FOVEATED SPLINES AND WAVELETS

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Abstract. Spline wavelets on a hybrid of uniform and geometric meshes that admits a natural dyadic multiresolution structure are constructed. The construction is extended to other scaling functions and wavelets. The wavelets provide good approximation of functions near singularities and efficient representation of images with high resolution around regions of interest.

1. Introduction

A foveated image is a nonuniform resolution image in which the resolution is highest at a point, called a *fovea,* or a region (foveation region), but falls off away from the fovea or the foveation region. Foveation is an imaging concept inspired by the human visual process that, when focussed on a region of interest, the perception declines away from the direction of gaze. A foveated image can have more than one fovea or foveation region. Although the initial motivation of foveated imaging is in image compression, which enables only high resolution information of the region of interest to be transmitted ([1, 5, 9]), the concept of foveation has wider applications, for instance in vision research, 3-D visualization and virtual reality ([4, 2, 14]).

A foveated image can be obtained from an image of uniform resolution through an adaptive scale-space smoothing process, where the width of the smoothing kernel is small near the fovea and gradually expanding as the distance from the fovea increases ([6, 5, 15]), which accounts for a better approximation and hence higher resolution in a neighbourhood of the fovea. Our object is to take a multiresolution approach to fovea imaging. The basic idea is to construct low-pass filters that can preserve the quality of an image at any points or regions of interest and at the same time control the quality and achieve a desired level of compression outside the regions. The underlying theory developed in this approach has applications that go beyond image foveation. In particular, it provides good representation of functions near singularities and is also useful in image compression that preserves edges. In this paper we start with multiresolution subspaces comprising polynomial spline spaces on a nested family of knot sequences in $(0, \infty)$. Each knot sequence comprises a geometric mesh near the origin that integrates with a uniform mesh away from the origin, and the

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multiresolution subspace is spanned by the corresponding B-splines. Polynomial splines on the geometric mesh $(q^i)_{i\in\mathbb{Z}}$ or geometric splines were first studied by Micchelli [16]. They were independently rediscovered by Mallat [15] in the study of image foveation. The foveated splines that we introduce here are a hybrid of geometric and uniform splines. The filter sequences are the corresponding discrete B-splines. Away from the fovea the filter sequence is the uniform B-spline filter, which changes smoothly towards a fovea to achieve a better approximation. The corresponding wavelets can be viewed as a "foveated version" of the compactly supported uniform spline wavelets of Chui and Wang ([7]). However, unlike the uniform case there is no Riesz basis of compactly supported semiorthogonal foveated spline wavelets.

More precisely, for an integer $\ell \in \mathbb{Z}$ the knot sequence, $\mathbf{x}_\ell := (x_{\ell j})_{j\in\mathbb{Z}}$, is defined by

$$x_{\ell j} := \begin{cases} 2^{-\ell}(j+1), & j \geq 0, \\ 2^{-\ell+j}, & j \leq -1. \end{cases} \tag{1.1}$$

For any $\ell \in \mathbb{Z}$,

$$\mathbf{x}_\ell \subset \mathbf{x}_{\ell+1} \tag{1.2}$$

and

$$x_{\ell+1,j} = 2^{-1} x_{\ell j}, \quad j \in \mathbb{Z}. \tag{1.3}$$

For an integer $n \geq 1$ the sequence of subspaces $\widetilde{V}_\ell \equiv \widetilde{V}_{n,\ell}$, $\ell \in \mathbb{Z}$, of piecewise polynomial functions of order n (degree $n-1$) in $L_2((0,\infty))$ with knots in $\mathbf{x}_\ell$, form a nested sequence with $\mathrm{cl}(\bigcup_{\ell=0}^{\infty} \widetilde{V}_\ell) = L_2((0,\infty))$. It is known ([3], [17]) that the B-splines $N_{n,j}$, $j \in \mathbb{Z}$, with knot sequence $\mathbf{x}_0$ and supported on $[x_{0\,j}, x_{0,j+n}]$ form a Riesz basis for $\widetilde{V}_0$. Therefore we have a multiresolution approximation of $L_2((0,\infty))$. Unlike the uniform case the orthogonal complement of $\widetilde{V}_0$ in $\widetilde{V}_1$ does not have a Riesz basis of compactly supported functions for $n \geq 2$. In Section 3 we construct a direct sum decomposition $\widetilde{V}_1 = \widetilde{V}_0 \oplus \widetilde{W}_0$ and a Riesz basis of compactly supported functions for $\widetilde{W}_0$. We also construct a Riesz basis for the orthogonal complement of $\widetilde{V}_0$ in $\widetilde{V}_1$ comprising functions with bounded support, in which all but $n-1$ of them are compactly supported.

The multiresolution approximation by the hybrid splines is nonstationary. The B-splines $N_{n,j}$, $j \in \mathbb{Z}$, with knot sequence $\mathbf{x}_0$ form a scaling sequence that satisfies the matrix two-scale relation

$$N_{n,i} = \sum_{j=-\infty}^{\infty} a_{ij} N_{n,j}(2\,\cdot), \tag{1.4}$$

where the transition matrix $(a_{ij})_{i,j\in\mathbb{Z}}$ is of the form

$$a_{ij} = \begin{cases} a_{j-2i-1}, & i \geq 0, \\ \delta_{i,j-1}, & i \leq -n, \end{cases} \tag{1.5}$$

$a_j = 2^{-n+1}\binom{n}{j}$, $j = 0, 1, \dots, n$, is the low-pass filter sequence for the uniform B-spline of order n and for $-n \le i \le 0$, a_{ij} are discrete B-splines on the hybrid of uniform and geometric knots with $a_{ij} = 0$ for $j \notin \{i+1, ..., 2i+n+1\}$. In Section 4 we extend this concept to the construction of hybrid scaling sequences from any given scaling function ϕ with finite low-pass filter (a_j). In the construction, the hybrid scaling functions inherit the smoothness of ϕ, and the entries a_{ij}, $-n \le i \le 0$, $j = i+1, ..., 2i+n+1$, of the transition matrix provide parameters that allow their construction with the desired properties, such as Riesz basis, orthonormality and polynomial reproduction. A necessary and sufficient condition for local linear independence of hybrid scaling functions is given in Section 4, from which a necessary condition for the stability of the hybrid scaling function basis is derived. An analysis of their polynomial reproduction properties and the construction of hybrid scaling functions that reproduce polynomials up to certain degree are given in Section 5.

We remark that the analysis and construction of normalized tight frames of hybrid splines is much simpler and neater than that of the Riesz bases of wavelets and is treated separately in [10].

2. Cardinal Splines on the Half Line

Take an integer $n \ge 1$. Let $\mathcal{S}_n$ denote the space of all spline functions of order n, i.e. C^{n-2} piecewise polynomials of degree $n-1$, with knots in $\mathbb{Z}$. Let $V_0 := \{f \in \mathcal{S}_n : \mathrm{supp}(f) \subset [0,\infty)\} \cap L^2((0,\infty))$, and $V_j := \{f(2^j\,\cdot) : f \in V_0\}$, $j \in \mathbb{Z}$. Let $N_n \in \mathcal{S}_n$ denote the B-spline with support $[0,n]$. Then it is known that $\{N_n(\,\cdot - j) : j = 0, 1, ...\}$ forms a Riesz basis for V_0. Moreover $\{V_j : j \in \mathbb{Z}\}$ forms a multiresolution approximation of $L^2((0,\infty))$, i.e. $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$, and $\mathrm{cl}(\bigcup_{j=0}^{\infty} V_j) = L^2((0,\infty))$.

Now let W_0 denote the orthogonal complement of V_0 in V_1, and W^* the orthogonal complement of $\mathcal{S}_n \cap L^2(\mathbb{R})$ in $\{f(2\,\cdot) : f \in \mathcal{S}_n \cap L^2(\mathbb{R})\}$. It is known ([7], [11]) that there is a function ψ in W^* with support $[0, 2n-1]$ such that $\{\psi(\,\cdot - j) : j \in \mathbb{Z}\}$ forms a Riesz basis for W^*, and there are no nontrivial functions in W^* with shorter support. We shall construct a Riesz basis for W_0. Let $\Phi := \{f \in W_0 : \mathrm{supp}(f) \subset [0, 2n-2]\}$.

Lemma 2.1.

$$dim(\Phi) = n-1.$$

Proof. Let $S = \{f \in V_1 : \mathrm{supp}(f) \subset [0, 2n-2]\}$. Then $\dim(S) = 3n-3$. It is known that

$$\{0\} = S \cap W^* = \{f \in S : f \perp N_n(\,\cdot - j),\ j = -n+1, ..., 2n-3\}.$$

So the linear transformation $(\lambda_j)_{j=-n+1}^{2n-3} : S \to \mathbb{R}^{3n-3}$, where $\lambda_j(f) = \int_{-\infty}^{\infty} f N_n(\,\cdot - j)$, $j = -n+1, ..., 2n-3$, has full rank. Hence $\Lambda := (\lambda_j)_{j=0}^{2n-3} : S \to \mathbb{R}^{2n-2}$ is of rank $2n-2$. Since $\Phi = \ker(\Lambda)$, it follows that $\dim(\Phi) = \dim(S) - (2n-2) = n-1$. ♠

Lemma 2.2. *W_0 is the closed linear span of Φ and $\psi(\,\cdot - j)$, $j \geq 0$, in $L^2((0,\infty))$.*

Proof. Let $\widetilde{W}$ denote the closed linear span of Φ and $\psi(\,\cdot - j)$, $j \geq 0$, in $L^2((0,\infty))$. Clearly $\widetilde{W} \subset W_0$. Take $N \geq 3n$. Let $U := \{f \in V_0 : \mathrm{supp}(f) \subset [0,N]\}$ and $\widehat{W} = \{f \in V_1|_{[0,N]} : f \perp U\}$. Since $\dim(U) = N - n + 1$ and $\dim(V_1|_{[0,N]}) = 2N$, we have $\dim(\widehat{W}) = N + n - 1$.

Note that Φ and $\psi(\,\cdot - j)$, $j \geq 0$, are locally linearly independent on $[0,N]$. For if $\phi \in \Phi$ and

$$\phi(x) + \sum_{j=0}^{N-1} a_j \psi(x-j) = 0, \quad 0 \leq x \leq N,$$

then

$$\sum_{j=0}^{N-1} a_j \psi(x-j) = 0, \quad 2n-2 \leq x \leq N.$$

But it is known (see [11]) that $\psi(\,\cdot - j)$, $j \in \mathbb{Z}$, are locally linearly independent on $[2n-2, N]$, and hence $a_j = 0$, $j = 0, ..., N-1$. Thus $\dim(\widetilde{W}|_{[0,N]}) = \dim(\Phi) + N = N + n - 1$, by Lemma 2.1. Since $\widetilde{W}|_{[0,N]} \subset \widehat{W}$, we have $\widetilde{W}|_{[0,N]} = \widehat{W}$.

Now take $f \in W_0$. Since $f|_{[0,N]} \in \widehat{W}$, there are unique $\phi \in \Phi$ and $c_0, ..., c_{N-1} \in \mathbb{R}$ with

$$f(x) = \phi(x) + \sum_{j=0}^{N-1} c_j \psi(x-j), \quad 0 \leq x \leq N.$$

Then letting $N \to \infty$ gives

$$f(x) = \phi(x) + \sum_{j=0}^{\infty} c_j \psi(x-j), \quad x \geq 0,$$

for $c_j \in \mathbb{R}$, $j \geq 0$. Thus $f \in \widetilde{W}$ and so $\widetilde{W} = W_0$. ♠

We shall need the following result in [18].

Lemma 2.3. *Let U and V be closed linear subspaces of a Hilbert space and let X and Y be Riesz bases for U and V respectively. Then $U + V$ is closed and $U \cap V = \{0\}$ if and only if $X \cup Y$ is a Riesz basis for $\overline{U \oplus V}$.*

Theorem 2.4. *Suppose that $\phi_1, ..., \phi_{n-1}$ is a basis for Φ. Then $\phi_1, ..., \phi_{n-1}$, $\psi(\,\cdot - j)$, $j \geq 0$, is a Riesz basis for W_0.*

Proof. Let V be the closed linear span of $\psi(\,\cdot - j)$, $j \geq 0$, in $L^2(\mathbb{R})$. By Lemma 2.2, $W_0 = \Phi + V$. It is known ([7], [11]) that $\Phi \cap V = \{0\}$, and $\{\psi(\,\cdot - j) : j \in \mathbb{Z}\}$ forms a Riesz basis for V. The result then follows from Lemma 2.3. ♠

For $j \in \mathbb{Z}$, let $W_j := \{f(2^j\ \cdot) : f \in W_0\}$. Then $W_j \perp W_k$, $j \neq k$, and for $k \leq \ell$, $V_\ell = V_k + \sum_{j=k}^{\ell-1} W_j$. It follows that

$$L^2((0,\infty)) = V_0 + \sum_{j=0}^{\infty} W_j = \sum_{j=-\infty}^{\infty} W_j.$$

From Theorem 2.4 we can then deduce the following

Corollary 2.5. *Take any basis* $\{\phi_1, ..., \phi_{n-1}\}$ *of* Φ. *Then the following are Riesz bases for* $L^2((0,\infty))$:

(a) $\{N(\ \cdot - j) :\ j \geq 0\} \cup \{2^{k/2}\phi_j(2^k\ \cdot) :\ j = 1, ..., n-1,\ k \geq 0\} \cup \{2^{k/2}\psi(2^k(\ \cdot - j)) : j, k \geq 0\}$,

(b) $\{2^{k/2}\phi_j(2^k\ \cdot) :\ j = 1, ..., n-1 :\ k \in \mathbb{Z}\} \cup \{2^{k/2}\psi(2^k(\ \cdot - j)) :\ j \geq 0,\ k \in \mathbb{Z}\}$.

3. Hybrid Uniform-Geometric Splines

Take an integer $n \geq 1$. Let $\widetilde{V}_0 \equiv \widetilde{V}_{n,0}$ denote the space of all spline functions of order n in $L^2((0,\infty))$ with knots $x_j \equiv x_{0\,j}$, $j \in \mathbb{Z}$, as in (1.1). For $j \in \mathbb{Z}$, let $N_{n,j}$ denote the B-spline in $\widetilde{V}_0$ with support $[x_j, x_{j+n}]$, normalized so that $\|N_{n,j}\|_2 = 1$. Since $x_j = j+1$, for $j \geq 0$,

$$N_{n,j} = N_n(\ \cdot - j - 1), \quad j \geq 0.$$

Note that here and elsewhere we identify a function on $\mathbb{R}$ with support in $(0,\infty)$ with its restriction to $(0,\infty)$. Also writing $\widetilde{N}_n := N_{n,-n}$ we see that

$$N_{n,j} = 2^{-(j+n)/2}\widetilde{N}_n(2^{-(j+n)}\ \cdot), \quad j \leq -n-1.$$

It is known that $\{N_{n,j} : j \in \mathbb{Z}\}$ forms a Riesz basis for $\widetilde{V}_0$.

Letting $\widetilde{V}_j \equiv \widetilde{V}_{n,j} := \{f(2^j\ \cdot) : f \in \widetilde{V}_0\}$, $j \in \mathbb{Z}$, we see that $\{\widetilde{V}_j : j \in \mathbb{Z}\}$ forms a multiresolution approximation for $L^2((0,\infty))$. Now write $\widetilde{W}_0 = \{f(\ \cdot - 1) : f \in W_0\}$ and take any basis $\{\phi_1, ..., \phi_{n-1}\}$ of Φ as in the preceding section.

Theorem 3.1. $\widetilde{V}_1 = \widetilde{V}_0 \oplus \widetilde{W}_0$, *and a Riesz basis for* $\widetilde{V}_1$ *is given by* $N_{n,j}$, $j \in \mathbb{Z}$, $\phi_1(\ \cdot - 1), ..., \phi_{n-1}(\ \cdot - 1)$, $\psi(\ \cdot - j)$, $j \geq 1$.

Proof. Take $f \in \widetilde{V}_1$. Then $f = f_1 + f_2 + f_3$, where $f_1,\ f_2,\ f_3 \in \widetilde{V}_1$ and $\mathrm{supp}(f_1) \subset (0,1]$, $\mathrm{supp}(f_2) \subset [x_{-n+1}, \frac{1}{2}x_n]$, $\mathrm{supp}(f_3) \subset [1,\infty)$. Let $X := \{f \in \widetilde{V}_1 : \mathrm{supp}(f) \subset [x_{-n+1}, x_{n-1}]\}$. Then $X = \mathrm{span}\{N_{n,-n+2}(2\ \cdot), \ldots, N_{n,n-1}(2\ \cdot)\}$ and so is of dimension $2n-2$. Further, the functions $N_{n,-n+1}, ..., N_{n,-1}, N_{n,1}(2\ \cdot), ..., N_{n,n-1}(2\ \cdot)$ lie in X and, since the left-hand ends of their supports are strictly increasing, are linearly independent, and so they span X. Since $f_2 \in X$ and $\mathrm{supp}(N_{n,j}(2\ \cdot)) \subset [1,\infty)$, $j = 1, ..., n-1$, we can write $f_2 = f_4 + f_5$, where f_4 lies in the span of $N_{n,-n+1}, ..., N_{n,-1}$, and $\mathrm{supp}(f_5) \subset [1,\infty)$.

Putting $g = f_1 + f_4$, $h = f_5 + f_3$, we have $f = g + h$, where $g \in \widetilde{V}_0$, $\mathrm{supp}(h) \subset [1, \infty)$. Now $h(\,\cdot + 1) \in V_1$ and so $h(\,\cdot + 1) = F + G$, where $F \in V_0$, $G \in W_0$. Thus $h = h_1 + h_2$, where $h_1 \in \widetilde{V}_0$, $h_2 \in \widetilde{W}_0$. Hence $f = (g + h_1) + h_2$, where $g + h_1 \in \widetilde{V}_0$, $h_2 \in \widetilde{W}_0$. So $\widetilde{V}_1 = \widetilde{V}_0 + \widetilde{W}_0$.

Now take $f \in \widetilde{V}_0 \cap \widetilde{W}_0$. Since $f \in \widetilde{W}_0$, $\mathrm{supp}(f) \subset [1, \infty)$ and $f(\,\cdot + 1) \in W_0$. Then $f(\,\cdot + 1) \perp V_0$ and so $f \perp \widetilde{V}_0$. Thus $f = 0$ and so $\widetilde{V}_1 = \widetilde{V}_0 \oplus \widetilde{W}_0$.

The final part of the result follows from Lemma 2.3 ♠

Note that $\widetilde{W}_0$ is not orthogonal to $\widetilde{V}_0$. Indeed we now show that if W denotes the orthogonal complement of $\widetilde{V}_0$ in $\widetilde{V}_1$, then for $n \geq 2$ it is not possible to construct a Riesz basis for W comprising compactly supported functions.

Take $j \leq 0$, $k \geq 2n$ and consider $X := \{f \in W : \mathrm{supp}(f) \subset [x_j, x_k]\} = \{f \in \widetilde{V}_1 : \mathrm{supp}(f) \subset [x_j, x_k], f \perp N_{n,\ell}, \ell = -n + j + 1, \ldots, k - 1\}$. From the relation

$$[x_\ell, ..., x_{\ell+n}]g = \frac{1}{n!}\int_{-\infty}^{\infty} N_{n,\ell}(x) g^{(n)}(x)dx, \quad \ell \in \mathbb{Z},$$

between the "normalized" divided difference on the left and the corresponding B-spline on the right, we know that $f \in X$ if and only if $f = g^{(n)}$, where g is a spline function of order $2n$ with knots at $x_{1,j}$, supported in $[x_j, x_k]$ and $g(x_{j+1}) = \cdots = g(x_{k-1}) = 0$. Thus $\dim(X) = (2k - 2n + 1 - j) - (k - j - 1) = k - 2n + 2$, which is independent of j. So elements of X have support in $[x_0, x_k]$. Thus any function in W with compact support must have support in $[1, \infty)$ and so lie in $\overline{\mathrm{span}}\{\psi(\,\cdot - j - 1) : j \geq 0\}$, where ψ is the uniform spline wavelet of order n with compact support on $[0, 2n - 1]$.

Now for $n \geq 2$ and for any basis $\{\phi_1, ..., \phi_{n-1}\}$ of Φ, $\phi_1(\,\cdot - 1), ..., \phi_{n-1}(\,\cdot - 1)$ lie in $\widetilde{V}_1$, but by Theorem 2.4, they do not lie in $\widetilde{V}_0 \oplus \overline{\mathrm{span}}\{\psi(\,\cdot - j - 1) : j \geq 0\}$. Thus W cannot be the closed linear span of compactly supported functions.

Our next result gives a Riesz basis for W of functions with bounded support.

Theorem 3.2. *Let $U = \{f \in W : supp(f) \subset (0, 2n - 1]\}$. Then $dim(U) = n - 1$ and $W = U \oplus \overline{span}\{\psi(\,\cdot - j) : j \geq 1\}$. If $\psi_1, \ldots, \psi_{n-1}$ comprise a basis for U, then a Riesz basis for W is given by $\psi_1, \ldots, \psi_{n-1}, \psi(\,\cdot - j),\ j \geq 1$.*

Proof. By Theorem 3.1, any function f in $\widetilde{V}_1$ can be written as $f = f_0 + f_1 + f_2$, where $f_0 \in \widetilde{V}_0$, $f_2 \in \overline{\mathrm{span}}\{\psi(\,\cdot - j) : j \geq 1\}$ and $f_1 \in \widetilde{V}_1$ satisfies $\mathrm{supp}(f_1) \subset [1, 2n - 1]$, $f_1 \perp N_{n,j},\ j \geq 0$. Now we can write $f_1 = g + h$, where $g \in \widetilde{V}_0$, $h \in U$. Thus

$$\widetilde{V}_1 = \widetilde{V}_0 \oplus U \oplus \overline{\mathrm{span}}\{\psi(\,\cdot - j) : j \geq 1\}.$$

Since $U \subset W$ and $\psi(\,\cdot - j) \in W,\ j \geq 1$,

$$W = U \oplus \overline{\mathrm{span}}\{\psi(\,\cdot - j) : j \geq 1\}.$$

Now let P denote the projection from $L^2((0,\infty))$ onto $\overline{\text{span}}\{\psi(\,\cdot - j) : j \le -n\}$. Let $S = \text{span}\{N_{n,j}(2\,\cdot) : 2-n \le j \le 3n-3\}$ and put $T = (I-P)S$. Since $N_{n,j}(2\,\cdot),\ j \le 3n-3$, are linearly independent and $N_{n,j}(2\,\cdot) = 2^{-1/2}N_{n,j-1},\ j \le 1-n$, a basis for T is $\{(I-P)N_{n,j}(2\,\cdot) : 2-n \le j \le 3n-3\}$.

Now $U = \{f \in T : f \perp N_{n,j},\ -n+1 \le j \le 2n-3\}$. For $-n+1 \le j \le 2n-3$, let $\lambda_j : L^2((0,\infty)) \to \mathbb{R}$ denote the linear functional $\lambda_j(f) = \int_0^\infty f N_{n,j}$. As in the proof of Lemma 2.1, the linear functionals $\lambda_{-n+1},\dots,\lambda_{2n-3}$ are linearly independent when restricted to the space $\{f \in \widetilde{V}_1 : \text{supp}(f) \subset [1, 2n-1]\}$, which equals $\text{span}\{N_{n,1}(2\,\cdot),\dots,N_{n,3n-3}(2\,\cdot)\} = \text{span}\{(I-P)N_{n,1}(2\,\cdot),\dots,(I-P)N_{n,3n-3}(2\,\cdot)\}$. Thus $\lambda_{-n+1},\dots,\lambda_{2n-3}$ are linearly independent when restricted to the larger space T, and hence $\dim(U) = \dim(T) - (3n-3) = n-1$.

Finally, let $\psi_1,\dots,\psi_{n-1}$ denote a basis for U. Then applying Lemma 2.3, as in the proof of Theorem 2.4, a Riesz basis for W is given by $\psi_1,\dots,\psi_{n-1},\psi(\,\cdot - j),\ j \ge 1$. ♠

As an example, take $n = 2$. Then $\dim(U) = 1$ and we may write

$$\psi_1 = \sum_{j=-\infty}^{-1} c_j N_{2,j} + \sum_{j=0}^{2} c_j N_{2,j+1}(2\,\cdot),$$

where $\psi_1 \perp N_{2,j},\ j \le 1$. Putting $\lambda = \int_0^\infty N_{n,-2}N_{n,-1}$, this gives

$$\lambda c_{j-1} + c_j + \lambda c_{j+1} = 0, \quad j \le -2.$$

Thus we may choose $c_j = \alpha^j,\ j \le -1$, where α denotes the root of $\lambda x^2 + x + \lambda = 0$ with $\alpha > -1$. The coefficients c_0, c_1, c_2 can then be determined from the three linear equations $\psi_1 \perp N_{2,j},\ j = -1, 0, 1$.

The Riesz bases in Theorem 3.1 for $\widetilde{V}_0$ and $\widetilde{W}_0$ are 'almost orthogonal' in the following sense. We have

$$\begin{aligned} \psi(\,\cdot - j) &\perp \widetilde{V}_0, \quad j \ge 1, \\ \phi_k(\,\cdot - 1) &\perp N_{n,j}, \quad k = 1,\dots,n-1,\ j \le -n \ \text{ and } \ j \ge 0. \end{aligned}$$

So the only functions which are not orthogonal are $\phi_1(\,\cdot - 1),\dots,\phi_{n-1}(\,\cdot - 1)$ and $N_{n,-n+1},\dots,N_{n,-1}$. By choosing a suitable basis $\phi_1,\dots,\phi_{n-1}$ for Φ, we can gain further orthogonalities among these functions, as we shall see later.

Now for $j \in \mathbb{Z}$, let $\widetilde{W}_j = \{f(2^j\,\cdot) : f \in \widetilde{W}_0\}$. Take integers $j < k$. Since $\widetilde{W}_0 \perp \{f \in \widetilde{V}_0 : \text{supp}(f) \subset [1,\infty)\}$, we have $\widetilde{W}_k \perp \{f \in \widetilde{V}_k : \text{supp}(f) \subset [2^{-k},\infty)\}$. Now for $g \in \widetilde{W}_j$, $\text{supp}(g) \subset [2^{-j},\infty) \subset [2^{-k},\infty)$ and $g \in \widetilde{V}_{j+1} \subset \widetilde{V}_k$. So $g \perp \widetilde{W}_k$. Thus

$$\widetilde{W}_j \perp \widetilde{W}_k, \quad j,k \in \mathbb{Z},\ j \ne k.$$

Since for $k \in \mathbb{Z}$, $2^{k/2}\phi_j(2^k \ \cdot -1)$, $j = 1,..,n-1$, $2^{k/2}\psi(2^k \ \cdot -j)$, $j \geq 1$, form a Riesz basis for $\widetilde{W}_k$, with Riesz bounds independent of k, a Riesz basis for $\oplus \sum_{k=-\infty}^{\infty} \widetilde{W}_k$ is given by $2^{k/2}\phi_j(2^k \ \cdot -1), j = 1,...,n-1$, $k \in \mathbb{Z}$, $2^{k/2}\psi(2^k \ \cdot \ -j)$, $j \geq 1$, $k \in \mathbb{Z}$.

From Theorem 3.1 we see that for any $k \in \mathbb{Z}$, $k \geq 1$,

$$\widetilde{V}_k = \widetilde{V}_0 \oplus \sum_{j=0}^{k-1} \widetilde{W}_j,$$

and a Riesz basis for $\widetilde{V}_k$ is $N_{n,j}$, $j \in \mathbb{Z}$, $2^{\ell/2}\phi_j(2^\ell \ \cdot -1)$, $j = 1,...,n-1$, $\ell = 0,...,k-1$, $2^{\ell/2}\psi(2^\ell \ \cdot -j)$, $j \geq 1$, $\ell = 0,...,k-1$. Letting $k \to \infty$ gives

$$L^2((0,\infty)) = \mathrm{cl}(\widetilde{V}_0 \oplus \sum_{j=0}^{\infty} \widetilde{W}_j).$$

We conjecture that

$$L^2((0,\infty)) = \widetilde{V}_0 \oplus \sum_{j=0}^{\infty} \widetilde{W}_j \tag{3.1}$$

and a Riesz basis for $L^2((0,\infty))$ is $N_{n,j}$, $j \in \mathbb{Z}$, $2^{k/2}\phi_j(2^k \ \cdot -1)$, $j = 1,...,n-1$, $k \geq 0$, $2^{k/2}\psi(2^k \ \cdot -j)$, $j \geq 1$, $k \geq 0$. If this is true, then letting $k \to -\infty$ in $L^2((0,\infty)) = \widetilde{V}_k \oplus \sum_{j=k}^{\infty} \widetilde{W}_j$ gives

$$L^2((0,\infty)) = \widetilde{V}_{-\infty} \oplus \sum_{j=-\infty}^{\infty} \widetilde{W}_j,$$

where $\widetilde{V}_{-\infty} = \bigcap_{k=-\infty}^{0} \widetilde{V}_k$, which is the space of all geometric spline functions of order n in $L^2((0,\infty))$ with knots 2^j, $j \in \mathbb{Z}$. A Riesz basis for $\widetilde{V}_{-\infty}$ is $2^{j/2}\widetilde{N}_n(2^j \ \cdot)$, $j \in \mathbb{Z}$. It would follow that a Riesz basis for $L^2((0,\infty))$ is $2^{j/2}\widetilde{N}_n(2^j \ \cdot)$, $j \in \mathbb{Z}$, $2^{k/2}\phi_j(2^k \ \cdot -1)$, $j = 1,...,n-1$, $k \in \mathbb{Z}$, $2^{k/2}\psi(2^k \ \cdot -j)$, $j \geq 1$, $k \in \mathbb{Z}$.

We now return to the choice of basis $\phi_1,...,\phi_{n-1}$ for Φ. For $j = 1,...,n-1$, let $\Phi_j = \{f \in \widetilde{V}_1 : \mathrm{supp}(f) \subset [1, n+j],\ f \perp N_{n,\ell}, \ell \geq 1-j\}$. Thus $\Phi_j \subset \Phi(\ \cdot -1) := \{f(\ \cdot -1) : f \in \Phi\}$, $j = 1,...,n-1$. For $j = 1,...,n-1$, $\ell \geq n+j-1$, $\mathrm{supp}(N_{n,\ell}) \subset [n+j, \infty)$, and so trivially $N_{n,\ell} \perp \Phi_j$. Thus Φ_j comprises all functions in the $(n+2j-1)$ dimensional space $\{f \in \widetilde{V}_1 : \mathrm{supp}(f) \subset [1, n+j]\}$ satisfying the $n+2j-2$ conditions $f \perp N_{n,\ell}$, $\ell = 1-j,...,n+j-2$. To show that $\dim(\Phi_j) = 1$ we shall use the following lemma.

Lemma 3.3. *Take $1 \leq j \leq n-1$ and let $f \in \widetilde{V}_1$ have support in $[1, n+j-\frac{1}{2}]$ and $f \perp N_{n,\ell}$, $\ell = 1-j,...,n+j-2$. Then $f = 0$.*

Proof. Define g so that $g^{(n)} = f$ and g has support in $(-\infty, n+j-\frac{1}{2}]$. Then

$$[x_\ell,...,x_{\ell+n}]g = \int_{-\infty}^{\infty} N_{n,\ell}\, g^{(n)} = 0, \quad \ell = 1-j,...,n+j-2.$$

Since $x_{n+j-1} = n+j > n+j-\frac{1}{2}$, $g(x_\ell) = 0$ for $\ell \geq n+j-1$. Since $[x_{n+j-2}, ..., x_{2n+j-2}]g = 0$, we have $g(x_{n+j-2}) = 0$. Continuing in this way gives $g(x_\ell) = 0,\ \ell = 1-j, ..., n+j-2$.

Now for $k \in \mathbb{Z}$, let M_k denote the B-spline of order $2n$ with knots $\frac{1}{2}k, \frac{1}{2}(k+1)..., \frac{1}{2}(k+2n)$. Then we can write

$$g = \sum_{k=-\infty}^{2j-1} c_k M_k,$$

for some constants c_k. First suppose that g does not vanish on any non-trivial interval in $(-\infty, n+j-\frac{1}{2}]$. Since g coincides with a polynomial of degree $n-1$ on $(-\infty, 1]$, there is a polynomial p of degree $n-1$ such that $c_k = p(k),\ k \leq 1$. So the number of sign changes in the sequence $\{c_k : k \leq 1\}$ is at most $n-1$. Hence the number of sign changes in the sequence $\{c_k : k \leq 2j-1\}$ is at most $n+2j-3$. Since g vanishes at $n+2j-2$ points, this contradicts the variation diminishing property of B-splines.

Now suppose that g is non-trivial but vanishes on some non-trivial internal in $(-\infty, n+j-\frac{1}{2}]$. We note that for $\ell = 0, ..., n+j-2$,

$$x_\ell = \ell + 1 \in \left(\frac{\ell+j+1-n}{2}, \frac{\ell+j+1+n}{2}\right),$$

while for $1-j \leq \ell \leq -1$, $x_\ell < 1 < \frac{\ell+j+1+n}{2}$ and $x_\ell \geq \frac{\ell}{2}+1 > \frac{\ell+j+1-n}{2}$. So for $\ell = 1-j, ..., n+j-2$, $M_{\ell+j+1-n}(x_\ell) > 0$. If g vanishes on some interval in $(-\infty, 1)$ then $g(x) = 0$ for $x \in (-\infty, 1)$. It follows from the Schoenberg-Whitney Theorem that $g = 0$, and hence $f = 0$. Suppose that g vanishes on $(\frac{k}{2}, \frac{k+1}{2})$ for some $k \in \mathbb{Z}$, $1 \leq \frac{k}{2} < n+j-\frac{1}{2}$, and does not vanish on any non-trivial interval in $(-\infty, \frac{k}{2}]$. Again from the Schoenberg-Whitney Theorem g vanishes on $(\frac{k}{2}, \infty)$. Then our previous argument leads to a contradiction, which shows that $g = 0$. Hence $f = 0$. ♠

Theorem 3.4. *There is a basis $\phi_1, ..., \phi_{n-1}$ for Φ, unique up to normalization, such that for $j = 1, ..., n-1$, supp$(\phi_j(\,\cdot - 1)) \subset [1, n+j]$ and $\phi_j(\,\cdot - 1) \perp N_{n,\ell}$, $\ell \geq 1-j$. Moreover, $\phi_j(\,\cdot - 1)$ does not have support in $[1, n+j-\frac{1}{2}]$ or in $[\frac{3}{2}, n+j]$.*

Proof. For $\ell \in \mathbb{Z}$, define $\lambda_\ell : \widetilde{V}_1 \to \mathbb{R}$ by $\lambda_\ell(f) = \int_{-\infty}^{\infty} f N_{n,\ell}$. Take $1 \leq j \leq n-1$. By Lemma 3.3, $\lambda_{1-j}, ..., \lambda_{n+j-2}$ are linearly independent over $\{f \in \widetilde{V}_1 : \mathrm{supp}(f) \subset [1, n+j-\frac{1}{2}]\}$. Thus they are linearly independent over $\{f \in \widetilde{V}_1 : \mathrm{supp}(f) \subset [1, n+j]\}$. Hence $\dim(\Phi_j) = 1$. By Lemma 3.3, any non-trivial element of Φ_j cannot have support in $[1, n+j-\frac{1}{2}]$, and similarly it cannot have support in $[\frac{3}{2}, n+j]$. Taking $\phi_j(\,\cdot - 1) \in \Phi_j$, $j = 1, ..., n-1$, gives the result. ♠

For $n = 2$, direct calculation shows that we can define ϕ_1 as the unique element of $\widetilde{V}_1$ with support $[0, 2]$ satisfying

$$\phi_1(1/2) = 9, \quad \phi_1(1) = -6, \quad \phi_1(3/2) = 1.$$

As an application, we consider the problem of smoothing by hybrid splines that preserves features. The mask $(a_{ij}) \equiv (a^n_{ij})$ in the two-scale relation (1.4) and (1.5) provides the low-pass filters for the smoothing process. They are discrete B-splines (see [8]) on the hybrid geometric-uniform mesh (1.1) and are generated by the recursive relation,

$$a^n_{ij} = \left(\frac{x_{0,i+n} - x_{1,j+n-1}}{x_{0,i+n} - x_{0,i+1}}\right) a^{n-1}_{i+1,j} + \left(\frac{x_{1,j+n-1} - x_{0,i}}{x_{0,i+n-1} - x_{0,i}}\right) a^{n-1}_{ij}, \tag{3.2}$$

with

$$a^1_{ij} = \begin{cases} 1, & x_{0,i} \le x_{1,j} < x_{0,i+1}, \\ 0, & \text{otherwise}, \end{cases} \qquad i, j \in \mathbb{Z}.$$

Given a one-dimensional signal, (s^0_i), we assume that a point of interest, s^0_{-n}, has been detected by some feature detector. The point is identified as a fovea and the low-pass filtering,

$$\begin{aligned} s^{\ell+1}_i &= \frac{\sum_{j=i+1}^{2i+n+1} a_{ij} s^\ell_{j-1}}{\sum_{j=i+1}^{2i+n+1} a_{ij}}, \quad i = -n, -n+1, \dots, \\ s^{\ell+1}_i &= \frac{\sum_{j=-2n-i+1}^{-3n-2i+1} a_{-i-2n,\, j} s^\ell_{-j-2n+1}}{\sum_{j=-2n-i+1}^{-3n-2i+1} a_{-i-2n,\, j}}, \quad i = -n, -n-1, \dots, \end{aligned}$$

is applied to obtain smooth versions (s^ℓ_i), $\ell = 1, 2, \dots$, that preserve the feature at s^0_{-n}.

The table below shows the entries, a_{ij}, $-5 \le i \le 0$, $-4 \le j \le 6$, of the transition matrix for the 5-th order hybrid B-splines, $N_{5,i}$, $i \in \mathbb{Z}$. The rows of the matrix are the corresponding low-pass filters.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{4}{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{11}{15} & \frac{6}{11} & \frac{6}{55} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{11} & \frac{51}{77} & \frac{5}{14} & \frac{1}{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{8}{35} & \frac{65}{112} & \frac{69}{112} & \frac{5}{16} & \frac{1}{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{5}{16} & \frac{10}{16} & \frac{10}{16} & \frac{5}{16} & \frac{1}{16} \end{pmatrix}.$$

Figure 1(a) below shows a one-dimensional digital signal with a singularity, and Figures 1(b), 1(c) and 1(d) show respectively the results of first, second and third level of smoothing that preserve the singularity.

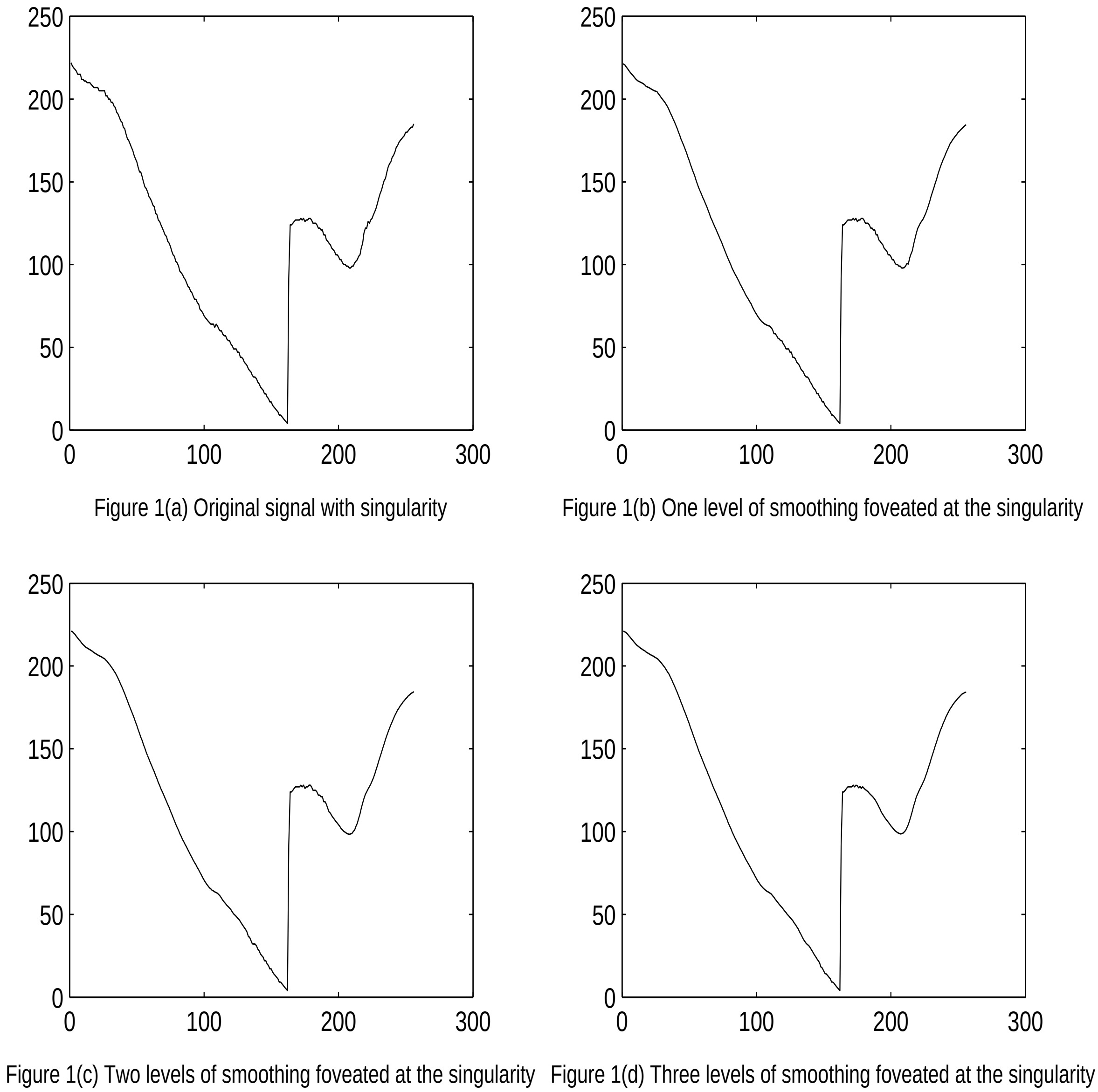

FIGURE 1. Smoothing that preserves singularity

Figure 2(a) is an image where the feature of interest is the lower edge of the middle beam across the building, and Figures 2(b), 2(c) and 2(d) show respectively the results of the first, second and third order of smoothing that preserve the edge.

Figure 2(a) Original image

Figure 2(b) First level smoothing

Figure 2(c) Second level smoothing

Figure 2(d) Third level smoothing

FIGURE 2. Smoothing that preserves edge

4. Hybrid Refinable Functions

Let ϕ be a refinable function with support on $[0, n]$ satisfying the equation

$$\phi(x) = \sum_{j=0}^{n} a_j \phi(2x - j), \quad x \in \mathbb{R}, \tag{4.1}$$

for numbers $a_0, ..., a_n$ with $a_0 a_n \neq 0$, and $\sum_{j\in\mathbb{Z}} a_{2j} = \sum_{j\in\mathbb{Z}} a_{2j+1} = 1$, where $a_j = 0$ for $j < 0$ or $j > n$. Now take numbers $a_{ij},\ i, j \in \mathbb{Z}$, where

$$a_{ij} = \begin{cases} a_{j-2i-1}, & i \geq 0, \\ \delta_{i,j-1}, & i \leq -n, \end{cases} \tag{4.2}$$

and for $-n \leq i \leq 0$, $a_{ij} = 0$ for $j \notin \{i+1, ..., 2i+n+1\}$. We shall consider the *hybrid refinement equation*

$$\phi_i(x) = \sum_{j=-\infty}^{\infty} a_{ij}\phi_j(2x), \quad x > 0, \ i \in \mathbb{Z}. \tag{4.3}$$

We first note that (4.3) is satisfied for $i \geq 0$ by

$$\phi_j = \phi(\,\cdot - j - 1), \quad j \geq 0, \tag{4.4}$$

since in this case, for $x > 0$,

$$\begin{aligned} \phi_i(x) &= \phi(x - i - 1) \\ &= \sum_{j=-\infty}^{\infty} a_j \phi(2x - 2i - 2 - j) \qquad \text{by (4.1)} \\ &= \sum_{j=-\infty}^{\infty} a_j \phi_{2i+j+1}(2x) \\ &= \sum_{j=-\infty}^{\infty} a_{ij}\phi_j(2x). \end{aligned}$$

Next we note that for $-n \leq i \leq -1$,

$$\phi_i(x) = \sum_{j=i+1}^{2i+n+1} a_{ij}\phi_j(2x), \quad x > 0.$$

Thus ϕ_i is defined in terms of $\phi_{i+1}(2\ \cdot), ..., \phi_{2i+n+1}(2\ \cdot)$ and so recursively ϕ_i is defined uniquely in terms of $\phi(2^j\ \cdot - k)$, $1 \leq j \leq -i$, $k \geq 1$. Also for $i \leq -n$, $\phi_i = \phi_{i+1}(2\ \cdot)$. In particular we see that for all $i \in \mathbb{Z}$, ϕ_i has the same regularity as ϕ. We shall say that the refinable functions $(\phi_i)_{i\in\mathbb{Z}}$ with mask $(a_{ij})_{i,j\in\mathbb{Z}}$ *extend* the refinable function ϕ with mask $(a_i)_0^n$.

Letting

$$x_j \equiv x_{0j} := \begin{cases} 2^j, & j \leq -1, \\ j+1, & j \geq 0, \end{cases}$$

as before, we see that ϕ_i has support $[x_i, x_{i+n}]$, $i \in \mathbb{Z}$.

Since

$$\sum_{j=-\infty}^{\infty} a_{2j} = \sum_{j=-\infty}^{\infty} a_{2j+1} = 1,$$

and for $i \leq -1$, $a_{ij} = 0$ for $j \geq n$, we have for $j \geq n$,

$$\sum_{i=-\infty}^{\infty} a_{ij} = \sum_{i=-\infty}^{\infty} a_{j-2i-1} = 1.$$

We shall assume

$$\sum_{i=-\infty}^{\infty} a_{ij} = 1, \quad j \in \mathbb{Z}. \tag{4.5}$$

It is known that we may choose a solution ϕ of (4.1) to satisfy

$$\sum_{j=-\infty}^{\infty} \phi(x-j) = 1, \quad x \in \mathbb{R}. \tag{4.6}$$

Now from (4.3), for $x > 0$,

$$\begin{aligned}
\sum_{i=-\infty}^{\infty} \phi_i(x) &= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_{ij}\phi_j(2x) \\
&= \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} a_{ij}\phi_j(2x) \\
&= \sum_{j=-\infty}^{\infty} \phi_j(2x), \qquad (4.7)
\end{aligned}$$

by (4.5). For $x \geq n$,

$$\begin{aligned}
\sum_{i=-\infty}^{\infty} \phi_i(x) &= \sum_{i=0}^{\infty} \phi(x-i-1) \\
&= \sum_{i=-\infty}^{\infty} \phi(x-i-1) = 1,
\end{aligned}$$

by (4.6). Then for any $x > 0$, we may choose $k \in \mathbb{Z}$ with $2^{-k}n \leq x$, and apply (4.7) iteratively to give

$$\sum_{i=-\infty}^{\infty} \phi_i(x) = \sum_{i=-\infty}^{\infty} \phi_i(2^k x) = 1.$$

Thus

$$\sum_{i=-\infty}^{\infty} \phi_i(x) = 1, \quad x > 0. \tag{4.8}$$

We also note that by (4.3)

$$\begin{aligned}
\sum_{i=-\infty}^{\infty} c_i\phi_i &= \sum_{i=-\infty}^{\infty} c_i \sum_{j=-\infty}^{\infty} a_{ij}\phi_j(2\ \cdot) \\
&= \sum_{j=-\infty}^{\infty} d_j\phi_j(2\ \cdot), \qquad (4.9)
\end{aligned}$$

where

$$d_j = \sum_{i=-\infty}^{\infty} c_i a_{ij}, \quad j \in \mathbb{Z}. \tag{4.10}$$

Recall that if $\{\phi(\ \cdot - i) : i \in \mathbb{Z}\}$ are linearly independent, then they are locally linearly independent. We shall always assume that $\{\phi(\ \cdot - i) : i \in \mathbb{Z}\}$ are linearly independent. Conditions on the mask for linear independence are given in [13].

Let U_0 denote the closed linear span in $L^2(\mathbb{R})$ of $\{\phi(\ \cdot - i) : i \in \mathbb{Z}\}$ and

$$U_j := \{f(2^j\ \cdot) : f \in U_0\}, \quad j \in \mathbb{Z}.$$

Similarly, let V_0 denote the closed linear span in $L^2((0,\infty))$ of $\{\phi_i : i \in \mathbb{Z}\}$ and

$$V_j = \{f(2^j\ \cdot) : f \in V_0\}.$$

Theorem 4.1. *If* (ϕ_i) *extend* ϕ *and* $\overline{\bigcup_{j=0}^{\infty} U_j} = L^2(\mathbb{R})$, *then* $\overline{\bigcup_{j=0}^{\infty} V_j} = L^2((0,\infty))$.

Proof. Since $\{\phi(\ \cdot - i) : i \in \mathbb{Z}\}$ are locally linearly independent, there are constants $A, B > 0$ such that for $f = \sum_{i=-\infty}^{\infty} a_i \phi(\ \cdot - i) \in U_0$,

$$A^2 \sum_{i=-n+1}^{0} a_i^2 \le \int_0^1 f^2 \le B^2 \sum_{i=-n+1}^{0} a_i^2.$$

Take $f \in L^2((0,\infty))$ and $\epsilon > 0$. Choose $m \in \mathbb{Z}$ so that $\int_0^{2^{-m}} f^2 < \epsilon^2$. Define $\widetilde{f} \in L^2(\mathbb{R})$ by

$$\widetilde{f}(x) = \begin{cases} f(x), & x > 2^{-m}, \\ 0, & \textit{otherwise}. \end{cases}$$

Since $\overline{\bigcup_{j=0}^{\infty} U_j} = L^2(\mathbb{R})$, we may choose $N \in \mathbb{Z}$ and $g = \sum_{i=-\infty}^{\infty} a_i\ 2^{-N/2}\phi(2^N\ \cdot - i)$ such that $2^{-N}n < 2^{-m}$ and $\|g - \widetilde{f}\|_2 < \epsilon$. Define $k \in \mathbb{Z}$, $k \ge 1$, by $2^{-N}(k+n) = 2^{-m}$. Let $h = \sum_{i=k}^{\infty} a_i\ 2^{-N/2}\phi(2^N\ \cdot - i)$. Then $h \in V_N$, has support in $(0,\infty)$, $h(x) = g(x)$ for $x \ge 2^{-m}$ and

$$\begin{aligned} \|h - f\|_2 &\le \left(\int_0^\infty (h-g)^2\right)^{1/2} + \left(\int_0^\infty (g-\widetilde{f})^2\right)^{1/2} + \left(\int_0^\infty (\widetilde{f}-f)^2\right)^{1/2} \\ &\le \left(\int_0^{2^{-m}} (h-g)^2\right)^{1/2} + \left(\int_{-\infty}^{\infty} (g-\widetilde{f})^2\right)^{1/2} + \left(\int_0^{2^{-m}} f^2\right)^{1/2} \\ &< \left(\int_0^{2^{-m}} h^2\right)^{\frac{1}{2}} + \left(\int_{-\infty}^{2^{-m}} g^2\right)^{\frac{1}{2}} + 2\epsilon < \left(\int_0^{2^{-m}} h^2\right)^{\frac{1}{2}} + 3\epsilon\ . \end{aligned}$$

Now

$$\begin{aligned} \int_0^{2^{-m}} h^2 &= \int_{2^{-N}k}^{2^{-m}} \left(\sum_{i=k}^{k+n-1} a_i\ 2^{-N/2}\phi(2^N\ \cdot - i)\right)^2 \\ &\le nB^2 \sum_{i=k}^{k+n-1} a_i^2 \le nB^2A^{-2} \int_{2^{-m}-2^{-N}}^{2^{-m}} g^2 \\ &< nB^2A^{-2}\epsilon^2\ . \end{aligned}$$

Thus $\|h - f\|_2 < \epsilon(3 + n^{1/2}BA^{-1})$ and the result follows. ♠

Theorem 4.2. *The functions $\{\phi_i : i \in \mathbb{Z}\}$ are locally linearly independent over (x_k, x_{k+1}), $k \in \mathbb{Z}$, if and only if $a_{ii+1} \neq 0$, $-n+2 \leq i \leq -1$, and for $k = 1, ..., n-3$, the matrix $(a_{ij})_{i=k-n+1, j=2k-n+2}^{k \quad 2k+2}$ has full rank.*

Proof. For $j \in \mathbb{Z}$, the functions whose supports overlap (x_j, x_{j+1}) are $\phi_{j-n+1}, ..., \phi_j$. Now for $k \geq 0$, define $A_k = (a_{ij})_{i=k-n+1, j=2k-n+2}^{k \quad 2k+2}$ while for $k \leq -1$, define $A_k = (a_{ij})_{i=k-n+1, j=k-n+2}^{k \quad k+1}$. First consider $k \geq 0$. We shall show that if $\phi_{2k-n+2}, ..., \phi_{2k+2}$ are locally linearly independent over $(x_{2k+1}, x_{2k+3}) = (2x_k, 2x_{k+1})$, then (ϕ_j) are locally linearly independent over (x_k, x_{k+1}) if and only if A_k has full rank.

For $x \in (x_k, x_{k+1})$, let $f(x) = \sum_{j=k-n+1}^{k} c_j \phi_j(x)$. By (4.9), $f(x) = \sum_{j=2k-n+2}^{2k+2} d_j \phi_j(2x)$, where $(d_j)_{2k-n+2}^{2k+2} = (c_j)_{k-n+1}^{k} A_k$. Suppose that A_k has full rank. If $f(x) = 0$ for $x_k < x < x_{k+1}$, then $d_j = 0$, $j = 2k-n+2, ..., 2k+2$, and so $c_j = 0$, $j = k-n+1, ..., k$. Thus (ϕ_j) are locally linearly independent over (x_k, x_{k+1}). Conversely suppose that (ϕ_j) are locally linearly independent over (x_k, x_{k+1}). If A_k does not have full rank, then there is a non-zero sequence $c = (c_j)_{k-n+1}^{k}$ such that $cA_k = 0$ and hence $f(x) = \sum_{j=k-n+1}^{k} c_j \phi_j(x) = 0$, for $x_k < x < x_{k+1}$, which is a contradiction. Thus (ϕ_j) are locally linearly independent over (x_k, x_{k+1}) if and only if A_k has full rank. Similarly, we may take $k \leq -1$ and suppose that $\phi_{k-n+2}, ..., \phi_{k+1}$ are locally linearly independent over $(x_{k+1}, x_{k+2}) = (2x_k, 2x_{k+1})$. Then as above we see that (ϕ_j) are locally linearly independent over (x_k, x_{k+1}) if and only if A_k is non-singular, i.e. $a_{ii+1} \neq 0$, $i = k-n+1, ..., k$.

In particular we see from the above argument that if (ϕ_j) are locally linearly independent over (x_k, x_{k+1}), $k \in \mathbb{Z}$, then A_k has full rank for $k = 1, ..., n-3$ and $a_{ii+1} \neq 0$, for $-n+2 \leq i \leq -1$. Conversely, we assume that these conditions hold and show that (ϕ_j) are locally linearly independent over (x_k, x_{k+1}), $k \in \mathbb{Z}$. Note that since $a_{ii+1} \neq 0$, for $i = -n+1, ..., -1$, and $a_{01} = a_0 \neq 0$, A_0 has full rank. For $i \geq 0$, $\phi_i = \phi(\,\cdot - i - 1)$ and so for any $k \geq n-1$, (ϕ_i) are locally linearly independent over (x_k, x_{k+1}). Thus A_k has full rank for $k \geq n-1$. It follows that A_{n-2} also has full rank.

Now suppose inductively that (ϕ_j) are locally linearly independent over $(x_\ell, x_{\ell+1})$ for some ℓ, $1 \leq \ell \leq n-1$. Then since $A_{\ell-1}$ has full rank, our above argument shows that (ϕ_j) are locally linearly independent over $(x_{\ell-1}, x_\ell)$. Thus (ϕ_j) are locally linearly independent over (x_k, x_{k+1}) for all $k \geq 0$. Now recall that $a_{ii+1} = 1$, $i \leq -n+1$. Thus A_k is non-singular for all $k \leq -1$. Repeating the above inductive argument then shows that (ϕ_j) are locally linearly independent over (x_k, x_{k+1}) for all integers k. ♠

For $n = 3$, the conditions of Theorem 4.2 are simply $a_{-10} \neq 0$. For $n = 4$ we can write (a_{ij}) as

$$\begin{pmatrix} \ddots & & & & & & & & \\ & 1 & \lambda & & & & & & \\ & & 1-\lambda & \mu & \nu & & & & \\ & & & 1-\mu & 1-\nu & a_3 & a_4 & & \\ & & & & a_0 & a_1 & a_2 & a_3 & a_4 \\ & & & & & & \ddots & & \end{pmatrix}$$

and the conditions become $\lambda \neq 1$, $\mu \neq 1$ and the matrix

$$\begin{pmatrix} \mu & \nu & & & \\ 1-\mu & 1-\nu & a_3 & a_4 & \\ & a_0 & a_1 & a_2 & a_3 \\ & & & a_0 & a_1 \end{pmatrix}$$

has full rank.

Now renormalize the functions ϕ_i, $i \in \mathbb{Z}$, by defining

$$\widetilde{\phi}_i = \begin{cases} \phi_i, & i \geq -n+1, \\ 2^{(1-i-n)/2}\phi_i, & i \leq -n. \end{cases}$$

Corollary 4.3. *Under the conditions of Theorem 4.2, the functions $\{\widetilde{\phi}_i : i \in \mathbb{Z}\}$ form a Riesz basis for V_0.*

Proof. By Theorem 4.2 there are constants $A_j, B_j > 0$, $j \in \mathbb{Z}$, so that for $f \in V_0$, $f = \sum_{j=-\infty}^{\infty} c_j \widetilde{\phi}_j$,

$$A_j \sum_{i=j-n+1}^{j} c_i^2 \leq \int_{x_j}^{x_{j+1}} f^2 \leq B_j \sum_{i=j-n+1}^{j} c_i^2, \quad j \in \mathbb{Z}.$$

Since $\widetilde{\phi}_i = \phi(\,\cdot - i - 1)$, $i \geq 0$, we may choose $A_j = A_{n-1}$, $B_j = B_{n-1}$ for $j \geq n-1$. Since $\widetilde{\phi}_i = 2^{\frac{1}{2}}\widetilde{\phi}_{i+1}(2\,\cdot)$, $i \leq -n$, we may choose $A_j = A_{-n+1}$, $B_j = B_{-n+1}$, $j \leq -n+1$. So taking $A = min\{A_j : -n+1 \leq j \leq n-1\}$, $B = max\{B_j : -n+1 \leq j \leq n-1\}$, we have

$$A \sum_{i=j-n+1}^{j} c_i^2 \leq \int_{x_j}^{x_{j+1}} f^2 \leq B \sum_{i=j-n+1}^{j} c_i^2, \quad j \in \mathbb{Z}.$$

Adding over $j \in \mathbb{Z}$ then gives

$$A \sum_{i=-\infty}^{\infty} c_i^2 \leq \int_0^{\infty} f^2 \leq B \sum_{i=-\infty}^{\infty} c_i^2.$$

♠

5. Polynomial reproduction

We shall say that the refinable function ϕ with mask $(a_i)_0^n$ *reproduces polynomials of degree* m if the space Π_m of all polynomials of degree m lies in the linear span of $\{\phi(\,\cdot - i) : i \in \mathbb{Z}\}$. Similarly we say that the refinable functions (ϕ_i) with mask (a_{ij}) reproduce polynomials of degree m if $\Pi_m\big|_{(0,\infty)}$ lies in the linear span of $\{\phi_i : i \in \mathbb{Z}\}$. Polynomial reproduction is an important property as it is related to approximation order of the subspace. If ϕ reproduces polynomials of degree $m \le n-1$, we would like to extend ϕ to (ϕ_i) which also reproduces polynomials of degree m. For $m = n-1$ and ϕ is the uniform B-spline of order n, we have already seen that we can extend ϕ to hybrid splines (ϕ_i). In what follows we shall give a general recursive construction.

For refinable functions (ϕ_i) with mask (a_{ij}), we shall write

$$I_i := \int_0^\infty \phi_i, \quad i \in \mathbb{Z}.$$

For $i \ge 0$,

$$I_i = \int_0^\infty \phi(\,\cdot - i - 1) = \int_{-\infty}^\infty \phi = 1.$$

For $i \in \mathbb{Z}$,

$$I_i = \sum_{j=-\infty}^{\infty} a_{ij} \int_0^\infty \phi_j(2\,\cdot) = \frac{1}{2} \sum_{j=-\infty}^{\infty} a_{ij} I_j, \tag{5.1}$$

which defines I_i recursively for $i \le -1$. Note that for $i \le -n$, $I_i = \frac{1}{2} I_{i+1}$.

Now given a mask (a_{ij}^{n-1}) which extends the mask $(a_i^{n-1})_0^{n-1}$, we define a new mask $(a_{ij}^n)_0^n$ by

$$a_{ij}^n = \frac{1}{2} \sum_{k=-\infty}^{j} \left(\frac{a_{ik}^{n-1}}{I_i^{n-1}} - \frac{a_{i+1k}^{n-1}}{I_{i+1}^{n-1}} \right) I_k^{n-1}, \quad i, j \in \mathbb{Z}, \tag{5.2}$$

where $I_i^{n-1} = \int_0^\infty \phi_i^{n-1}$, $i \in \mathbb{Z}$. From (5.2) we have

$$a_{ij}^n - a_{ij-1}^n = \frac{I_j^{n-1}}{2} \left(\frac{a_{ij}^{n-1}}{I_i^{n-1}} - \frac{a_{i+1j}^{n-1}}{I_{i+1}^{n-1}} \right), \quad i, j \in \mathbb{Z}. \tag{5.3}$$

By (5.1), we can rewrite (5.2) as

$$a_{ij}^n = \frac{1}{2} \sum_{k=j+1}^{\infty} \left(\frac{a_{i+1k}^{n-1}}{I_{i+1}^{n-1}} - \frac{a_{ik}^{n-1}}{I_i^{n-1}} \right) I_k^{n-1}, \quad i, j \in \mathbb{Z}. \tag{5.4}$$

Note that for $i \ge 0$, $j \in \mathbb{Z}$, (5.2) becomes

$$a_{ij}^n = \frac{1}{2}(a_{ij-1}^{n-1} + a_{ij}^{n-1}). \tag{5.5}$$

Also for $i \le -1$, we can use (5.3) to define a_{ij}^n recursively on j, e.g. for increasing j, starting with

$$a_{ii+1}^n = \frac{I_{i+1}^{n-1}}{2 I_i^{n-1}} a_{ii+1}^{n-1}.$$

Theorem 5.1. *Suppose that* (ϕ_i^{n-1}) *with mask* (a_{ij}^{n-1}) *extend* ϕ^{n-1} *with mask* $(a_i^{n-1})_0^{n-1}$ *and reproduces polynomials of degree* m, $0 \le m \le n-2$. *Let*

$$\sum_{i=0}^{n} a_i^n z^i = \frac{1}{2}(1+z)\sum_{i=0}^{n-1} a_i^{n-1} z^i, \quad z \in \mathbb{C}. \tag{5.6}$$

Assume $I_i^{n-1} = \int_0^\infty \phi_i^{n-1} \neq 0$, $i \in \mathbb{Z}$, *and define* (a_{ij}^n) *by (5.2). Then* (a_{ij}^n) *extends* $(a_i^n)_0^n$. *Letting* (ϕ_i^n) *be the corresponding refinable functions, they reproduce polynomials of degree* $m+1$, *and for some constant* $c \neq 0$,

$$c\phi_i^{n\,\prime} = \frac{\phi_i^{n-1}}{I_i^{n-1}} - \frac{\phi_{i+1}^{n-1}}{I_{i+1}^{n-1}}, \quad i \in \mathbb{Z}. \tag{5.7}$$

Moreover if (ϕ_i^{n-1}) *are locally linearly independent over* (x_k, x_{k+1}), $k \in \mathbb{Z}$, *then so are* (ϕ_i^n).

Proof. From (5.5), for $i \ge 0$, $j \in \mathbb{Z}$,

$$a_{ij}^n = \frac{1}{2}(a_{j-2i-2}^{n-1} + a_{j-2i-1}^{n-1}) = a_{j-2i-1}^n,$$

by (5.6). For $i \le -n$, $j \in \mathbb{Z}$, (5.2) gives

$$\begin{aligned} a_{ij}^n &= 0, \quad j \le i, \\ a_{ii+1}^n &= \frac{I_{i+1}^{n-1}}{2I_i^{n-1}} = 1, \\ a_{ij}^n &= \frac{I_{i+1}^{n-1}}{2I_i^{n-1}} - \frac{I_{i+2}^{n-1}}{2I_{i+1}^{n-1}} = 0, \ j \ge i+2. \end{aligned}$$

For $-n \le i \le -1$, (5.2) and (5.4) give $a_{ij}^n = 0$ for $j < i+1$ and $j > 2i+n+1$ respectively. Thus (a_{ij}^n) extends $(a_i^n)_{i=0}^n$.

Also for $j \in \mathbb{Z}$, (5.3) gives for $j \in \mathbb{Z}$,

$$\sum_{i=-\infty}^{\infty} a_{ij}^n - \sum_{i=-\infty}^{\infty} a_{ij-1}^n = \frac{I_j^{n-1}}{2} \sum_{i=-\infty}^{\infty} \left(\frac{a_{ij}^{n-1}}{I_i^{n-1}} - \frac{a_{i+1,j}^{n-1}}{I_{i+1}^{n-1}} \right) = 0,$$

and so $\sum\limits_{i=-\infty}^{\infty} a_{ij}^n$ is independent of j. But

$$\begin{aligned} \sum_{i=-\infty}^{\infty} a_{in}^n &= \sum_{i=0}^{\infty} \frac{1}{2}(a_{in-1}^{n-1} + a_{in}^{n-1}) \\ &= \frac{1}{2}\sum_{i=-\infty}^{\infty} a_{in-1}^{n-1} + \frac{1}{2}\sum_{i=-\infty}^{\infty} a_{in} = 1, \end{aligned}$$

and hence

$$\sum_{i=-\infty}^{\infty} a_{ij}^n = 1, \quad j \in \mathbb{Z}.$$

So, as before, we may choose corresponding refinable functions (ϕ_i^n) with $\sum_{i=-\infty}^{\infty} \phi_i^n(x) = 1$ for $x > 0$.

Since

$$\phi_i^n = \sum_{j=-\infty}^{\infty} a_{ij}^n \phi_j^n(2\,\cdot), \quad i \in \mathbb{Z},$$

we have

$$\phi_i^{n\,\prime} = 2\sum_{j=-\infty}^{\infty} a_{ij}^n \phi_j^{n\,\prime}(2\,\cdot), \quad i \in \mathbb{Z}.$$

Also for $i \in \mathbb{Z}$,

$$\begin{aligned}
\frac{\phi_i^{n-1}}{I_i^{n-1}} - \frac{\phi_{i+1}^{n-1}}{I_{i+1}^{n-1}} &= \frac{1}{I_i^{n-1}} \sum_{j=-\infty}^{\infty} a_{ij}^{n-1}\phi_j^{n-1}(2\,\cdot) - \frac{1}{I_{i+1}^{n-1}} \sum_{j=-\infty}^{\infty} a_{i+1j}^{n-1}\phi_j^{n-1}(2\,\cdot) \\
&= \sum_{j=-\infty}^{\infty} \phi_j^{n-1}(2\,\cdot)\left(\frac{a_{ij}^{n-1}}{I_i^{n-1}} - \frac{a_{i+1j}^{n-1}}{I_{i+1}^{n-1}}\right) \\
&= 2\sum_{j=-\infty}^{\infty} \frac{\phi_j^{n-1}(2\,\cdot)}{I_j^{n-1}}(a_{ij}^n - a_{ij-1}^n) \quad \text{by (5.3)} \\
&= 2\sum_{j=-\infty}^{\infty} a_{ij}^n \left(\frac{\phi_j^{n-1}(2\,\cdot)}{I_j^{n-1}} - \frac{\phi_{j+1}^{n-1}(2\,\cdot)}{I_{j+1}^{n-1}}\right),
\end{aligned}$$

and so both sides of (5.7) satisfy the same refinement equation. Also for $i \geq 0$, $\phi_i^n = \phi^n(\,\cdot - i - 1)$, where

$$\phi^n(x) = \sum_{j=0}^{n} a_j^n \phi^n(2x - j), \quad x \in \mathbb{R},$$

and so, from (5.6), $c\phi^{n\,\prime} = \phi^{n-1} - \phi^{n-1}(\,\cdot - 1)$, for some constant $c \neq 0$. Thus for $i \geq 0$,

$$\begin{aligned}
c\phi_i^{n\,\prime}(x) &= c\phi^{n\,\prime}(x - i - 1) \\
&= \phi^{n-1}(x - i - 1) - \phi^{n-1}(x - i - 2) \\
&= \phi_i^{n-1}(x) - \phi_{i+1}^{n-1}(x)
\end{aligned}$$

and so (5.7) holds for $i \geq 0$. Hence (5.7) holds for all $i \in \mathbb{Z}$.

Now take p, $1 \leq p \leq m+1$, and write

$$x^{p-1} = \sum_{i=-\infty}^{\infty} b_i \phi_i^{n-1}(x), \quad x > 0.$$

Let $c_0 = 0$ and define

$$c_i = \begin{cases} cp\sum_{j=1}^{i} b_j I_j^{n-1}, & i \geq 1, \\ -cp\sum_{j=i+1}^{0} b_j I_j^{n-1}, & i \leq -1. \end{cases}$$

Define

$$f(x) = \sum_{i=-\infty}^{\infty} c_i \phi_i^n(x), \quad x > 0.$$

Then by (5.7), for $x > 0$,

$$\begin{aligned} f'(x) &= \sum_{i=-\infty}^{\infty} c^{-1} c_i \left(\frac{\phi_i^{n-1}(x)}{I_i^{n-1}} - \frac{\phi_{i+1}^{n-1}(x)}{I_{i+1}^{n-1}} \right) \\ &= \sum_{i=-\infty}^{\infty} c^{-1}(c_i - c_{i-1}) \frac{\phi_i^{n-1}(x)}{I_i^{n-1}} \\ &= \sum_{i=-\infty}^{\infty} p b_i \phi_i^{n-1}(x) = p x^{p-1}. \end{aligned}$$

Thus $f(x) = x^p + k$, $x > 0$, for some constant k, and so

$$x^p = \sum_{i=-\infty}^{\infty} (c_i - k)\phi_i^n(x), \quad x > 0.$$

So (ϕ_i^n) reproduces polynomials of degree $m + 1$.

Finally we suppose that (ϕ_i^{n-1}) are locally linearly independent over (x_k, x_{k+1}), $k \in \mathbb{Z}$ and let

$$\sum_{i=k-n+1}^{k} A_i \phi_i^n(x) = 0, \quad x_k < x < x_{k+1}. \tag{5.8}$$

Then for $x_k < x < x_{k+1}$,

$$\begin{aligned} 0 &= \sum_{i=k-n+1}^{k} A_i \phi_i^{n\,\prime}(x) \\ &= c^{-1} \sum_{i=k-n+1}^{k} A_i \left(\frac{\phi_i^{n-1}(x)}{I_i^{n-1}} - \frac{\phi_{i+1}^{n-1}(x)}{I_{i+1}^{n-1}} \right) \\ &= c^{-1} \sum_{i=k-n+2}^{k} (A_i - A_{i-1}) \frac{\phi_i^{n-1}(x)}{I_i^{n-1}}, \end{aligned}$$

since $\phi_{k-n+1}^{n-1}(x) = \phi_{k+1}^{n-1}(x) = 0$. Thus

$$A_i - A_{i-1} = 0, \quad i = k - n + 2, ..., k,$$

and so

$$A_i = A_k, \quad i = k - n + 1, ..., k.$$

Therefore (5.8) becomes

$$A_k \sum_{i=k-n+1}^{k} \phi_i^n(x) = 0, \quad x_k < x < x_{k+1},$$

so $A_i = 0$, $i = k - n + 1, ..., k$ and (ϕ_i^n) are locally linearly independent on (x_k, x_{k+1}). ♠

Now suppose that the refinable function ϕ with mask $(a_i)_0^n$ reproduces polynomials of degree m, $1 \leq m \leq n - 1$. Then we have the following procedure for constructing an extension (ϕ_i) with mask (a_{ij}) which also reproduces polynomials of degree m. We must have

$$\sum_{i=0}^{n} a_i z^i = 2^{-m}(1+z)^{m+1}p(z), z \in \mathbb{C},$$

for a polynomial p of degree $n-m-1 \geq 0$ with $p(1)=1$. Let

$$\sum_{i=0}^{n-m} b_i z^i = (1+z)p(z),\ \ z \in \mathbb{C}.$$

Then there is a refinable function $\overline{\phi}$ with mask $(b_i)_0^{n-m}$ which reproduces constants and so has an extension $(\overline{\phi}_i)$ with mask (b_{ij}) which also reproduces constants. Theorem 5.1 then gives a recursive construction to give the desired mask (a_{ij}). We note that if $\{\phi(\,\cdot - i) : i \in \mathbb{Z}\}$ are linearly independent, then it follows that $\{\overline{\phi}(\,\cdot - i) : i \in \mathbb{Z}\}$ are linearly independent. Then we can construct (b_{ij}) to satisfy the conditions of Theorem 4.2 to ensure that $(\overline{\phi}_i)$ are locally linearly independent over (x_k, x_{k+1}), $k \in \mathbb{Z}$. It then follows from Theorem 5.1 that (ϕ_i) are locally linearly independent over (x_k, x_{k+1}), $k \in \mathbb{Z}$. The only difficulty with the above procedure is that for each recursive step applying Theorem 5.1, we need to ensure that the integrals of the refinable functions are non-zero.

As an example, take the Daubechies 6-tap mask $(a_i)_0^5$, where

$$\sum_{i=0}^{5} a_i z^i = 2^{-2}(1+z)^3 p(z), \quad z \in \mathbb{C}.$$

Then the mask $(b_i)_0^3$, given by

$$\sum_{i=0}^{3} b_i z^i = (1+z)p(z), \quad z \in \mathbb{C},$$

where

$$b_0 = 1.881869, \quad b_1 = 0.800730, \quad b_2 = -0.881869, \quad b_3 = 0.199270\ ,$$

can be extended to the following matrix (b_{ij}) with one degree of freedom

$$\begin{pmatrix} \ddots & & & & & & \\ & 1 & & & & & \\ & & 1 & \lambda & & & \\ & & & 1-\lambda & b_2 & b_3 & & \\ & & & & b_0 & b_1 & b_2 & b_3 \\ & & & & & & \ddots & \end{pmatrix}.$$

For local linear independence we require $\lambda \neq 1$. Then one can choose λ so that the integrals of the refinable functions in the two recursive steps are non-zero, though we do not know which value of λ is most suitable.

The recursive construction in Theorem 5.1 then gives a foveated extension (a_{ij}) of Daubechies 6-tap mask. The following matrix shows the entries, a_{ij}, $-5 \le i \le 0$, $-4 \le j \le 6$, of the extension corresponding to $\lambda = \frac{1}{2}$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1.528106 & -0.630018 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.528106 & 0.133336 & 0.132090 & -0.054459 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.496682 & 0.397443 & -0.086658 & -0.120832 & 0.049818 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.470467 & 1.141117 & 0.650365 & -0.190934 & -0.120832 & 0.049817 \end{pmatrix}.$$

The entries in the last row are the Daubechies 6-tap mask.

Similarly, we can extend the Daubechies 8-tap mask with three degrees of freedom.

Now suppose that ϕ with mask $(a_i)_0^n$ extends to (ϕ_i) with mask (a_{ij}). It is known (see for instance [12]) that ϕ reproduces polynomials of degree m if and only if $\frac{1}{2^p}$, $p = 0, 1, \ldots, m$, are left eigenvalues of of the matrix (a_{j-2i}). Further, if $C_p := (c_{pi})_{i\in\mathbb{Z}}$ is a normalized left eigenvector with eigenvalue $\frac{1}{2^p}$, then

$$x^p = \sum_{i=-\infty}^{\infty} c_{pi}\phi(x-i), \quad x \in (0, \infty), \; p = 0, 1, \ldots, m. \tag{5.9}$$

We shall assume that ϕ reproduces polynomials of degree m. The following theorem gives the corresponding necessary and sufficient condition for (ϕ_i) to reproduce polynomials of degree m.

Theorem 5.2. *Suppose that conditions of Theorem 4.2 are satisfied. Then (ϕ_i) reproduces polynomials of degree m if and only if $\frac{1}{2^p}$, $p = 0, 1, \ldots, m$, are left eigenvalues of (a_{ij}). Further, if $K_p := (k_{pi})_{i\in\mathbb{Z}}$ is a normalized left eigenvector with eigenvalue $\frac{1}{2^p}$, then*

$$x^p = \sum_{i=-\infty}^{\infty} k_{pi}\phi_i(x), \quad x \in (0, \infty), \; p = 0, 1, \ldots, m. \tag{5.10}$$

Proof. Assume that (ϕ_i) reproduces polynomials of degree m, and suppose that (5.10) holds. Then for $p = 0, 1, \ldots, m$ by (4.3),

$$x^p = \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} k_{pi} a_{ij} \phi_j(2x), \quad x \in (0, \infty).$$

This together with (5.10) give

$$\frac{1}{2^p} \sum_{j=-\infty}^{\infty} k_{pj}\phi_j(x) = \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} k_{pi} a_{ij} \phi_j(x), \quad x \in (0, \infty),$$

and hence

$$\frac{1}{2^p} k_{pj} = \sum_{i=-\infty}^{\infty} k_{pi} a_{ij},$$

which shows that $\frac{1}{2^p}$ is a left eigenvalue of (a_{ij}).

Conversely, suppose that $\frac{1}{2^p}$ is a left eigenvalue of (a_{ij}). Let K_p be a corresponding eigenvector. Then

$$\frac{1}{2^p}k_{pj} = \sum_{i=-\infty}^{\infty} k_{pi}a_{ij}, \quad j \in \mathbb{Z} \tag{5.11}$$

and

$$\frac{1}{2^p}k_{pj} = \sum_{i=0}^{\infty} k_{pi}a_{j-2i-1}, \quad j \geq n.$$

Recall that

$$\frac{1}{2^p}c_{pj+1} = \sum_{i=-\infty}^{\infty} c_{pi+1}a_{j-2i-1}, \quad j \in \mathbb{Z}.$$

Therefore we can choose K_p so that

$$k_{pi} = c_{i+1}, \quad i \geq 0. \tag{5.12}$$

Let

$$f(x) = \sum_{i=-\infty}^{\infty} k_{pi}\phi_i(x), \quad x > 0.$$

Then for $x > 0$, by (4.3) and (5.11),

$$f(x) = \sum_{j=-\infty}^{\infty} 2^{-p}k_{pj}\phi_j(2x) = 2^{-p}f(2x),$$

and for $x \geq n$,

$$f(x) = \sum_{i=0}^{\infty} k_{p,i}\phi(x-i-1) = \sum_{i=-\infty}^{\infty} c_{p,i}\phi(x-i) = x^p. \tag{5.13}$$

Therefore, for any $x > 0$, choosing $k \in \mathbb{Z}$ with $2^{-k}n \leq x$ gives

$$f(x) = 2^{-kp}f(2^k x) = 2^{-kp}(2^k x)^p = x^p.$$

Thus

$$\sum_{i=-\infty}^{\infty} k_{pi}\phi_i(x) = x^p, \quad x > 0.$$

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