathcal{P}^0} fg\, dS = \sum_{T\in F(\mathcal{P}^0)} \int_T fg\, dS,$$

and let $L^2(\mathcal{P}^0)$ be the space of all surface integrable functions with respect to this inner product. Let V_0 be the space of linear spline functions in $L^2(\mathcal{P}^0)$, i.e. the space of continuous L^2-functions on $\mathcal{P}^0$ that are linear on each of its triangular faces. The space V_0 is spanned by nodal functions ϕ_p^0, $p \in P(\mathcal{P}^0)$, where

$$\phi_p^0(q) = \delta_{pq}, \quad q \in P(\mathcal{P}^0).$$

Subdivision of $\mathcal{P}^0$ at the mid-point of each of its edges gives a refinement $\mathcal{P}^1$ of $\mathcal{P}^0$, whose vertices are the set $P(\mathcal{P}^1)$, which is the union of $P(\mathcal{P}^0)$ and the midpoints of all the edges of $\mathcal{P}^0$. Similarly we define the space V_1 of linear splines in $L^2(\mathcal{P}^0)$. Continuing in this way gives a nested sequence of subspaces $V_0 \subset V_1 \subset V_2 \subset \cdots$, with $\overline{\cup_{j=0}^{\infty} V_j} = L^2(\mathcal{P}^0)$. The general method of Section 2 for the construction of tight frames for $L^2(\mathcal{P}^0)$ is applicable at each resolution level, which gives tight frame elements in V_j, $j = 1, 2, \dots$, whose union together with $\{\phi_p^0 : p \in P(\mathcal{P}^0)\}$ form a tight frame for $L^2(\mathcal{P}^0)$.

We now construct tight frame elements in V_1. Take the nodal basis functions $\phi_p^0 \in V_0$, $p \in P(\mathcal{P}^0)$. For each vertex $p \in P(\mathcal{P}^0)$ where m edges meet, let its neighboring vertices be p_j, $j = 1, \dots, m$, arranged in the anticlockwise direction, and let p_j^1 be the mid-point of the edge $[p, p_j]$. Let ϕ_p^1, $\phi_j^1 \equiv \phi_{p_j^1}^1$ be the nodal basis functions at p, p_j^1 respectively in V_1. Then (2.1) and (2.2) are satisfied with

$$\phi_p^0 = \phi_p^1 + \sum_{j=1}^{m} a_{pj} \phi_j^1,$$

where $a_{pj} = \frac{1}{2}$, $j = 1, \ldots, m$. Letting

$$
\begin{aligned}
I_p &:= \int_{\mathcal{P}^0} \phi_p^0 \, dS = \sum_{T \in F(\mathcal{P}^0)} \int_T \phi_p^0 \, dS, \quad p \in P(\mathcal{P}^0), \\
I_q^1 &:= \int_{\mathcal{P}^0} \phi_q^1 \, dS = \sum_{T \in F(\mathcal{P}^1)} \int_T \phi_q^1 \, dS, \quad q \in P(\mathcal{P}^1), \\
\Phi_p^0 &:= I_p^{-1/2} \phi_p^0 \ , \quad \Phi_q^1 := (I_q^1)^{-1/2} \phi_q^1 \ ,
\end{aligned}
$$

(2.4) becomes

$$
\Phi_p^0 = (I_p^1/I_p)^{1/2} \Phi_p^1 + \sum_{j=1}^m \frac{1}{2} (I_j^1/I_p)^{1/2} \Phi_j^1,
$$

where $I_j^1 \equiv I_{p_j}^1$, $\Phi_j^1 \equiv \Phi_{p_j}^1$ and

$$
v := \frac{1}{\sqrt{2 I_p}} \left(\sqrt{2 I_p^1}, \sqrt{I_1^1}, \ldots, \sqrt{I_m^1} \right)
$$

is a $1 \times (m+1)$ unit vector. Extending v to an orthogonal matrix, the construction (2.5) gives m frame elements $\psi_j^p \in V_1$, $j = 1, 2, \ldots, m$, with support equals that of ϕ_p^0.

To compute the components of v explicitly, we note that for any triangle $T = [a, b, c]$ with vertices $a, b, c \in \mathbb{R}^3$ and any linear function $f : T \to \mathbb{R}$ taking value 1 on one vertex and 0 on the other two vertices, $\int_T f \, dS = |T|/3$, where $|T|$ denotes the area of T. Let $T_j = [p, p_j, p_{j+1}]$ be the triangle in $\mathcal{P}^0$with vertices p, p_j, p_{j+1}, where p_j are defined cyclically, and let $T_j^1 = [p, p_j^1, p_{j+1}^1]$ be the corresponding triangle in $\mathcal{P}^1$. Then

$$
\begin{aligned}
I_p &= \frac{1}{3} \sum_{i=1}^m |T_i|, \\
I_p^1 &= \frac{1}{3} \sum_{i=1}^m |T_i^1| = \frac{1}{12} \sum_{i=1}^m |T_i|, \\
I_j^1 &= |T_{j-1}^1| + |T_j^1| = \frac{1}{4} \left(|T_{j-1}| + |T_j| \right), \quad j = 1, \ldots, m,
\end{aligned}
$$

and hence

$$
\frac{I_p^1}{I_p} = \frac{1}{4}, \quad \frac{I_j^1}{2 I_p} = \frac{3}{4} \lambda_j, \text{ where } \lambda_j := \frac{|T_{j-1}| + |T_j|}{2 \sum_{i=1}^m |T_i|}.
$$

Therefore,

$$
v = \frac{1}{2} (1, \sqrt{3\lambda_1}, \ldots, \sqrt{3\lambda_m}).
$$

A natural way to extend v to an orthogonal matrix is to find an orthogonal matrix V that maps the unit vector $e_1 := (1, 0, \ldots, 0)$ to v, and this is provided by the Householder transformation [13]:

$$
V = I - \frac{2 (e_1 - v)^T (e_1 - v)}{||e_1 - v||^2},
$$

which is a $(m+1)\times(m+1)$ matrix whose $(k+1)^{th}$ row, $k=1,\dots,m$, is

$$\left(\frac{1}{2}\sqrt{3\lambda_k},\frac{-3}{2}\sqrt{\lambda_k\lambda_1},\dots,\frac{-3}{2}\sqrt{\lambda_k\lambda_{k-1}},1-\frac{3}{2}\lambda_k,\frac{-3}{2}\sqrt{\lambda_k\lambda_{k+1}},\dots,\frac{-3}{2}\sqrt{\lambda_k\lambda_m}\right). \quad (4.1)$$

Other orthogonal extensions of v can be obtained by transforming any orthogonal matrix with first row equals e_1 by V.

Using the orthogonal extension V, (2.5) and (4.1) give the following frame elements associated with a vertex $p\in P(\mathcal{P}^0)$:

$$\psi_k^p=\frac{\lambda_k}{\sqrt{2(|T_{k-1}|+|T_k|)}}\left\{6\phi_p^1+(2/\lambda_k-3)\phi_k^1-3\sum_{\substack{j=1\\ j\neq k}}^{m}\phi_j^1\right\},\quad k=1,\dots,m.$$

5. Tight frames from Powell-Sabin elements on a six-direction mesh

Let T denote a regular triangulation of $\mathbb{R}^2$ into equilateral triangles and P be the set of their vertices. To be definite we assume elements of T have sides of length 2, choose the origin as one of the vertices and the x-axis parallel to one of their sides. Let $\mathcal{T}$ denote the triangulation gained by inserting the medians of all elements of T, and let S denote the space of all C^1 piecewise quadratic functions on $\mathcal{T}$. From the well-known Powell-Sabin '6-split' elements [16], we know that each function in S is determined uniquely by its values and first order derivatives at P.

Now let $\phi_1\in S$ have support in the hexagon centered at the origin and have non-zero triangular Beźier coefficients as shown in Figure 1. Note that $\phi_1(\cdot\,,y)=\phi_1(\cdot\,,-y)$, $y\in\mathbb{R}$, and the support of ϕ_1 is actually on the pentagon BCDEF. Let R denote a rotation through $\frac{2\pi}{3}$ clockwise and write $\phi_2:=\phi_1(R\cdot\,)$, $\phi_3:=\phi_1(R^2\cdot\,)$. Their supports are illustrated in Figure 2. Note that $\phi_3(\cdot\,,y)=\phi_2(\cdot\,,-y)$, $y\in\mathbb{R}$. Also their derivatives at the origin are:

$$\triangledown\phi_1(0,0)=(-2,0),\quad \triangledown\phi_2(0,0)=(1,-\sqrt{3}),\quad \triangledown\phi_3(0,0)=(1,\sqrt{3}). \qquad (5.1)$$

Clearly ϕ_1,ϕ_2,ϕ_3 are linearly independent and form a basis of the subspace of S comprising all functions with support in the hexagon. Moreover, $\{\phi_i(\cdot\;-p):p\in P,\ i=1,2,3\}$ span all of S and

$$\sum_{p\in P}\sum_{i=1}^{3}\phi_i(\cdot\;-p)=3. \qquad (5.2)$$

Integrating (5.2) over a region Ω formed by two adjacent triangles in T gives

$$3|\Omega|=\sum_{p\in P}\sum_{j=1}^{3}\int_\Omega\phi_j(\cdot\;-p)=\sum_{j=1}^{3}\int_{\mathbb{R}^2}\phi_j=3\int_{\mathbb{R}^2}\phi_i,\quad i=1,2,3,$$

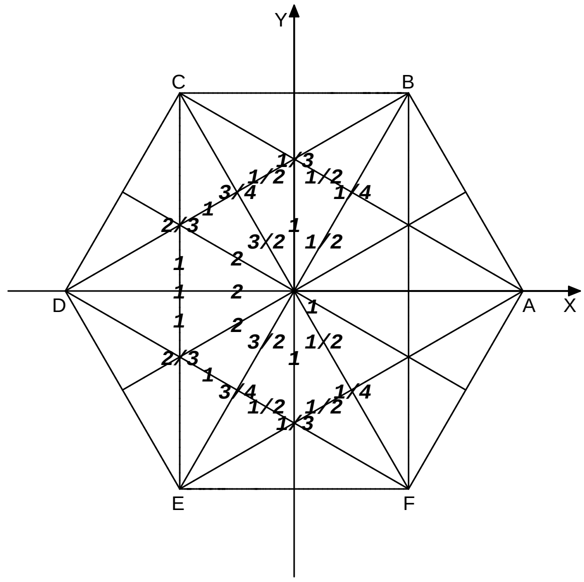

FIGURE 1. Beźier coefficients of ϕ_1.

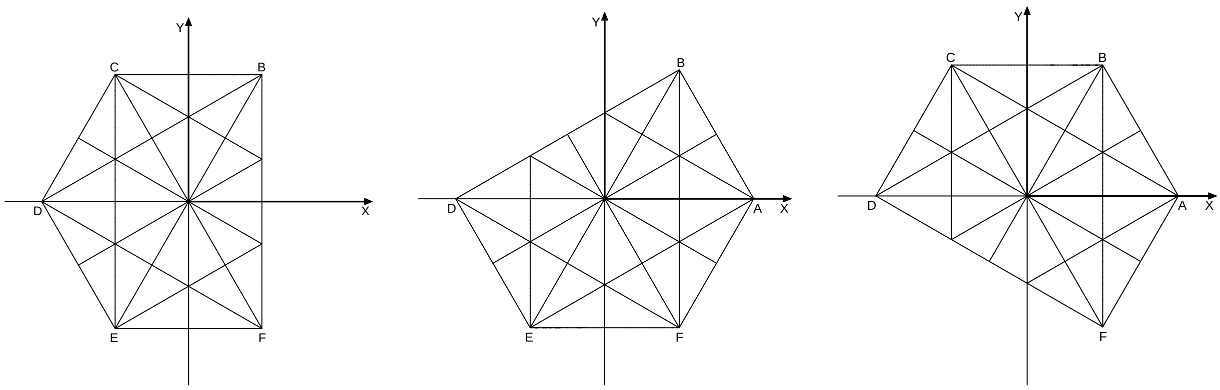

FIGURE 2. Support of (a) ϕ_1 (left), (b) ϕ_2 (middle), (c) ϕ_3 (right).

and so

$$\int_{\mathbb{R}^2} \phi_i = |\Omega| = 2\sqrt{3}, \quad i = 1, 2, 3.$$

In order to apply the theory of Section 2 we must satisfy (2.9) and so rescale our functions by defining $\phi_i^0 := \phi_i/3$, $i = 1, 2, 3$. Then for $i = 1, 2, 3$, $I_i^0 = 2/\sqrt{3}$ and $\Phi_i := (I_i^0)^{-1/2}\phi_i^0$ is given by

$$\Phi_i = 2^{-1/2}3^{-3/4}\phi_i, \quad i = 1, 2, 3. \tag{5.3}$$

Now let $\widetilde{T}$ denote the mesh of equilateral triangles gained by dividing each element of T into four by joining the mid-points of its edges. Let $\widetilde{\mathcal{T}}$ denote the triangulation gained by inserting the medians of elements of $\widetilde{T}$. Let $\widetilde{S}$ denote the space of all C^1 piecewise quadratic functions on $\widetilde{\mathcal{T}}$, i.e. $\widetilde{S} = \{f(2\cdot) : f \in S\}$. Now $\widetilde{\mathcal{T}}$ is a refinement of $\mathcal{T}$, see

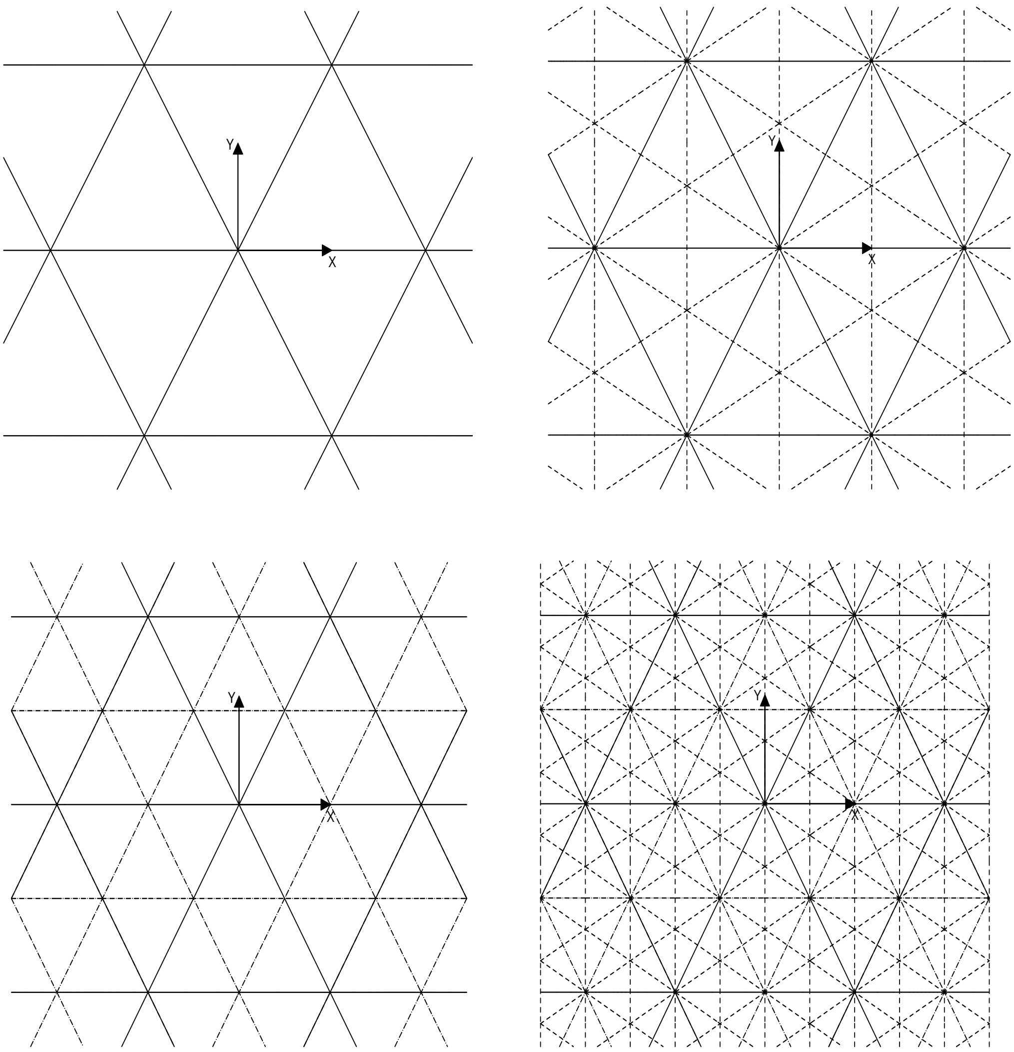

FIGURE 3. (a) Regular triangular mesh T (top left), (b) median subdivision $\mathcal{T}$ of T (top right), (c) edge mid-point subdivision $\widetilde{T}$ of T (bottom left), (d) median subdivision $\widetilde{\mathcal{T}}$ of $\widetilde{T}$ (bottom right).

Figure 3, and so $S \subset \widetilde{S}$. Thus we have refinement equations

$$\phi_i = \sum_{p \in P} \sum_{j=1}^{3} a_{ij}^p \phi_j(2 \cdot \ - p), \quad i = 1, 2, 3. \tag{5.4}$$

To describe this in detail, let $p_0 = 0$ and $p_1, \ldots, p_6$ denote respectively the vertices A, B, C, D, E, F of the hexagon shown in Figure 1. Then with $\phi = (\phi_1, \phi_2, \phi_3)^T$, (5.4) becomes

$$\phi = \sum_{k=0}^{6} A(k) \phi(2 \cdot \ - p_k), \tag{5.5}$$

for 3×3 matrices $A(k)$, $k = 0, \ldots, 6$. Then

$$[\phi(p_k/2), \partial_x \phi(p_k/2), \partial_y \phi(p_k/2)] = A(k) C, \quad k = 0, \ldots, 6, \tag{5.6}$$

where $C := [\phi(0), 2\partial_x\phi(0), 2\partial_y\phi(0)]$ and by (5.1),

$$C = \begin{bmatrix} 1 & -4 & 0 \\ 1 & 2 & -2\sqrt{3} \\ 1 & 2 & 2\sqrt{3} \end{bmatrix}. \tag{5.7}$$

The matrices $A(k)$ can be calculated from (5.6) and (5.7) but first we shall rearrange (5.5) to take account of symmetry.

Define P_j, $j \in \mathbb{Z}$, by $P_j = p_j$, $j = 1, \ldots, 6$, and so that (P_j) is periodic of period 6, i.e. $P_{j+6} = P_j$, $j \in \mathbb{Z}$. Also we extend the definition of ϕ_j to j in $\mathbb{Z}$ so that (ϕ_j) has period 3. Then for $j \in \mathbb{Z}$,

$$RP_j = P_{j-2}, \qquad \phi_j(R\cdot\,) = \phi_{j+1}.$$

Letting $\phi_{jk} = \phi_j(2\cdot\, - P_k)$, $j, k \in \mathbb{Z}$, we have for $x \in \mathbb{R}^2$,

$$\begin{aligned} \phi_{jk}(Rx) &= \phi_j(2Rx - P_k) = \phi_j(R(2x - P_{k+2})) \\ &= \phi_{j+1}(2x - P_{k+2}) = \phi_{j+1,k+2}(x). \end{aligned}$$

We now arrange the functions on the right-hand side of (5.5) into cosets under the operation of R, i.e. we define

$$\begin{aligned} &\psi^0 := \phi(2\cdot\,), \quad \psi^1 := (\phi_{11}, \phi_{23}, \phi_{35})^T, \quad \psi^2 := (\phi_{15}, \phi_{21}, \phi_{33})^T, \\ &\psi^3 := (\phi_{13}, \phi_{25}, \phi_{31})^T, \quad \psi^4 := (\phi_{12}, \phi_{24}, \phi_{36})^T, \\ &\psi^5 := (\phi_{16}, \phi_{22}, \phi_{34})^T, \quad \psi^6 := (\phi_{14}, \phi_{26}, \phi_{32})^T. \end{aligned}$$

Then for $k = 0, \ldots, 6$, writing $\psi^k = (\psi^k_1, \psi^k_2, \psi^k_3)^T$,

$$\psi^k_j(R\cdot\,) = \psi^k_{j+1}, \quad j \in \mathbb{Z} \pmod 3. \tag{5.8}$$

Thus (5.5) becomes

$$\phi = \sum_{k=0}^{6} B(k)\psi^k, \tag{5.9}$$

for 3×3 matrices $B(k)$, $k = 0, \ldots, 6$. Defining $B(k)_{ij}$, $i, j \in \mathbb{Z}$, periodic with period 3 in i and j, we have for $i \in \mathbb{Z}$, $x \in \mathbb{R}^2$,

$$\phi_i(x) = \phi_{i-1}(Rx) = \sum_{k=0}^{6}\sum_{j=1}^{3} B(k)_{i-1,j}\psi^k_j(Rx) = \sum_{k=0}^{6}\sum_{j=1}^{3} B(k)_{i-1,j-1}\psi^k_j(x),$$

by (5.8). Since $B(k)$, $k = 0, \ldots, 6$, are uniquely determined by (5.9) we have

$$B(k)_{ij} = B(k)_{i-1,j-1}, \quad i, j \in \mathbb{Z} \pmod 3,$$

i.e. $B(k)$, $k = 0, \ldots, 6$, are circulant matrices. We shall denote by $\mathcal{C}$ the space of all 3×3 real circulant matrices.

Now suppose that

$$f = \sum_{k=0}^{6} C(k)\psi^k,$$

for $C(k) \in \mathcal{C}$, $k = 0, \ldots, 6$. Writing $f = (f_1, f_2, f_3)^T$, clearly $f_2 = f_1(R\cdot\ )$, $f_3 = f_1(R^2\cdot\ )$. We shall consider $f(S\cdot\ )$, where $S(x, y) := (x, -y)$, $x, y \in \mathbb{R}$. Defining the permutations $\sigma : (1, 2, 3) \to (1, 3, 2)$, $\tau : (1, \ldots, 6) \to (1, 6, 5, 4, 3, 2)$, we have for $j = 1, 2, 3$, $k = 1, \ldots, 6$,

$$\phi_{jk}(S\cdot\ ) = \phi_j(2S\cdot\ - P_k) = \phi_j(2S\cdot\ - SP_{\tau(k)}) = \phi_{\sigma(j)}(2\cdot\ - P_{\tau(k)}) = \phi_{\sigma(j),\tau(k)}.$$

Denoting $\eta(g_1, g_2, g_3)^T := (g_1, g_3, g_2)^T$, it follows that

$$\eta f = \sum_{k=0}^{6} C^T(k)\eta\psi^k,$$

and $\psi^0(S\cdot\ ) = \eta\psi^0$, $\psi^1(S\cdot\ ) = \eta\psi^1$, $\psi^2(S\cdot\ ) = \eta\psi^3$, $\psi^3(S\cdot\ ) = \eta\psi^2$, $\psi^4(S\cdot\ ) = \eta\psi^5$, $\psi^5(S\cdot\ ) = \eta\psi^4$, $\psi^6(S\cdot\ ) = \eta\psi^6$, so that

$$\eta f(S\cdot\ ) = C^T(0)\psi^0 + C^T(1)\psi^1 + C^T(3)\psi^2 + C^T(2)\psi^3 + C^T(5)\psi^4 + C^T(4)\psi^5 + C^T(6)\psi^6.$$

In particular,

$$f_1(S\cdot\ ) = \pm f_1, \quad f_2(S\cdot\ ) = \pm f_3,$$

when $C^T(0) = \pm C(0)$, $C^T(1) = \pm C(1)$, $C^T(2) = \pm C(3)$, $C^T(4) = \pm C(5)$, $C^T(6) = \pm C(6)$.

We shall now calculate the matrices $B(k)$, $k = 0, \ldots, 6$, in (5.9). From (5.6), for $k = 0, \ldots, 6$,

$$[A(k)_{11}, A(k)_{12}, A(k)_{13}] = [\phi_1(p_k/2), \partial_x\phi_1(p_k/2), \partial_y\phi_1(p_k/2)]C^{-1},$$

where by (5.7),

$$C^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/6 & 1/12 & 1/12 \\ 0 & -\sqrt{3}/12 & \sqrt{3}/12 \end{bmatrix}.$$

Then from (5.1),

$$[A(0)_{11}, A(0)_{12}, A(0)_{13}] = [1, -2, 0]C^{-1} = [2/3, 1/6, 1/6],$$

and further calculations give

$$[A(1)_{11}, A(1)_{12}, A(1)_{13}] = [0, 0, 0]C^{-1} = [0, 0, 0],$$
$$[A(2)_{11}, A(2)_{12}, A(2)_{13}] = [1/4, -1, 0]C^{-1} = [1/4, 0, 0],$$
$$[A(3)_{11}, A(3)_{12}, A(3)_{13}] = [3/4, 0, -\sqrt{3}]C^{-1} = [1/4, 1/2, 0],$$
$$[A(4)_{11}, A(4)_{12}, A(4)_{13}] = [1, 2, 0]C^{-1} = [0, 1/2, 1/2],$$
$$[A(5)_{11}, A(5)_{12}, A(5)_{13}] = [3/4, 0, \sqrt{3}]C^{-1} = [1/4, 0, 1/2],$$
$$[A(6)_{11}, A(6)_{12}, A(6)_{13}] = [1/4, -1, 0]C^{-1} = [1/4, 0, 0].$$

By (5.5) and (5.9),

$$\sum_{k=0}^{6}\sum_{j=1}^{3} A(k)_{ij}\phi_{jk} = \sum_{k=0}^{6}\sum_{j=1}^{3} B(k)_{ij}\psi_j^k, \quad i = 1, 2, 3.$$

Since $\psi^1 \equiv (\psi_1^1, \psi_2^1, \psi_3^1)^T = (\phi_{11}, \phi_{23}, \phi_{35})^T$, it follows from the linear independence of ϕ_{jk} that

$$[B(1)_{11}, B(1)_{12}, B(1)_{13}] = [A(1)_{11}, A(3)_{12}, A(5)_{13}] = [0, 1/2, 1/2].$$

Since $B(1) \in \mathcal{C}$,

$$B(1) = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

Similarly,

$$B(0) = \begin{bmatrix} 2/3 & 1/6 & 1/6 \\ 1/6 & 2/3 & 1/6 \\ 1/6 & 1/6 & 2/3 \end{bmatrix}, \quad B(2) = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}, \quad B(4) = \begin{bmatrix} 1/4 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 1/2 & 0 & 1/4 \end{bmatrix},$$

$B(3) = B(2), \quad B(5) = B(4)^T, \quad B(6) = 0.$

We now apply our method in Section 2 to construct a frame. We need to consider the 3×18 matrix $U = [U(0)\dots U(5)]$ where for $k = 0, \dots, 5$, $U(k)$ is gained from $B(k)$ by dividing each column by twice the square root of the sum of the entries in the column. Then $U(0) = \frac{1}{2}B(0), \quad U(1) = \frac{1}{2}B(1), \quad U(2) = U(3) = B(2), \quad U(4) = \frac{1}{\sqrt{3}}B(4), \quad U(5) = U(4)^T.$

Clearly, $U(k) \in \mathcal{C}$, $k = 0, \dots, 5$. A straightforward computation shows that

$$UU^T = \frac{1}{24}\begin{bmatrix} 14 & 5 & 5 \\ 5 & 14 & 5 \\ 5 & 5 & 14 \end{bmatrix},$$

which has eigenvalues $1, 3/8, 3/8$. We shall extend U to a matrix V with orthonormal columns by a modification of the method for establishing Theorem 4.1 of [12], which ensures that V comprises blocks in $\mathcal{C}$, and hence the frame elements will have appropriate symmetry.

We first note that any 3×3 circulant matrix M can be written as $M = F^* D F$, where D is a diagonal matrix and for $\omega := \exp(2\pi i/3)$, F is the unitary Fourier matrix given by

$$F^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{bmatrix},$$

(see [10], p. 73). Then a matrix M is in $\mathcal{C}$ if and only if $M = F^* D F$, for $D \in \mathcal{D} := \{\mathrm{diag}(d, e, \overline{e}) : d \in \mathbb{R}, e \in \mathbb{C}\}$. Note that in this case $M^T = F^* \overline{D} F$.

Now let $\Lambda := \mathrm{diag}(1, \sqrt{3/8}, \sqrt{3/8})$, so that

$$UU^T = F^* \Lambda^2 F. \tag{5.10}$$

For some $D(k) \in \mathcal{D}$, $k = 0, \ldots, 5$,

$$\begin{aligned} U = [U(0) \ldots U(5)] &= F^*[D(0) \ldots D(5)]\mathrm{diag}(F, F, \ldots, F) \\ &= F^* \Lambda X \mathrm{diag}(F, F, \ldots, F), \end{aligned} \tag{5.11}$$

where $\mathrm{diag}(F, F, \ldots, F)$ is a 18×18 block diagonal Fourier matrix and

$$X := [\Lambda^{-1} D(0) \ldots \Lambda^{-1} D(5)].$$

Now from (5.11),

$$X = \Lambda^{-1} F U \mathrm{diag}(F^*, F^*, \ldots, F^*),$$

and so by (5.10),

$$XX^* = \Lambda^{-1} F U U^T F^* \Lambda^{-1} = \Lambda^{-1} F F^* \Lambda^2 F F^* \Lambda^{-1} = I.$$

Thus X has orthonormal rows. We shall extend X to a unitary matrix $Y = [D_{ij}]_{i,j=0}^5$, where $D_{ij} \in \mathcal{D}$, $i, j = 0, \ldots, 5$, and $D_{0j} = \Lambda^{-1} D(j)$, $j = 0, \ldots, 5$. We then define $\widetilde{\Lambda} \in \mathcal{D}$

by $\widetilde{\Lambda} := \mathrm{diag}(0, \sqrt{5/8}, \sqrt{5/8})$, and define V as the 7×6 matrix of 3×3 blocks, viz.

$$V := \mathrm{diag}(F^*, F^*, \ldots, F^*) \begin{bmatrix} \Lambda & 0 \\ \widetilde{\Lambda} & 0 \\ 0 & I \end{bmatrix} Y \mathrm{diag}(F, F, \ldots, F) \tag{5.12}$$

$$= \begin{bmatrix} U(0) & \cdots & U(5) \\ F^*\widetilde{\Lambda}D_{00}F & \cdots & F^*\widetilde{\Lambda}D_{05}F \\ F^*D_{10}F & \cdots & F^*D_{15}F \\ \vdots & & \vdots \\ F^*D_{50}F & \cdots & F^*D_{55}F \end{bmatrix}$$

$$= \begin{bmatrix} U(0) & \cdots & U(5) \\ V_{00} & \cdots & V_{05} \\ \vdots & & \vdots \\ V_{50} & \cdots & V_{55} \end{bmatrix},$$

where $V_{ij} \in \mathcal{C}$, $i, j = 0, \ldots, 5$. Since the second matrix on the right-hand side of (5.12) has orthonormal columns and the other matrices in the product are unitary, V has orthonormal columns.

It remains to give a detailed construction of Y. For $j = 0, \ldots, 5$, write $\Lambda^{-1}D(j) = \mathrm{diag}(d_j, e_j, \bar{e}_j)$. Note that $e_0, e_1 \in \mathbb{R}$, $d_2 = d_3$, $e_2 = e_3 \in \mathbb{R}$, $d_4 = d_5$, $e_4 = \bar{e}_5$. We construct an orthogonal matrix $(d_{ij})_{i,j=0}^{5}$, where $d_{0j} = d_j$, $j = 0, \ldots, 5$, and

$$\begin{aligned} &d_{12} = \frac{\sqrt{2}}{2}, \quad d_{13} = -\frac{\sqrt{2}}{2}, \quad d_{1j} = 0 \text{ otherwise}, \\ &d_{24} = \frac{\sqrt{2}}{2}, \quad d_{25} = -\frac{\sqrt{2}}{2}, \quad d_{2j} = 0 \text{ otherwise}, \\ &d_{i2} = d_{i3}, \quad d_{i4} = d_{i5}, \ 3 \le i \le 5. \end{aligned}$$

Similarly we construct a unitary matrix $(e_{ij})_{i,j=0}^{5}$, where $e_{0j} = e_j$, $j = 0, \ldots, 5$, and

$$\begin{aligned} &e_{12} = \frac{\sqrt{2}}{2}, \quad e_{13} = -\frac{\sqrt{2}}{2}, \quad e_{1j} = 0 \text{ otherwise}, \\ &e_{24} = \frac{\sqrt{2}\bar{e}_4}{2|e_4|}, \quad e_{25} = -\frac{\sqrt{2}e_4}{2|e_4|}, \quad e_{2j} = 0 \text{ otherwise}, \end{aligned}$$

and $e_{ij} \in \mathbb{R}$, $3 \le i \le 5$, $0 \le j \le 5$, satisfy

$$e_{i2} = e_{i3}, \quad e_{i4} = e_{i5}, \; 3 \le i \le 5.$$

Then for $i, j = 0, \dots, 5$, let $D_{ij} = \mathrm{diag}(d_{ij}, e_{ij}, \bar{e}_{ij})$. By construction Y is unitary and extends X. Moreover

$$\begin{aligned} V_{12} &= -V_{13}^T, \quad V_{1j} = 0 \text{ otherwise}, \\ V_{24} &= -V_{25}^T, \quad V_{2j} = 0 \text{ otherwise}, \end{aligned} \tag{5.13}$$

and for $3 \le i \le 5$,

$$V_{i0} = V_{i0}^T, \quad V_{i1} = V_{i1}^T, \quad V_{i2} = V_{i2}^T = V_{i3} = V_{i3}^T, \quad V_{i4} = V_{i5}^T. \tag{5.14}$$

In this case, as noted in (5.3), the normalized functions Φ_i are given by $\Phi_i = c^{-1}\phi_i$, $i = 1, 2, 3$, where $c = 2^{1/2}3^{3/4}$. Then from our earlier theory the frame elements are given by $\Psi^i = (\Psi_1^i, \Psi_2^i, \Psi_3^i)$, where for $i = 0, \dots, 5$,

$$\Psi^i = \sum_{k=0}^{5} E_i(k)\psi^k,$$

and

$$\begin{aligned} &E_i(0) = cV_{i0}, \quad E_i(1) = cV_{i1}, \qquad E_i(2) = \tfrac{c}{2}V_{i2}, \\ &E_i(3) = \tfrac{c}{2}V_{i3}, \quad E_i(4) = \tfrac{c\sqrt{3}}{2}V_{i4}, \quad E_i(5) = \tfrac{c\sqrt{3}}{2}V_{i5}. \end{aligned}$$

From our earlier discussion

$$\Psi_2^i = \Psi_1^i(R\cdot\,), \quad \Psi_3^i = \Psi_1^i(R^2\cdot\,),$$

and from (5.13), (5.14),

$$\begin{aligned} &\Psi_1^i(S\cdot\,) = \Psi_1^i, \quad \Psi_2^i(S\cdot\,) = \Psi_3^i, \quad i = 0, 3, 4, 5, \\ &\Psi_1^i(S\cdot\,) = -\Psi_1^i, \quad \Psi_2^i(S\cdot\,) = -\Psi_3^i, \quad i = 1, 2. \end{aligned}$$

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