

# Refinable Shift Invariant Spaces of Spline Functions

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**Abstract.** We study spaces of piecewise polynomials on regular meshes which are refinable. General properties of such spaces are considered, in particular stability and linear independence, orthogonality and symmetry. Then a more detailed study is undertaken of classes of such spaces in the univariate case. Finally, after considering box splines, we study further examples of such spaces in two dimensions.

## §1. Introduction

As this proceedings is in honour of Tom Lyche's 60th birthday, it is appropriate to devote this paper to the study of spline functions. Moreover, while we recognise that Tom has also studied other types of spline functions, the term here will refer only to piecewise algebraic polynomials (with a finite number of polynomial pieces on any bounded set). We study spaces  $S$  of spline functions defined on  $\mathbb{R}^d$ , where we concentrate on the cases  $d = 1$  or  $2$ . Moreover, the polynomial pieces will be on a regular mesh, which is most easily expressed by stating that  $S$  is **shift-invariant**, i.e.,

$$f \in S \implies f(\cdot - j) \in S, \quad j \in \mathbb{Z}^d.$$

Our final major requirement is that the space  $S$  is **refinable**, i.e.,

$$f \in S \implies f(M^{-1}\cdot) \in S,$$

where  $M$  is a  $d \times d$  **dilation matrix**, i.e., it has integer entries and  $M^{-n}x \rightarrow 0$  as  $n \rightarrow \infty$  for all  $x \in \mathbb{R}^d$ . Because of the structure of polynomials, refinable shift-invariant spaces of spline functions occur naturally, and refinability has numerous applications such as subdivision in computer-aided design, signal processing, data compression and multi-scale methods

in numerical analysis. A common consequence of studying refinable spaces is the construction of wavelets, but this we do not consider here.

In Section 2 we introduce those properties of shift-invariant spaces which will be required later, considering generators and properties of stability and linear independence, orthogonality, and symmetry. Section 3 is devoted to spaces of univariate spline functions, while Section 4 considers the multivariate case, concentrating on general constructions rather than isolated examples, and on spaces which are intrinsically uniform rather than those which are special cases of spaces which are not shift-invariant.

## §2. Shift-Invariant Spaces

Suppose that  $S$  is a shift-invariant space of functions from  $\mathbb{R}^d$  to  $\mathbb{R}$ . For integrable  $\phi_1, \dots, \phi_r \in S$  with compact support,  $\phi = (\phi_1, \dots, \phi_r)$  is called a **generator (of  $S$ )** if  $S$  comprises all linear combinations of shifts of  $\phi$ , i.e.,  $f \in S$  if and only if

$$f = \sum_{j \in \mathbb{Z}^d} \phi(\cdot - j)c(j), \quad (1)$$

for a sequence  $c$  of  $r \times 1$  matrices  $c(j)$ ,  $j \in \mathbb{Z}^d$ . In this case we write  $S = S(\phi)$  and call  $S$  a local finitely generated shift-invariant (local FSI) space. We shall be concerned with spline functions  $f$  of compact support and, for such functions, we wish the summation in (1) to comprise a finite number of terms. It will therefore be convenient to call  $\phi$  a **local generator** of  $S$  if every  $f$  in  $S$  with compact support satisfies (1) for  $c$  with finite support.

Now suppose that  $\phi$  is a local generator of a refinable local FSI space  $S$ . Since  $\phi(M^{-1}\cdot) \in S$ , it is a finite linear combination of shifts of  $\phi$ , i.e.,

$$\phi = \sum_{j \in \mathbb{Z}^d} \phi(M \cdot - j)a(j), \quad (2)$$

for a sequence  $a$  with finite support of  $r \times r$  matrices  $a(j)$ ,  $j \in \mathbb{Z}^d$ . Such an equation (2) is called a **refinement equation** (or two-scale equation) and a vector  $\phi$  of functions satisfying such an equation is called **refinable**. Conversely, if a generator  $\phi$  of a local FSI space  $S$  is refinable, then for any  $f \in S$ ,  $f(M^{-1}\cdot)$  can be expressed as a linear combination of shifts of  $\phi$ , i.e.,  $S$  is refinable.

We see that a generator for a local FSI space is analogous to a set spanning a finite dimensional vector space. To obtain analogues of basis and dimension we need a concept of ‘linear independence’ for generators. We say that a generator  $\phi$  is **linearly independent** if its shifts are linearly independent, i.e.,

$$\sum_{j \in \mathbb{Z}^d} \phi(\cdot - j)c(j) = 0 \implies c = 0.$$

It is shown in [29] that for  $d = 1$ , every local FSI space has a linearly independent generator. It is also known [2, 29, 34] that for  $d = 1$  any linearly independent generator is local.

An elegant characterisation of linearly independent generators is given in [30], in terms of Fourier transforms. For a generator  $\phi$ , its Fourier transform  $\hat{\phi}$  is analytic in  $\mathbb{R}^d$  and so can be extended to  $\mathbb{C}^d$ . It is shown in [30] that  $\phi$  is linearly independent if and only if for each  $z \in \mathbb{C}^d \setminus \{0\}$ , there are  $b_1, \dots, b_r$  in  $\mathbb{Z}^d$  for which the matrix

$$A := [\hat{\phi}_j(z + 2\pi b_k)]_{j,k=1}^r \tag{3}$$

is non-singular.

It is further shown in [30] that the shifts of  $\phi$  form a Riesz basis in  $L^2(\mathbb{R}^d)$  if the above condition holds for each  $z \in \mathbb{R}^d \setminus \{0\}$ . In this case we shall say that  $\phi$  is **stable**. Thus linear independence of  $\phi$  implies its stability. The converse is not true: an example is given in [20] of a refinable local FSI space of univariate spline functions with a generator  $\phi = (\phi_1, \phi_2)$  which is stable but not linearly independent.

Taking Fourier transforms of (1) shows that for  $f \in S(\phi)$ ,

$$\hat{f}(u) = \hat{\phi}(u)P(e^{-iu}), \quad u \in \mathbb{R}^d,$$

where  $P$  is the  $r \times 1$  matrix of Laurent polynomials,

$$P(z) = \sum_{j \in \mathbb{Z}^d} c(j)z^j, \quad z \in (\mathbb{C} \setminus \{0\})^d.$$

It follows that if  $\phi = (\phi_1, \dots, \phi_r)$  is a stable, local generator of  $S$ , then  $\psi = (\psi_1, \dots, \psi_r)$  is also a stable, local generator of  $S$  if and only if  $s = r$  and

$$\hat{\psi}(u) = \hat{\phi}(u)A(e^{-iu}), \quad u \in \mathbb{R}^d, \tag{4}$$

where  $A$  is an  $r \times r$  matrix of Laurent polynomials which is **unimodular**, i.e.,  $\det A(z)$  is a non-trivial monomial. This allows us to define the following analogy of dimension of a vector space.

If  $S$  is a local FSI space with a stable, local generator  $\phi = (\phi_1, \dots, \phi_r)$ , then we say  $S$  has **multiplicity  $r$** .

We now extend the concept of linear independence of generators to that of local linear independence. For a non-empty open subset  $U$  of  $\mathbb{R}^d$ , a generator  $\phi$  is said to be **linearly independent over  $U$**  if

$$\sum_{j \in \mathbb{Z}^d} \phi(x - j)c(j) = 0, \quad x \in U,$$

for  $r \times 1$  matrices  $c(j)$ ,  $j \in \mathbb{Z}^d$ , implies that  $c(j)_i = 0$  whenever  $\phi_i(\cdot - j) \not\equiv 0$  on  $U$ . Clearly, if  $\phi$  is linearly independent over  $(0, 1)^d$ , then  $\phi$  is linearly

independent. The converse is not true; indeed it is shown in [33] that there is a local FSI space on  $\mathbb{R}$  with a linearly independent generator  $\phi = (\phi_1, \phi_2)$  of continuous functions, which has no generator which is linearly independent over  $(0, 1)$ .

In fact a local FSI space  $V$  has a generator which is linearly independent over  $(0, 1)$  if and only if it can be characterized in terms of generalized “join-up” conditions. For a local FSI space let  $V| := \{f\chi_{[0,1]} : f \in V\}$  (where  $\chi_{[0,1]}$  denotes the characteristic function of  $[0, 1)$ ) and let  $\psi = (\psi_1, \dots, \psi_q)$  be a generator whose components form a basis for  $V|$ . Then  $V \subset S(\psi)$  and any element  $f \in V$  can be written in the form  $f = \psi *' c$  for some sequence of  $q \times 1$  matrices  $c(j)$ ,  $j \in \mathbb{Z}$ . In [25] it is shown that a local FSI space  $V$  has a generator which is linearly independent over  $(0, 1)$  if and only if

$$V = \{\psi *' c : s(0)c(j) + s(1)c(j-1) = 0, j \in \mathbb{Z}\}, \quad (5)$$

for some  $p \times q$  matrices  $s(0)$  and  $s(1)$  of rank  $p$  such that the rank of  $\bar{s}(z) := s(0) + zs(1)$  equals  $p$  for all  $z \in \mathbb{C}$ . Furthermore, if  $V$  is of the form (5) then  $V$  has multiplicity  $r = q - p$ . The cardinal spline space  $V = \mathcal{S}_n^{p-1}$ ,  $n \geq p$ , comprising all piecewise polynomial functions of degree  $n$  in  $C^{p-1}(\mathbb{R})$  with integer knots is an important example of a local FSI space on  $\mathbb{R}$  which has a generator that is linearly independent over  $(0, 1)$ . Here one may choose the components of  $\psi = (\psi_1, \dots, \psi_{n+1})$  as  $\psi_k = \pi_k \chi_{[0,1]}$  where  $\pi_k$  denotes the monomial of degree  $k$ . In this case the rows of the matrices  $s(0)$  and  $s(1)$  represent continuity conditions at the integers.

As for bases of finite dimensional vector spaces, it is useful for generators to have orthogonality properties. We say a generator  $\phi = (\phi_1, \dots, \phi_r)$  is orthogonal if the shifts of its components are orthonormal, i.e.,

$$\int_{\mathbb{R}^d} \phi_i \phi_j(\cdot - k) = \delta_{ij} \delta_{k0}, \quad i, j = 1, \dots, r, k \in \mathbb{Z}^d.$$

Clearly an orthogonal generator is linearly independent and local. We say a local FSI space is orthogonal if it has an orthogonal generator.

In considering orthogonality, it is convenient to define the Grammian matrix  $G$  of a generator  $\phi = (\phi_1, \dots, \phi_r)$  as the  $r \times r$  matrix of Laurent polynomials defined by

$$G_{ij}(z) = \sum_{k \in \mathbb{Z}^d} z^k \int_{\mathbb{R}^d} \phi_i \phi_j(\cdot - k), \quad i, j = 1, \dots, r, z \in (\mathbb{C} \setminus \{0\})^d.$$

Then  $\phi$  is orthogonal if and only if its Grammian matrix is the identity.

Now if generators  $\phi$  and  $\psi$  satisfy (4) and have Grammian matrices  $G$  and  $H$  respectively, then it is easily seen that

$$H = A_* G A,$$

where  $A_*(z) := \overline{A^T}(1/z)$ . Suppose  $\phi$  is a local generator of a local FSI space  $S$ . Then it follows that if  $S$  has an orthogonal generator  $\psi$ , then the determinant of  $G$  must be a non-zero constant. In the case  $d = 1$ , the converse is also true: if the determinant of  $G$  is a non-zero constant then  $S$  is orthogonal (see [16] and [27]).

More generally, for generators  $\phi = (\phi_1, \dots, \phi_r)$ ,  $\psi = (\psi_1, \dots, \psi_s)$ , we may define the Grammian matrix  $G$  of  $(\phi, \psi)$  by

$$G_{ij}(z) = \sum_{k \in \mathbb{Z}^d} z^k \int_{\mathbb{R}^d} \phi_i \psi_j(\cdot - k), \quad i, j = 1, \dots, r, z \in (\mathbb{C} \setminus \{0\})^d.$$

We say that  $\phi$  and  $\psi$  are **biorthogonal** if  $r = s$  and their Grammian matrix is the identity. We say that local FSI spaces  $S$  and  $T$  are **biorthogonal** if  $S = S(\phi)$ ,  $T = T(\psi)$ , where  $\phi$  and  $\psi$  are biorthogonal.

Now suppose that local FSI spaces  $S$  and  $T$  have local, stable generators  $\phi = (\phi_1, \dots, \phi_r)$  and  $\psi = (\psi_1, \dots, \psi_r)$  respectively. Then, as above, we can see that  $S$  and  $T$  are biorthogonal if and only if the Grammian matrix  $G$  of  $(\phi, \psi)$  is unimodular. Moreover,, in this case, for any local, stable generator  $\eta$  of  $S$ , there is a unique local, stable generator  $\xi$  of  $T$  such that  $\eta$  and  $\xi$  are biorthogonal.

The final property of shift-invariant spaces which we shall consider is symmetry. We say a shift-invariant space  $S$  is **symmetric** if

$$f \in S \implies f(-\cdot) \in S,$$

though for  $d > 1$  one can also consider other symmetries of the form

$$f \in S \implies f(B\cdot) \in S,$$

for some matrices  $B$ , e. g.  $d = 2$  and  $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Just as refinability and orthogonality can be considered either for the local FSI space  $S$  or for its generators, so we can consider symmetry of generators. For a function  $f$  in  $S$  to be symmetric we require

$$f(-\cdot) = \sigma f(\cdot + k),$$

for some  $k$  in  $\mathbb{Z}^d$ , where  $\sigma = \pm 1$ , i.e.,  $f$  is even or odd about  $\frac{1}{2}k$ . By replacing  $f$  by a shift we may assume  $\frac{1}{2}k \in [0, 1)^d$ , i.e.,  $k \in \{0, 1\}^d$ . Thus we define a generator  $\phi = (\phi_1, \dots, \phi_r)$  to be **symmetric** if for  $j = 1, \dots, r$ , there are  $\sigma_j = \pm 1$ ,  $\alpha_j \in \{0, 1\}^d$ , with

$$\phi_j(-\cdot) = \sigma_j \phi_j(\cdot + \alpha_j). \tag{6}$$

It is shown in [18] that, for  $d = 1$ , any local FSI space  $S$  has a symmetric, linearly independent (and hence local) generator. Moreover,, if such

$S$  has a generator which is linearly independent over  $(0,1)$ , then it has a symmetric generator which is linearly independent over  $(0,1)$ .

Now suppose that a local FSI space  $S$  on  $\mathbb{R}$  has a symmetric, linearly independent generator  $\phi = (\phi_1, \dots, \phi_r)$  satisfying (6). (Since  $d = 1$ ,  $\alpha_j = 0$  or  $1$ ,  $j = 1, \dots, r$ .) Denote by  $e(\phi)$  the number of even functions in  $\phi$ , i.e.,

$$e(\phi) := |\{1 \leq j \leq r : \sigma_j = 1\}|,$$

and let  $d(\phi)$  denote the number of functions in  $\phi$  which are even about 0 minus the number which are odd about 0, i.e.,

$$d(\phi) := \sum_{j=1}^n \sigma_j(1 - \alpha_j).$$

Then it is shown in [18] that  $e(\phi)$  and  $d(\phi)$  are independent of the choice of generator. Thus we can define  $e(S) := e(\phi)$ ,  $d(S) := d(\phi)$ , depending only on the space  $S$ .

Now if  $S$  and  $T$  are local FSI spaces on  $\mathbb{R}$  with  $S \cap T = \{0\}$ , then clearly

$$e(S + T) = e(S) + e(T), \quad d(S + T) = d(S) + d(T).$$

Using this, the following is shown in [18]. Suppose that  $V$  is a symmetric, local FSI space on  $\mathbb{R}$  with multiplicity  $r$ , which is refinable, i.e., for an integer  $m \geq 2$ ,

$$f \in V \implies f(m^{-1}\cdot) \in V.$$

Let  $V_1 = \{f(m\cdot) : f \in V\}$ . Then  $V \subset V_1$ . If  $V_1 = V + W$ , where  $V \cap W = \{0\}$  and  $W$  is a symmetric local FSI space, then

$$\begin{aligned} d(W) &= 0 \\ e(W) &= \begin{cases} \ell r + d(V) - e(V), & m = 2\ell, \ell \in \mathbb{Z}, \\ \ell r, & m = 2\ell + 1, \ell \in \mathbb{Z}. \end{cases} \end{aligned}$$

This holds, in particular, when  $W$  is the orthogonal complement of  $V$  in  $V_1$ .

### §3. Univariate Splines

Throughout this section  $S$  will denote a local FSI space of univariate spline functions which is refinable with dilation 2, i.e.,

$$f \in S \implies f\left(\frac{1}{2}\cdot\right) \in S. \quad (7)$$

We recall that  $S$  has a linearly independent (and hence local) generator. The following result extends to dilation  $m \in \mathbb{Z}$ ,  $m \geq 2$ , but we are restricting to  $m = 2$  for simplicity.

**Theorem 1.** [32] *If  $S$  has multiplicity 1, then for some integer  $n \geq 0$ ,  $S$  comprises all  $C^{n-1}$  spline functions of degree  $n$  with knots in  $\mathbb{Z}$ .*

Here a spline function  $f$  of degree  $n$  means a piecewise polynomial of degree  $n$ , and a knot is a point where some derivative of  $f$  (in this case  $f^{(n)}$ ) has a discontinuity. We shall denote  $S$  as in Theorem 1 by  $V$ . A linearly independent generator for  $V$  comprises the B-spline  $N_n$  of degree  $n$  with support  $[0, n + 1]$ , introduced by Schoenberg [36], which may be defined by its Fourier Transform:

$$\hat{N}_n(u) = \left( \frac{1 - e^{-iu}}{iu} \right)^{n+1}, \quad u \in \mathbb{R}. \quad (8)$$

We see from (4) that any linearly independent generator for  $V$  must be of the form

$$\phi = cN_n(\cdot - j) \quad (9)$$

for some constant  $c \neq 0$  and integer  $j$ . Indeed it is shown in [32] that any stable generator of  $V$  must be of the form (9) and hence any stable generator is linearly independent.

In order to define more general spaces of spline functions we introduce some notation. Take  $n \geq 0$ . Let  $S_n$  be a finite subset of  $[0, 1]$  and for  $k = 0, \dots, n - 1$ , let  $S_k$  be a subset of  $S_n$ . We may then define the space  $\mathcal{S}$  of all piecewise polynomials of degree  $n$  whose only discontinuities are in  $f^{(k)}$  at  $S_k + \mathbb{Z}$ ,  $k = 0, \dots, n$ . We shall describe any such space  $\mathcal{S}$  by specifying the non-empty sets  $S_0, \dots, S_n$ . Thus the space  $V$  above is given by  $S_n = \{0\}$ .

Note that for  $f \in \mathcal{S}$ ,  $f$  has a discontinuity in a derivative at  $\alpha$  if and only if

$$\alpha \in S_k \implies 2\alpha \pmod{\mathbb{Z}} \in S_k, \quad k = 0, \dots, n.$$

A detailed analysis of refinable spline spaces  $\mathcal{S}$  is given in [19], from which the following derives.

**Theorem 2.** [19] *Let  $\mathcal{S}$  be a refinable spline space as above with a total of  $r$  knots  $\pmod{\mathbb{Z}}$ , i.e.,  $\sum_{k=0}^n |S_k| = r$ . Then  $\mathcal{S}$  has multiplicity  $r$ .*

In [19], an explicit description is given of a generator for  $\mathcal{S}$  as in Theorem 2 in terms of Fourier transforms.

We may now extend the classification of Theorem 1 to multiplicity 2.

**Theorem 3.** [20] *If  $S$  has multiplicity 2, then  $S$  is of the form  $\mathcal{S}$  for one of the following.*

- (a) For some  $n \geq 0$ ,  $S_n = \{0, \frac{1}{2}\}$ .
- (b) For some  $0 \leq k < n$ ,  $S_k = S_n = \{0\}$ .
- (c) For some  $n \geq 0$ ,  $S_n = \{\frac{1}{3}, \frac{2}{3}\}$ .

For case (a), a linearly independent generator comprises  $(N_n(2\cdot), N_n(2\cdot - 1))$ . For (b), a linearly independent generator was considered in [10]. In (c) a linearly independent generator  $\phi = (\phi_1, \phi_2)$  is given by usual B-splines with consecutive knots, e.g. for  $n = 1$  we may take  $\phi_1$  to have knots  $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}$  and  $\phi_2$  to have knots  $\frac{2}{3}, \frac{4}{3}, \frac{5}{3}$ .

In [20] conditions are given on the knots under which  $S$  must be of the form  $\mathcal{S}$ . As these are rather technical, we give here a special case.

**Theorem 4.** [20] *If  $S$  comprises spline functions of degree  $n \geq 0$  with knots having odd denominators, then  $S$  is of the form  $\mathcal{S}$  for some choice of  $S_0, \dots, S_n$ .*

An FSI space  $S$  of multiplicity  $r \geq 3$  need not be of the form  $\mathcal{S}$ . It is shown in [20] that for  $r = 3$ ,  $S$  must be of the form  $\mathcal{S}$  or a subspace of one of the following spaces  $\mathcal{S}$  of multiplicity 4.

- (a)  $S_n = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ .
- (b) For some  $0 \leq k < n$ ,  $S_k = S_n = \{0, \frac{1}{2}\}$ .
- (c)  $S_n = \{\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}\}$ .

Let us consider case (a) above in more detail. First, define the multiresolution  $(V_j)_{j \in \mathbb{Z}}$  associated with  $V$  by

$$V_j = \{f(2^j \cdot) : f \in V\} \quad (j \in \mathbb{Z}), \quad (10)$$

i.e.,  $V_j$  is the space of all  $C^{n-1}$  spline functions of degree  $n$  with knots in  $2^{-j}\mathbb{Z}$ . A generator for  $V_1$  is  $(\phi_1, \phi_2)$ , where

$$\phi_1 = N_n(2\cdot), \quad \phi_2 = N_n(2\cdot - 1). \quad (11)$$

Now let  $\phi_3$  be any element of  $V_2$  which is not in  $V_1$ . Then  $\phi = (\phi_1, \phi_2, \phi_3)$  is linearly independent and is a generator of a space  $S$  with  $V_1 \subset S \subset V_2$ . Since  $\phi(\frac{1}{2}\cdot)$  lies in  $V_1 \subset S$ ,  $\phi$  is refinable, and so  $S$  is refinable. Since we have a large choice of possible  $\phi_3$ , we can use this flexibility to derive spaces  $S$  with other desirable properties, and this we proceed to discuss.

The first property we consider is orthogonality. The uniform B-splines  $N_n$  are the most basic refinable spline functions and have many elegant properties, see e.g. [1]. However their shifts are clearly not orthogonal for  $n \geq 1$ . It is therefore natural to try to construct refinable spline functions which are biorthogonal to them. As discussed above, for sufficient flexibility we take  $r = 3$ , and for a generator  $\psi = (\psi_1, \psi_2, \psi_3)$  of uniform B-splines we take

$$\psi_j = N_n(3\cdot - j + 1), \quad j = 1, 2, 3,$$

which generates the space  $T$  of all  $C^{n-1}$  spline functions of degree  $n$  with knots in  $\frac{1}{3}\mathbb{Z}$ . In [13] a space  $S$  was constructed, with  $V_1 \subset S \subset V_2$  as

above, which is biorthogonal to  $T$ . As seen above,  $S$  is generated by  $\phi = (\phi_1, \phi_2, \phi_3)$ , where  $\phi_1, \phi_2$  are given by (11). Now we saw in Section 2 that  $S$  and  $T$  are biorthogonal if and only if the Grammian matrix  $G$  of  $(\phi, \psi)$  is unimodular. Setting  $\det G = 1$  gives a set of linear equations for the unknown function  $\phi_3$  and it is shown in [13] that this has a unique solution (modulo  $V_1$ ) of minimal support  $[\frac{1}{4}, 2n - \frac{1}{4}]$ .

In order to solve equations such as that above for  $\phi_3$ , it is useful to express  $\phi_3$  as a linear combination of shifts of some canonical functions. This can be achieved as follows. It is shown in [15] that  $V_1 = V + W$ ,  $V \cap W = \{0\}$ , where  $W$  is a local FSI space generated by a function  $M_n$ , where  $M_n$  is chosen to have the smallest possible support, which is  $[0, n]$ . Then  $V_2 = V_1 + W_1$ ,  $V_1 \cap W_1 = \{0\}$ , where  $W_1$  is a local FSI space with linearly independent generator  $(M_n(2 \cdot), M_n(2 \cdot - 1))$ . Thus we can write

$$\phi_3 = \sum_{j=-\infty}^{\infty} M_n(2 \cdot - j)\alpha(j), \tag{12}$$

where the sequence  $\alpha$  has finite support.

Now suppose again that  $S$  is generated by  $\phi = (\phi_1, \phi_2, \phi_3)$  as in (11) and (12) and define the Laurent polynomial

$$p(z) := \sum_{j=-\infty}^{\infty} \alpha(j)z^j, \quad z \in \mathbb{C} \setminus \{0\}.$$

It is shown in [15] that if  $p$  has no real negative zeros and  $p(z), p(-z)$  have no common zeros, then there is a space  $\tilde{S}$  such that  $S$  and  $\tilde{S}$  are biorthogonal, where  $\tilde{S}$  is of the same form as  $S$ , i.e.,  $\tilde{S}$  is generated by  $\tilde{\phi} = (\phi_1, \phi_2, \tilde{\phi}_3)$ , where

$$\tilde{\phi}_3 = \sum_{j=-\infty}^{\infty} M_n(2 \cdot - j)\tilde{\alpha}(j),$$

for a sequence  $\tilde{\alpha}$  with finite support. Indeed it is shown that for  $S, \tilde{S}$  of this form, the Grammian matrix  $G$  of  $(\phi, \tilde{\phi})$  satisfies

$$z^6 \det G(z^2) = E(-z)F(z)p(z)\tilde{p}(z^{-1}) + E(z)F(-z)p(-z)\tilde{p}(-z^{-1}), \tag{13}$$

$$z \in \mathbb{C} \setminus \{0\},$$

where for  $z \in \mathbb{C} \setminus \{0\}$ ,

$$\tilde{p}(z) = \sum_{j=-\infty}^{\infty} \tilde{\alpha}(j)z^j,$$

$$E(z) = \sum_{k=-\infty}^{\infty} z^k \int N_n N_n(\cdot - k),$$

$$F(z^2) = E(z)E(-z). \tag{14}$$

It is known [37] that  $E$  (the Euler-Frobenius polynomial) has all negative real zeros, and so  $F$  has all positive real zeros. From the conditions on  $p$ , it follows from (13) that there is a Laurent polynomial  $\tilde{p}$  such that

$$\det G(z^2) = 1, \quad z \in \mathbb{C} \setminus \{0\},$$

which ensures that  $S$  and  $\tilde{S}$  are biorthogonal.

It is further shown in [16] that the function  $\phi_3$  in (12) can be chosen so that  $p = \tilde{p}$  and hence  $S = \tilde{S}$ . Thus in this case  $S$  is an orthogonal, refinable, local FSI space of spline functions. To show this is equivalent, from (13), to finding  $p$  so that for  $z \in \mathbb{C} \setminus \{0\}$ ,

$$E(-z)F(z)P(z) + E(z)F(-z)P(-z) = 2^6, \quad (15)$$

$$P(z) = p(z)p(z^{-1}). \quad (16)$$

By a Lemma in [21], (15) has a solution  $P$  which is non-negative for  $|z| = 1$ , and it follows from the Riesz Lemma that (16) has a solution  $p$ .

In the above construction,  $V_1$  has multiplicity 2 and  $V_2$  has multiplicity 4 and so there is significant freedom to choose a space  $S$  with multiplicity 3 that lies between  $V_1$  and  $V_2$ . Furthermore, as we pointed out, any such space must be refinable. More generally, suppose that  $V$  is a refinable FSI space and that  $S$  is a local FSI space such that  $V_k \subset S \subset V_{k+1}$  for some  $k \in \mathbb{Z}$  where  $(V_j)_{j \in \mathbb{Z}}$  is defined by (10). It follows that  $S$  is refinable since if  $f \in S$  and therefore in  $V_{k+1}$ , then  $f(\cdot/2) \in V_k \subset S$ . For  $j \in \mathbb{Z}$ , let  $\tilde{V}_j := \{f(2^j \cdot) : f \in S\}$ . If  $T$  is an FSI space such that  $\tilde{V}_\ell \subset T \subset \tilde{V}_{\ell+1}$  for some  $\ell \in \mathbb{Z}$ , then the same argument implies that  $T$  is refinable. In this case we have  $V_m \subset T \subset V_{m+2}$  where  $m = k + \ell$ . Clearly  $T$  inherits many of the properties of  $V$ . For example, if  $V$  is a spline space then  $T$  is also a spline space with the same polynomial degree, smoothness, and approximation order. In [7] such constructions were considered and the following existence theorem was proved.

**Theorem 5.** [7] *Suppose  $V$  is a local refinable FSI space of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Let  $(V_j)_{j \in \mathbb{Z}}$  be the multiresolution associated with  $V$ . Then there are integers  $k$  and  $m$  and a local orthogonal refinable FSI space  $S$  such that  $V_k \subset S \subset V_{k+m}$ .*

The final property which we considered in Section 2 was symmetry, and this is often desirable in applications. In our example above, if  $S$  is symmetric then the biorthogonal space  $\tilde{S}$  can also be chosen to be symmetric. However for the case  $S = \tilde{S}$ , the orthogonal space  $S$  will not in general be symmetric, for while the solution  $P$  of (15) will be reciprocal, i.e.,  $P(z^{-1}) = P(z)$ , the solution for (16) will not.

In order to gain both orthogonality and symmetry we require more degrees of freedom. For this reason [18] consider a space  $S$  with multiplicity

5 satisfying  $V_2 \subset S \subset V_3$ . As in the previous case,  $S$  is refinable. Since a generator for  $V_2$  is  $(\phi_1, \dots, \phi_4)$ , where

$$\phi_j = N_n(4 \cdot -j + 1), \quad j = 1, \dots, 4, \quad (17)$$

a linearly independent generator for  $S$  is of the form  $\phi = (\phi_1, \dots, \phi_5)$  where  $\phi_5$  is any element of  $V_3$  not in  $V_2$ . The space  $S$  will be symmetric if  $\phi_5$  is symmetric.

Now suppose that  $V_1 = V + W$ ,  $V \cap W = \{0\}$ , where  $W$  is a symmetric, local FSI space. If  $n$  is odd, then  $e(V) = d(V) = 1$  and so, for the formulae at the end of Section 2,  $e(W) = 1$  and  $d(W) = 0$ . Thus any symmetric function generating  $W$  must be even about  $\frac{1}{2} \pmod{\mathbb{Z}}$ . If  $n$  is even, then  $e(V) = 1$ ,  $d(V) = 0$  and so  $e(W) = d(W) = 0$ . Thus any symmetric function generating  $W$  must be odd about  $\frac{1}{2} \pmod{\mathbb{Z}}$ .

Now define

$$\tilde{M}_n = \begin{cases} M_n(\cdot + \frac{1}{2}n - \frac{1}{2}), & n \text{ odd,} \\ M_n(\cdot + \frac{1}{2}n) - M_n(1 - \cdot - \frac{1}{2}n), & n \text{ even.} \end{cases}$$

Then  $\tilde{M}_n$  generates a space  $W$ , as above, and  $\tilde{M}_n$  is even (respectively odd) about  $\frac{1}{2}$  as  $n$  is odd (respectively even). We write

$$\phi_5 = \sum_{j=-\infty}^{\infty} M_n(4 \cdot -j)\alpha(j), \quad (18)$$

where the sequence  $\alpha$  has finite support and

$$\alpha(-j) = -\alpha(j-1), \quad j \in \mathbb{Z}.$$

Then  $\phi_5$  is even (respectively odd) about 0 as  $n$  is even (respectively odd). We note that

$$\sum_{j=-\infty}^{\infty} \alpha(j)z^j = (z^{-1} - 1)p(z), \quad z \in \mathbb{C} \setminus \{0\},$$

for a Laurent polynomial  $p$  which is reciprocal.

In [18] it is shown that the Grammian matrix  $G$  of  $\phi$  (given by (17) and (18)) satisfies

$$2^8 \det G(z^4) = \sum_{j=0}^3 Q(i^j z), \quad z \in \mathbb{C} \setminus \{0\},$$

where

$$Q(z) = E(-z)E(iz)E(-iz)F(z)(z-1)(z^{-1}-1)p(z)^2, \quad z \in \mathbb{C} \setminus \{0\},$$

for  $E, F$  as before. Thus for  $S$  to be a symmetric, orthogonal, local FSI space, we need  $p$  to satisfy

$$\sum_{j=0}^3 Q(i^j z) = 2^8, \quad z \in \mathbb{C} \setminus \{0\}.$$

In [18] it is shown that such  $p$  exists for low values of  $n$ , and numerical evidence suggests that it is true for any reciprocal polynomial  $E$  with negative real zeros (and  $F$  given by (14)), but no such results are proved for general  $n$ .

To finish this section we consider interpolation. A generator  $\phi = (\phi_1, \dots, \phi_r)$  is said to be **fundamental for  $r$ -Hermite interpolation** if

$$\phi_i^{(j-1)}(k) = \delta_{ij} \delta_{k0}, \quad i, j = 1, \dots, r, k \in \mathbb{Z}.$$

Clearly, such  $\phi$  is linearly independent. Moreover, if the space generated by  $\phi$  is symmetric, then  $\phi$  is symmetric with  $\sigma_j = (-1)^{j-1}$ ,  $\alpha_j = 0$ ,  $j = 1, \dots, r$ , in (6).

Now take integers  $n, r \geq 3$  and let  $U$  denote the space of all  $C^{n-1}$  spline functions of degree  $n$  with knots in  $\frac{1}{r-1}\mathbb{Z}$ , and  $U_1 := \{f(2\cdot) : f \in U\}$ . It is shown in [14] that for any  $r \geq 3$ ,  $n \geq r+1$ , there is a symmetric, refinable, local FSI space  $S$  with  $U \subset S \subset U_1$  such that  $S$  has generator which is fundamental for  $r$ -Hermite interpolation. In [22] a space  $S$  with the same properties is constructed for  $n = r = 3$ , where the support of  $\phi$  is  $[-1, 1]$ , and this is used to derive shape-preserving subdivision schemes which simultaneously provide the function and its first and second derivatives.

#### §4. Multivariate Splines

Throughout this section  $S$  will denote a local FSI space of spline functions on  $\mathbb{R}^d$ ,  $d \geq 2$ , which is refinable with dilation matrix  $M = 2I$ . Whereas a univariate spline function coincides with a polynomial on certain intervals separated by points called knots, a multivariate spline function of degree  $n$  will coincide with a polynomial of degree  $n$  on certain regions, which are unions of a finite number of simplices, separated by subsets of  $(d-1)$ -dimensional hyperplanes which we refer to as **cut regions**. In order to extend the characterisation of Theorem 1, we shall need a generalisation of the B-spline  $N_n$ .

Take  $n \geq 0$  and non-trivial column vectors  $v_1, \dots, v_{n+d}$  in  $\mathbb{Z}^d$  which span  $\mathbb{R}^d$ , where for  $j = 1, \dots, n+d$ , the components of  $v_j$  are coprime. We then define the **box spline**  $B_n = B(\cdot | v_1, \dots, v_{n+d})$  by its Fourier Transform

$$\hat{B}_n(u) = \prod_{j=1}^{n+d} \frac{1 - e^{-iuv_j}}{iuv_j}, \quad u \in \mathbb{R}^d. \quad (19)$$

Putting  $d = 1$ ,  $v_1 = \dots = v_{n+1} = 1$ , (19) reduces to the definition (8) of  $N_n$ . Box splines were introduced in [3]. The function  $B_n$  is a spline function of degree  $n$  with support  $\{\sum_{j=1}^{n+d} t_j v_j : 0 \leq t_j \leq 1, j = 1, \dots, n+d\}$  and cut regions in  $\{\sum_{v_j \in T} t_j v_j : t_j \in \mathbb{R}\}$  for subsets  $T \subset \{v_1, \dots, v_{n+d}\}$  comprising  $d - 1$  linearly independent elements. Since

$$\hat{B}_n(2u) = \prod_{j=1}^{n+d} \left( \frac{1 + e^{-iuv_j}}{2} \right) \hat{B}_n(u), \quad u \in \mathbb{R}^d,$$

$B_n$  is refinable. It was shown in [6, 28] that  $(B_n)$  is linearly independent if and only if for any elements  $w_1, \dots, w_d$  in  $\{v_1, \dots, v_{n+d}\}$  which are linearly independent, the  $d \times d$  matrix  $[w_1, \dots, w_d]$  has determinant 1 or -1, and in this case  $(B_n)$  is linearly independent over any non-empty open set  $U \subset \mathbb{R}^d$ . Moreover,  $B_n$  is symmetric with  $\sigma = 1$ ,  $\alpha = \sum_{j=1}^{n+d} v_j$  as in (6). For many further elegant properties of box splines, see [4].

In [38] a characterisation was given of any compactly supported, refinable spline function on  $\mathbb{R}^d$  as a linear combination of shifts of derivatives of box splines. Further consideration of stability gives the following. (Both of these results actually extend to dilation matrices  $mI$ ,  $m \geq 2$ .)

**Theorem 6.** [23] *If  $S$  has multiplicity 1, then  $S$  has as generator a box spline  $B_n$ , where  $(B_n)$  is linearly independent.*

Henceforward we assume  $d = 2$ . For vectors  $v_1, \dots, v_m$  in  $\mathbb{Z}^2$  we denote by  $M = M(v_1, \dots, v_m)$  the set of all lines through points in  $\mathbb{Z}^2$  in the direction of  $v_1, \dots, v_m$ , i.e., of the form  $\{k + tv_j : t \in \mathbb{R}\}$  for  $k \in \mathbb{Z}^2$ ,  $1 \leq j \leq m$ . We note that the space  $S$  as in Theorem 6 comprises spline functions with cut lines  $M(v_1, \dots, v_{n+2})$ .

As another example, we consider the space  $S$  of all  $C^1$  spline functions of degree 3 with cut lines the ‘4-direction mesh’  $M(v_1, \dots, v_4)$ , where  $v_1 = (1, 0)$ ,  $v_2 = (0, 1)$ ,  $v_3 = (1, 1)$ ,  $v_4 = (1, -1)$ .

Clearly  $S$  is refinable. A generator  $\phi = (\phi_1, \dots, \phi_5)$  of  $S$  is given by the following classical finite element basis [5, 9, 35]. For  $k \in \mathbb{Z}^2$ , consider the linear functionals on  $S$ :

$$\begin{aligned} \sigma_{1k} f &= f(k), & \sigma_{2k} f &= D_{x_1} f(k), & \sigma_{3k} f &= D_{x_2} f(k), \\ \sigma_{4k} f &= D_{x_1} f(k + (0, \frac{1}{2})), & \sigma_{5k} f &= D_{x_2} f(k + (\frac{1}{2}, 0)). \end{aligned}$$

Then for  $i = 1, \dots, 5$ ,  $\phi_i$  is defined as the unique function in  $S$  satisfying

$$\sigma_{jk} \phi_i = \delta_{ij} \delta_{k0}, \quad j = 1, \dots, 5, \quad k \in \mathbb{Z}^2.$$

The generator  $\phi$  is local and linearly independent on  $[0, 1]^2$ . We have the following symmetries for  $x \in \mathbb{R}^2$ :

$$\begin{aligned}\phi_1(x_1, x_2) &= \phi_1(x_2, x_1) = \phi_1(|x_1|, |x_2|), \\ \phi_2(x_1, x_2) &= -\phi_2(-x_1, x_2) = \phi_2(x_1, -x_2), \\ \phi_3(x_1, x_2) &= \phi_2(x_2, x_1), \\ \phi_4(x_1, x_2) &= -\phi_4(-x_1, x_2) = \phi_4(x_1, 1 - x_2), \\ \phi_5(x_1, x_2) &= \phi_4(x_2, x_1).\end{aligned}$$

In the notation of (6),

$$\begin{aligned}\alpha_j &= 0, \quad j = 1, 2, 3, \quad \alpha_4 = (0, 1), \alpha_5 = (1, 0), \\ \sigma_1 &= 1, \quad \sigma_j = -1, \quad j = 2, \dots, 5.\end{aligned}$$

For  $j = 1, 2, 3$ ,  $\phi_j$  has support  $[-1, 1]^2$ , and  $\phi_4$  has support  $[-1, 1] \times [0, 1]$ . The Bernstein-Bézier representation of  $\phi$  is given in [31].

To finish the paper we consider a family of spaces  $S$  of  $C^{n-1}$  spline functions of degree  $n$ , for any  $n \geq 0$ . Despite their many elegant properties, the space  $S$  in Theorem 1, generated by a box spline  $B_n$ , fails to generalise some of the basic properties of the space in Theorem 1, generated by a B-spline  $N_n$ . Except for the cases  $n = 0$  and 1, the elements of  $S$  do not have maximal continuity  $C^{n-1}$ . Moreover,  $S$  does not comprise all spline functions of degree  $n$  subject to some continuity conditions. Indeed the restriction of  $S$  to any region not intersected by cut lines coincides with a space  $T$  which is a proper subspace of polynomials of degree  $n$  comprising the common null space of certain differential operators [3, 6]. Moreover,  $S$  does not comprise all piecewise polynomials with pieces coinciding with such spaces  $T$  subject to certain continuity conditions. In contrast we now consider spaces of the following form.

Take integers  $n \geq 0$ ,  $r \geq 1$ , and pairwise linearly independent column vectors  $v_0, \dots, v_{n+r} \in \mathbb{Z}^2$ . Let  $S_n = S_n(v_0, \dots, v_{n+r})$  comprise all functions  $f$  with continuous Fourier transform of the form

$$\hat{f}(u) = \frac{\sum_{|\alpha|=r-1} P_\alpha(e^{-iu}) u^\alpha}{(uv_0) \cdots (uv_{n+r})}, \quad u \in \mathbb{R}^2,$$

where  $P_\alpha$  is a Laurent polynomial for  $\alpha \in \mathbb{Z}^2$ ,  $|\alpha| = r - 1$ .

**Theorem 7.** [17] *The space  $S_n$  comprises all  $C^{n-1}$  spline functions of degree  $n$  with compact support and cut lines in  $M(v_0, \dots, v_{n+r})$  such that the jump of any derivative of order  $n$  across a cut line changes only at points in  $\mathbb{Z}^2$ . Moreover, there is a generator  $\phi = (\phi_1, \dots, \phi_r)$  such that any element of  $S$  is a finite linear combination of shifts of  $\phi$ , and any such generator is stable if and only if at most  $r$  cut lines intersect except in  $\mathbb{Z}^2$ .*

For  $r = 1$ , a generator  $(\phi)$  comprises the box spline  $B_n(\cdot|v_0, \dots, v_{n+1})$ . Here the condition for stability of  $(B_n)$  is that cut lines intersect only in  $\mathbb{Z}^2$ , which is equivalent to the condition for linear independence of  $(B_n)$  given earlier, a condition that is satisfied only for  $n = 0$  or  $1$ . The case  $r = 2$  was introduced in [11], where the generator  $\phi$  was called a multi-box spline, and this case was studied further in [12]. Here the condition for stability can be satisfied for degree  $n \leq 3$ . The extension  $r \geq 3$  allows stability for higher degree.

Before giving some examples, we discuss the possible symmetry of multi-box splines  $\phi$  for all  $r \geq 1$ . Recalling the notation of (6) we have the following.

**Theorem 8.** [17] *If  $\phi$  as in Theorem 7 is symmetric, then*

$$\sigma_1 \cdots \sigma_r = (-1)^{(n+1)(r+1)}, \quad \alpha_1 + \cdots + \alpha_r = v_0 + \cdots + v_{n+r}, \quad \text{mod } \mathbb{Z}^2.$$

For  $r = 1$ , this corresponds to the well-known fact that the box spline  $B_n(\cdot|v_0, \dots, v_{n+1})$  is even about  $\frac{1}{2}(v_0 + \cdots + v_{n+1})$ .

Next consider symmetry under the operator  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $Tx = (x_2, x_1)$ . If we have

$$\{Tv_0, \dots, Tv_{n+r}\} = \{t_0v_0, \dots, t_{n+r}v_{n+r}\}, \quad (20)$$

where  $t_j = 1$  or  $-1$ ,  $j = 0, \dots, n+r$ , then the set  $M(v_0, \dots, v_{n+r})$  of cut lines is invariant under  $T$  and it is natural to require a corresponding multi-box spline to be symmetric under  $T$ , i.e., for  $j = 1, \dots, r$ ,

$$\phi_j(T\cdot) = \tau_j \phi_j, \quad \tau_j = 1 \text{ or } -1. \quad (21)$$

**Theorem 9.** [17] *Suppose that (20) holds and  $\phi$  as in Theorem 7 satisfies (21). Then*

$$\tau_1 \cdots \tau_r = (-1)^s (t_0 \cdots t_{n+r})^{r-1}, \quad (22)$$

when  $r = 2s$  or  $2s + 1$ .

It may be that (21) is not satisfied for  $j = k, \ell$ , some  $k \neq \ell$ , but instead  $\phi_k = \phi_\ell(T\cdot)$ . In this case (22) holds with  $\tau_k \tau_\ell$  replaced by  $-1$ .

Another possible symmetry is under reflection in the  $x_2$ -axis, i.e.,  $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $Rx = (-x_1, x_2)$ . If we have

$$\{Rv_0, \dots, Rv_{n+r}\} = \{r_0v_0, \dots, r_{n+r}v_{n+r}\}, \quad (23)$$

where  $r_j = 1$  or  $-1$ ,  $j = 0, \dots, n+r$ , then  $M(v_0, \dots, v_{n+r})$  is invariant under  $R$  and it is natural to require for  $j = 1, \dots, r$ ,

$$\phi_j(Rx) = \rho_j \phi_j(x_1 + (\alpha_j)_1, x_2), \quad x \in \mathbb{R}^2, \rho_j = 1 \text{ or } -1. \quad (24)$$

**Theorem 10.** [17] *Suppose that (23) holds and  $\phi$  as in Theorem 8 satisfies (24). Then for  $s$  as in Theorem 9,*

$$\rho_1 \cdots \rho_r = (-1)^s (r_0 \cdots r_{n+r})^{r-1}. \quad (25)$$

As for Theorem 9 we replace  $\rho_k \rho_\ell$  in (25) by -1 if  $\phi_k = \phi_\ell(R \cdot)$ . Clearly a corresponding result to Theorem 10 holds for reflection in the  $x_1$ -axis.

It is suggested in [17], from examples considered, that there is always a symmetric multi-box spline which satisfies the above symmetry conditions, where appropriate, but this is not proved in general.

We finish by considering two examples of multi-box splines. First we take  $n = 1$ ,  $r = 2$ ,  $v_0 = (1, 0)$ ,  $v_1 = (0, 1)$ ,  $v_2 = (1, 1)$ ,  $v_3 = (1, -1)$ . Here there is a choice of symmetric generator  $\phi = (\phi_1, \phi_2)$ , where  $\phi_1$  has support  $[0, 1]^2$  and

$$\phi_2(x) = \phi_1 \left( \frac{x_1 + x_2 + 1}{2}, \frac{x_1 - x_2 + 1}{2} \right), \quad x \in \mathbb{R}^2.$$

Since at most two cut lines in  $M(v_0, \dots, v_3)$  intersect other than in  $\mathbb{Z}^2$ ,  $\phi$  is stable. Equations (6), (21) and (24) hold with

$$\alpha_1 = (1, 1), \alpha_2 = (0, 0), \sigma_j = \tau_j = \rho_j = 1, \quad j = 1, 2.$$

Next we take  $n = 3$ ,  $r = 4$ ,  $v_j$  as above for  $j = 0, \dots, 3$ , and

$$v_4 = (2, 1), v_5 = (1, 2), v_6 = (2, -1), v_7 = (1, -2).$$

Here there is a choice of symmetric generator  $\phi = (\phi_1, \dots, \phi_4)$ , satisfying the following. The support of  $\phi_1$  is bounded by the lines  $x_1 = \pm 1$ ,  $x_2 = \pm 1$ ,  $x_2 - x_1 = \pm 1$  while that of  $\phi_3$  is bounded by  $x_1 = \pm 1$ ,  $x_2 - x_1 = -1$  or  $2$ ,  $x_1 + x_2 = -1$  or  $2$ . Moreover,

$$\begin{aligned} \phi_2(x) &= \phi_1(-x_1, x_2), \quad \phi_4(x) = \phi_3(x_2, x_1), \quad x \in \mathbb{R}^2, \\ \alpha_1 &= \alpha_2 = (0, 0), \quad \alpha_3 = (0, 1), \quad \alpha_4 = (1, 0), \\ \sigma_j &= 1, \quad j = 1, \dots, 4, \quad \tau_1 = \tau_2 = \rho_3 = \rho_4 = 1. \end{aligned}$$

Since at most three cut lines in  $M(v_0, \dots, v_7)$  intersect other than in  $\mathbb{Z}^2$ ,  $\phi$  is stable.

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