Multi-Box Splines

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Abstract. We construct local generators, comprising r functions, for refinable spaces of bivariate C^{n-1} spline functions of degree n on meshes comprising all lines through points of the integer lattice in the directions of $n+r+1$ pairwise linearly independent vectors with integer components. The generators are characterised by their Fourier transforms. Their shifts are shown to form a Riesz basis if and only if at most r lines in the mesh intersect other than in the integer lattice, which can occur for $n \leq 2r-1$. The symmetry of these generators is studied and examples are given.

§1. Introduction

Box splines were introduced by de Boor and Höllig [1] as multivariate generalisations of uniform B-splines, and with further work by many authors, e.g. Dahmen and Micchelli [3], Jia [9], have become, to quote [4], 'one of the most dramatic successes of multivariate splines'. For a comprehensive account of box splines, see [2]. Yet despite their many elegant properties, box splines fail to generalise some of the basic properties of uniform B-splines, as we shall now discuss.

Let N_n denote the B-spline of degree n with integer knots and support $[0, n+1]$ and consider the bivariate box spline $B_n = B_n(.|v_0, \ldots, v_{n+1})$, where $v_j \in \mathbb{Z}^2$, $j = 0, \ldots, n+1$. We recall that B_n is a piecewise polynomial of degree n on the mesh $M = M(v_0, \ldots, v_{n+1})$ formed by all lines through points in $\mathbb{Z}^2$ in the directions of $v_0, \ldots, v_{n+1}$. As their name suggests, B-splines form a basis. More precisely, the shifts (i.e. integer translates) of N_n form a Riesz basis for the space of all square integrable C^{n-1} spline functions of degree n on $\mathbb{R}$ with integer knots. In order for the shifts of B_n to form a Riesz basis, the mesh M can comprise lines in at most three distinct directions. In this case B_n can have continuity C^{n-1} only for the simple cases $n = 0$ and 1. For $n \geq 2$, not only does B_n fail to be C^{n-1}, but the space $\mathcal{S}_n$ spanned by the shifts of B_n does not comprise all piecewise polynomials of degree n subject to some continuity conditions. Indeed the restriction of $\mathcal{S}_n$ to any region bounded but not intersected by mesh lines coincides not with the space of polynomials of degree n but with a proper subspace P comprising the common null space of certain differential operators [1,3]. Furthermore no characterisation is known

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of $\mathcal{S}_n$ as piecewise polynomials on M with pieces in P subject to certain continuity conditions.

In contrast to the complex structure of the space $\mathcal{S}_n$ above, we shall in this paper study certain spaces $S_n = S_n(v_0, \ldots, v_{n+r})$, $r \geq 1$, of C^{n-1} piecewise polynomials of degree n on the mesh $M = M(v_0, \ldots, v_{n+r})$, where $v_0, \ldots, v_{n+r} \in \mathbb{Z}^2$ are pairwise linearly independent. For simplicity we restrict elements of S_n to have compact support. In most cases of interest, S_n comprises *all* C^{n-1} piecewise polynomials of degree n on M with compact support, though in certain cases there is a simple restriction on the jumps of the nth order derivatives. We shall show that there are functions $\phi_1, \ldots, \phi_r$ in S_n such that $\phi = (\phi_1, \ldots, \phi_r)$ is a local generator of S_n, i.e. every element of S_n is a finite linear combination of shifts of $\phi_1, \ldots, \phi_r$. For $r = 1$, ϕ_1 is the box spline B_n and for this reason we call ϕ a **multi-box spline**. Whereas for the case $r = 1$ of box splines, the shifts of ϕ_1 form a Riesz basis only for $n = 0$ and 1, for general $r \geq 1$ the shifts of $\phi_1, \ldots, \phi_r$ can form a Riesz basis for degree n up to $2r - 1$.

A multi-box spline $\phi = (\phi_1, \ldots, \phi_r)$ can thus generate a space S_n comprising all compactly supported C^{n-1} piecewise polynomials of degree n on a certain mesh for any $n \geq 0$. A disadvantage is that for $n \geq 2$, we require $r \geq 2$ and in this case the choice of ϕ is not unique. Moreover unlike the box spline case $r = 1$ we know of no general simple formula for the Fourier transform $\hat{\phi}$.

For the case $r = 2$, multi-box splines were introduced in [7] and they were further studied in [8]. The work in [7] allowed the construction of Riesz bases of C^{n-1} multi-box splines for degree $n \leq 3$ but the construction did not result in functions with pleasant symmetry properties for $n = 2$ or 3. In order to gain Riesz bases of multi-box splines for degree greater than 3, and to gain 'good' examples for $n = 2$ and 3, we need to extend the construction to $r \geq 3$. This is what we have achieved here.

In Section 2 we define the space S_n in terms of Fourier transforms and then prove that it comprises certain C^{n-1} piecewise polynomials of degree n. In Section 3 we prove that the space S_n has a local generator $\phi = (\phi_1, \ldots, \phi_r)$ and characterise all such generators in terms of Fourier transforms. We prove in Section 4 that the shifts of such a local generator ϕ form a Riesz basis if and only if at most r lines in the mesh M intersect other than in points in $\mathbb{Z}^2$. It is then shown that the maximum degree n for which this condition can hold is precisely $2r - 1$. Much of the work in Sections 2 to 4 extends that for the case $r = 2$ in [7], but the greater complexity of the general case requires not only more work but often a simplification of the approach in [7] in order to make the generalisation practicable.

The existence of a local generator ϕ is shown in Section 3 by a recursive construction, but the functions so constructed need not satisfy any symmetry properties. While we conjecture that symmetric choices of ϕ always exist, we have not been able to prove this. Instead we study, in Section 5, the nature of the possible symmetries. Finally, Section 6 illustrates the theory with three examples of symmetric multi-box splines: piecewise constant on a four-direction mesh (which illustrates the unusual case where there is a restriction on the jumps of the functions across mesh lines), quadratic spline functions on a five direction mesh, and cubic spline functions on a highly symmetric eight-direction mesh.

There is clearly much work left to do, both in constructing specific examples of 'nice' multi-box splines, and in developing general results for multi-box splines analogous to the many elegant properties and applications of box splines.

§2. The Spline Space

For $m \geq 1$, let $v_0, \ldots, v_m$ be pairwise linearly independent vectors in $\mathbb{Z}^2$. We denote by $M = M(v_0, \ldots, v_m)$ the set of lines

$$\{k + tv_j : t \in \mathbb{R}\}, \quad k \in \mathbb{Z}^2, \quad j = 0, \ldots, m,$$

i.e. all lines through points of the integer lattice in the directions of $v_0, \ldots, v_m$. By a spline function of degree n over M we mean a function $f : \mathbb{R}^2 \to \mathbb{R}$ which coincides with a polynomial of degree n on any open region which is bounded by, but not intersected by, lines in M. (Note that this does not involve the values of f on the set M itself, which is of course of measure zero.)

We now consider the jump of a spline function f across a line in M. For any vector $v = (v_1, v_2)$ in $\mathbb{R}^2$, we define $v^\perp := (-v_2, v_1)$. Now for any point $x \in \mathbb{R}^2$ we define

$$f(x; v) := \lim_{h \to 0+} \lim_{\epsilon \to 0+} (f(x + hv + \epsilon v^\perp) - f(x + hv - \epsilon v^\perp)).$$

Roughly speaking, $f(x; v)$ denotes the jump of f at x in crossing, in direction $v^\perp$, the line segment $\{x + tv : t \geq 0\}$.

As an example, take $v_0 = (1,0)$, $v_1 = (0,1)$, $v_2 = (1,1)$, for which $M = M(v_0, v_1, v_2)$ is called a type-1 triangulation. An example of a spline function over M is the linear box spline B_1 which is the unique continuous spline function of degree 1 over M which satisfies

$$B_1(j) = \delta_{j0}, \quad j \in \mathbb{Z}^2.$$

It can be seen that with f equal to the partial derivative $\partial_1 B_1$:

$$f(0; v_1) = 1, \quad f(0; -v_1) = -1, \quad f((1,0); v_1) = -1, \quad f((1,0); -v_1) = 0.$$

Now take $n \geq 0$, $r \geq 1$ and pairwise linearly independent vectors $v_0, \ldots, v_{n+r}$ in $\mathbb{Z}^2$. We shall denote by $S_n = S_n(v_0, \ldots, v_{n+r})$ the space of all functions f with continuous Fourier transforms of form

$$\hat{f}(u) = \frac{\sum_{|\alpha|=r-1} P_\alpha(e^{-iu}) u^\alpha}{(iuv_0) \ldots (iuv_{n+r})}, \quad u \in \mathbb{R}^2, \tag{2.1}$$

where for any multi-index $\alpha \in \mathbb{N}^2$ of order $|\alpha| = r - 1$, P_α is a Laurent polynomial with real coefficients, and for $u, v \in \mathbb{R}^2$, uv denotes the scalar product uv^T. Without loss of generality, we may suppose that for $j = 0, \ldots, n+r$, the components of v_j are co-prime.

Theorem 1. *A function f lies in S_n if and only if it is a C^{n-1} spline function of degree n over $M(v_0,\ldots,v_{n+r})$ with compact support such that any nth order derivative $D_n f$ of f satisfies, for all $x \in \mathbb{R}^2 \setminus \mathbb{Z}^2$,*

$$D_n f(x;-v_j) = -D_n f(x;v_j), \quad j=0,\ldots,n+r. \tag{2.2}$$

Since $D_n f$ in (2.2) is a piecewise constant, (2.2) states that the jump of $D_n f$ across any line in $M = M(v_0,\ldots,v_{n+r})$ can change only at points of $\mathbb{Z}^2$. We shall see later that for many choices of $v_0,\ldots,v_{n+r}$, condition (2.2) is automatically satisfied and so S_n comprises *all* C^{n-1} spline functions of degree n over M with compact support.

The proof of Theorem 1 will be done by induction on n. In order to establish this for $n=0$, we shall need the following result. Although this is a special case of Theorem 1 in [6], we include a proof of this simple case for completeness. For $v \in \mathbb{R}^2$, D_v will denote the directional derivative $v_1\partial_1 + v_2\partial_2$.

Lemma 1. *If f satisfies the conditions of Theorem 1 for $n=0$, then*

$$\hat{f}(u) = \sum_{j\in\mathbb{Z}^2} \frac{e^{-iju}}{iv_0u} \sum_{k=1}^{r} (v_k^\perp v_0) \frac{f(j;v_k)+f(j;-v_k)}{iv_ku}, \quad u \in \mathbb{R}^2. \tag{2.3}$$

Proof: Let $\Omega \subset \mathbb{R}^2$ be a polygonal region with vertices $P_1,\ldots,P_m$, described counter-clockwise, and for $j=1,\ldots,m$, let w_j denote a vector in the direction from P_j to P_{j+1}, where $P_{m+1} = P_1$. Note that for $j=1,\ldots,m$, $w_j^\perp$ is normal to the edge P_jP_{j+1}, pointing into the region Ω. Then for u, w in $\mathbb{R}^2$, integration by parts gives

$$\int_\Omega e^{-ixu}\,dx = \int_\Omega D_w\left(-\frac{1}{iwu}e^{-ixu}\right)dx = \frac{1}{iwu}\sum_{j=1}^{m}\int_{P_j}^{P_{j+1}} \frac{w_j^\perp w}{|w_j|} e^{-ixu}\,ds(x),$$

where the last integral denotes the line integral along the edge P_jP_{j+1}. Thus

$$\begin{aligned}\int_\Omega e^{-ixu}\,dx &= \frac{1}{iwu}\sum_{j=1}^{m}\frac{w_j^\perp w}{iw_ju}(e^{-iP_ju} - e^{-iP_{j+1}u}) \\ &= \sum_{j=1}^{m}\frac{e^{-iP_ju}}{iwu}\left(\frac{w_j^\perp w}{iw_ju}g(P_j;w_j) + \frac{w_{j-1}^\perp w}{iw_{j-1}u}g(P_j;-w_{j-1})\right),\end{aligned} \tag{2.4}$$

where g denotes the characteristic function of Ω.

Now let f satisfy the conditions of Theorem 1 for $n=0$. Since f is piecewise constant, applying (2.4) over each region bounded by but not intersected by lines in $M(v_0,\ldots v_r)$, and adding, gives

$$\int_{\mathbb{R}^2} f(x)e^{-ixu}\,dx = \sum_{v\in V}\frac{e^{-ivu}}{iwu}\sum_{k=0}^{r}(v_k^\perp w)\frac{f(v;v_k)+f(v;-v_k)}{iv_ku}, \quad u \in \mathbb{R}^2, \tag{2.5}$$

where V denotes the set of vertices of M. By (2.2), $f(v;v_k)+f(v;-v_k)=0$, $k=0,\ldots,r$, except for $v \in \mathbb{Z}^2$. Putting $w=v_0$ then gives (2.3). □

Lemma 2. *Theorem 1 holds for $n = 0$.*

Proof: If f satisfies the conditions of Theorem 1, then (2.3) is satisfied, which is of form (2.1). Thus f lies in S_0.

We now assume f lies in S_0 and must show that f is a piecewise constant function with compact support over $M = M(v_0, \dots, v_r)$ such that

$$f(x; -v_j) = -f(x; v_j), \quad x \in \mathbb{R}^2 \setminus \mathbb{Z}^2, \quad j = 0, \dots, r. \tag{2.6}$$

From (2.1), with $n = 0$, $\hat{f}$ may be written

$$\hat{f}(u) = \sum_{j \in N} \frac{e^{-iju}}{iuv_0} \sum_{k=1}^{r} \frac{a_{jk}}{iuv_k}, \quad u \in \mathbb{R}^2, \tag{2.7}$$

for constants a_{jk}, $j \in \mathbb{Z}^2$, $k = 1, \dots, r$, where N is a finite subset of $\mathbb{Z}^2$.

Since S_0 is unchanged by replacing v_j by $-v_j$, any $0 \le j \le r$, we may order $v_0, \dots, v_r$ so that $v_0, \dots, v_r, -v_0, \dots, -v_r, v_0$ are in order clockwise. Choose $\alpha \in \mathbb{Z}^2$ so that N lies in the interior of the sector bounded by the line segments $L_0 := \{\alpha + tv_0 : t \ge 0\}$ and $M_0 := \{\alpha + tv_r : t \ge 0\}$. Similarly choose $\beta \in \mathbb{Z}^2$ so that N lies in the interior of the sector bounded by the line segments $L_1 := \{\beta - tv_0 : t \ge 0\}$ and $M_1 := \{\beta - tv_r : t \ge 0\}$. Let D denote the bounded open region bounded by L_0, M_0, L_1, M_1. Thus N lies in D. Let V denote the set of vertices of M and write

$$(V \cap L_1 \cap \bar{D}) \setminus \{\beta\} = \{w_1, \dots, w_T\}, \quad V \cap M_1 \cap \bar{D} = \{z_1, \dots, z_U\}.$$

Now take $j \in N$. We shall define a piecewise constant function ϕ_j on the mesh M as follows. For $k = 1, \dots, r$, let T_k denote the intersection of D with the sector bounded by the line segments $\{j + tv_{k-1} : t \ge 0\}$ and $\{j + tv_k : t \ge 0\}$. We define ϕ_j to be a constant c_k on T_k, $k = 1, \dots r$, and zero elsewhere. Thus

$$\phi_j(j; v_k) = c_k - c_{k+1}, \quad k = 0, \dots, r,$$

where $c_0 = c_{r+1} = 0$. Now define $c_1, \dots, c_r$ so that $c_k = c_{k+1} + a_{jk}$, $k = 1, \dots, r$. Since $\phi_j(j; -v_k) = 0$, $k = 0, \dots, r$, we have

$$\phi_j(j; v_k) + \phi_j(j; -v_k) = a_{jk}, \quad k = 1, \dots, r. \tag{2.8}$$

Moreover

$$\phi_j(v; v_k) + \phi_j(v; -v_k) = 0, \quad v \in V, \quad v \notin \{j, w_1, \dots w_T, z_1, \dots, z_U\}. \tag{2.9}$$

Now let $\phi = \sum_{j \in N} \phi_j$. Then by (2.5) with $w = v_0$,

$$\hat{\phi}(u) = \sum_{v \in V} \frac{e^{-ivu}}{iv_0 u} \sum_{k=1}^{r} (v_k^{\perp} v_0) \frac{\phi(v; v_k) + \phi(v; -v_k)}{iv_k u}, \quad u \in \mathbb{R}^2, \tag{2.10}$$

and by (2.7)-(2.9),

$$\hat{\phi}(u) = \hat{f}(u) + \sum_{l=1}^{T} \frac{e^{-iw_l u}}{iuv_0} \sum_{k=1}^{r} \frac{b_{lk}}{iuv_k} + \sum_{m=1}^{U} \frac{e^{-iz_m u}}{iuv_0} \sum_{k=1}^{r} \frac{c_{mk}}{iuv_k}, \quad u \in \mathbb{R}^2, \quad (2.11)$$

for some numbers b_{lk}, c_{mk}, $l = 1, \ldots, T$, $m = 1, \ldots, U$, $k = 1 \ldots, r$.

Take k, $1 \leq k \leq r$. Since ϕ has compact support, $\hat{\phi}$ is continuous, and hence $\hat{\phi} - \hat{f}$ is continuous. Putting $u = \lambda v_k^\perp$ in (2.11) then shows that

$$\sum_{l=1}^{T} e^{-i\lambda w_l v_k^\perp} b_{lk} + \sum_{m=1}^{U} e^{-i\lambda z_m v_k^\perp} c_{mk} = 0, \quad \lambda \in \mathbb{R}. \qquad (2.12)$$

First suppose $1 \leq k \leq r-1$. Note that any line in M parallel to v_k cannot intersect both $L_1 \cap \bar{D}$ and $M_1 \cap \bar{D}$, except at β. Thus $w_l v_k^\perp$, $l = 1, \ldots, T$ and $z_m v_k^\perp$, $m = 1, \ldots, U$, are distinct. It follows from (2.12) that

$$b_{lk} = 0, \quad l = 1, \ldots T, \quad c_{mk} = 0, \quad m = 1, \ldots U.$$

Next take $k = r$. Since $w_l v_r^\perp$, $l = 1, \ldots, T$ are distinct and also distinct from $z_m v_r^\perp$, $m = 1, \ldots, U$, it follows from (2.12) that

$$b_{lr} = 0, \quad l = 1 \ldots, T.$$

Thus from (2.11),

$$\hat{\phi}(u) - \hat{f}(u) = \sum_{m=1}^{U} \frac{e^{-iz_m u}}{iuv_0} \frac{c_{mr}}{iuv_r}, \quad u \in \mathbb{R}^2. \qquad (2.13)$$

Putting $u = \lambda v_0^\perp$ in (2.13) gives

$$\sum_{m=1}^{U} \frac{c_{mr} e^{-i\lambda z_m v_0^\perp}}{v_0^\perp v_r} = 0, \quad \lambda \in \mathbb{R},$$

and since $z_m v_0^\perp$, $m = 1 \ldots, U$ are distinct,

$$c_{mr} = 0, \quad m = 1, \ldots, U.$$

Thus $f = \phi$ almost everywhere. So f is a piecewise constant function with compact support over M and it follows from (2.7) and (2.10) that (2.6) is satisfied. □

Proof of Theorem 1: To complete the proof by induction we assume that Theorem 1 holds for $n = m - 1 \geq 0$ and prove that it holds for $n = m$.

First take f in $S_m(v_0, \ldots, v_{m+r})$. Then $\hat{f}$ is of form (2.1) with $n = m$. So

$$(D_{v_{m+r}} f)\hat{}(u) = \frac{\sum_{|\alpha|=r-1} P_\alpha(e^{-iu}) u^\alpha}{(iuv_0) \ldots (iuv_{m+r-1})}, \quad u \in \mathbb{R}^2.$$

Thus $D_{v_{m+r}}f$ lies in $S_{m-1}(v_0, \ldots, v_{m+r-1})$ and so, by our inductive hypothesis, $D_{v_{m+r}}f$ is a C^{m-2} spline function of degree $m-1$ with compact support over $M(v_0, \ldots, v_{m+r-1})$ and hence over $M(v_0, \ldots, v_{m+r})$. Similarly $D_{v_0}f$ is a C^{m-2} spline function of degree $m-1$ over $M(v_0, \ldots, v_{m+r})$ with compact support. Thus f is a C^{m-1} spline function of degree m over $M(v_0, \ldots, v_{m+r})$, and outside a compact set f is a polynomial. Since $\hat{f}$ is continuous, this polynomial must vanish and so f has compact support.

Now take j, $0 \le j \le n+r$, and choose k with $k \ne j$. By our above argument, $D_{v_k}f$ lies in $S_{m-1}(v_0, \ldots, \hat{v}_k, \ldots, v_{m+r})$, where $\hat{v}_k$ denotes that v_k is omitted. So by our inductive hypothesis,

$$D_{m-1}D_{v_k}f(x; -v_j) = -D_{m-1}D_{v_k}f(x; v_j), \quad x \notin \mathbb{Z}^2,$$

where D_{m-1} is any $(m-1)$th order derivative. Since $D_{v_j}D_{m-1}f$ is continuous across any line in direction v_j, and v_k and v_j are linearly independent, it follows that

$$D_m f(x; -v_j) = -D_m f(x; v_j), \quad x \notin \mathbb{Z}^2,$$

for any mth order derivative D_m. Thus (2.2) is satisfied and f satisfies the conditions of Theorem 1.

Next we assume that f satisfies the conditions of Theorem 1 for $n = m$ and prove that f is in $S_m(v_0, \ldots, v_{m+r})$. Let $g = D_{v_{m+r}}f$. Then g is a C^{m-2} spline function of degree $m-1$ over $M(v_0, \ldots, v_{m+r})$ with compact support. For any derivative D_{m-1} of order $m-1$, $D_{m-1}g = D_{v_{m+r}}D_{m-1}f$, which is continuous across any line in direction v_{m+r}. Thus g is a spline function over $M(v_0, \ldots, v_{m+r-1})$. By (2.2) for $n = m$,

$$D_{m-1}D_{v_{m+r}}f(x; -v_j) = -D_{m-1}D_{v_{m+r}}f(x; v_j), \quad x \notin \mathbb{Z}^2, \quad j = 0, \ldots, m+r,$$

and hence g satisfies (2.2) for $n = m-1$. So by our inductive hypothesis, $D_{v_{m+r}}f$ lies in $S(v_0, \ldots, v_{m+r-1})$ and we can write

$$(iuv_{m+r})\hat{f}(u) = \hat{g}(u) = \frac{\sum_{|\alpha|=r-1} P_\alpha(e^{-iu})u^\alpha}{(iuv_0)\ldots(iuv_{m+r-1})} \quad u \in \mathbb{R}^2,$$

for Laurent polynomials P_α, $|\alpha| = r-1$. Thus f lies in $S_m(v_0, \ldots, v_{m+r})$.

So Theorem 1 holds for $n = m$ and the inductive proof is complete. □

We complete this section by showing that condition (2.2) is automatically satisfied if no more than $n+1$ lines in $M(v_0, \ldots, v_{n+r})$ intersect at points other than $\mathbb{Z}^2$.

Lemma 3. *For $s \ge 1$, suppose distinct lines $l_1, \ldots l_s$ in $\mathbb{R}^2$ intersect in a point. If $n \ge s-1$ and ϕ is a C^{n-1} spline function of degree n on the mesh formed by $l_1, \ldots, l_s$, then for any derivative D_n of order n, and $1 \le j \le s$, the jump of $D_n\phi$ across l_j is constant along l_j.*

Proof: The proof is by induction on s. For $s = 1$ the result is trivially true. Suppose it is true for $s = m \ge 1$. Take $n \ge m$ and lines $l_1, \ldots, l_{m+1}$ and ϕ as in the Lemma. For $j = 1, \ldots, m+1$, let v_j be a vector in direction of l_j. Then since ϕ is continuous, $D_{v_{m+1}}\phi$ is a C^{n-2} spline function of degree $n-1$ on the mesh formed by $l_1, \ldots, l_m$. So by our inductive hypothesis, for any derivative D_{n-1} of order $n-1$, and any $1 \le j \le m$, the jump of $D_{n-1}D_{v_{m+1}}\phi$ across l_j is constant along l_j. Also the jump of $D_{n-1}D_{v_{m+1}}\phi$ across l_{m+1} is zero along l_{m+1}. Similarly the jump of $D_{n-1}D_{v_1}\phi$ across l_j is constant, $j = 1, \ldots, m+1$. Thus for any derivative D_n of order n, the jump of $D_n\phi$ across l_j is constant along l_j, $j = 1, \ldots, m+1$. So the result holds for $s = m+1$ and the proof is complete. □

Theorem 1 and Lemma 3 immediately give the following.

Corollary 1. *If at most $n+1$ lines in $M(v_0, \ldots, v_{n+r})$ intersect at points not in $\mathbb{Z}^2$, then $S_n(v_0, \ldots, v_{n+r})$ comprises all C^{n-1} spline functions of degree n over $M(v_0, \ldots, v_{n+r})$ with compact support.*

§3. Generators

From the definition of $S_n = S_n(v_0, \ldots, v_{n+r})$ in the previous section, or from Theorem 1, we can see immediately that S_n is **shift-invariant**, i.e.

$$f \in S_n \Rightarrow f(\cdot - j) \in S_n, \quad j \in \mathbb{Z}^2,$$

and S_n is **refinable**, i.e.

$$f \in S_n \Rightarrow f(\frac{\cdot}{2}) \in S_n.$$

We shall say that $\phi = (\phi_1, \ldots, \phi_s) \in S_n^s$ is a **local generator** of S_n if every element of S_n is a finite linear combination of shifts (i.e. integer translates) of the components of ϕ, i.e. $f \in S_n$ if and only if

$$f = \sum_{j \in \mathbb{Z}^2} \phi(\cdot - j)c(j), \tag{3.1}$$

for a sequence c of $r \times 1$ matrices $c(j)$, $j \in \mathbb{Z}^2$, where c has finite support. Equations such as (3.1) can be neatly written in terms of Fourier transforms. For $s, t \in \mathbb{Z}$, $s, t \ge 1$, we shall denote by $L_{s \times t}$ the set of all $s \times t$ matrices of bivariate Laurent polynomials with real coefficients. Then (3.1) is equivalent to

$$\hat{f}(u) = \hat{\phi}(u)C(e^{-iu}), \quad u \in \mathbb{R}^2, \tag{3.2}$$

where $C \in L_{r \times 1}$ is given by

$$C(z) = \sum_{j \in \mathbb{Z}^2} c(j)z^j, \quad z \in (\mathbb{C} \setminus \{0\})^2.$$

We shall say that a matrix M in $L_{r \times r}$ is **invertible** if $M(z)$ is non-singular for all $z \in \mathbb{C}^2$, $z \ne 0$. Clearly this is equivalent to $\det M(z) = cz^k$ for some $k \in \mathbb{Z}^2$, $c \in \mathbb{R}$, $c \ne 0$. If M in $L_{r \times r}$ is invertible, then it has an inverse M^{-1} in $L_{r \times r}$.

Theorem 2. *The space $S_n = S_n(v_0, \ldots, v_{n+r})$ has a local generator $\phi = (\phi_1, \ldots, \phi_r)$. Moreover $\psi = (\psi_1, \ldots, \psi_r) \in S_n^r$ is a local generator for S_n if and only if*

$$\hat{\psi}(u) = \frac{\tilde{u}M(e^{-iu})}{(iuv_0)\ldots(iuv_{n+r})}, \quad u \in \mathbb{R}^2, \tag{3.3}$$

where $\tilde{u} = (u_1^{r-1}, u_1^{r-2}u_2, \ldots, u_2^{r-1})$ and $M \in L_{r\times r}$ satisfies

$$\det M(z) = cz^k \prod_{j=0}^{n+r}(1 - z^{v_j}), \quad z \in (\mathbb{C} \setminus \{0\})^2, \tag{3.4}$$

for some $k \in \mathbb{Z}^2$, $c \in \mathbb{R}$, $c \neq 0$.

For the proof of Theorem 2 we shall need the following.

Lemma 4. *If $v \in \mathbb{Z}^2$ has co-prime components, then the Laurent polynomial $z^v - 1$ on $(\mathbb{C} \setminus \{0\})^2$ is irreducible.*

Proof: Let $v = (j, k)$. By replacing z_1 by z_1^{-1} or z_2 by z_2^{-1}, if necessary, we may assume $j, k \geq 0$. We shall suppose $z^v - 1 = p(z)q(z)$, $z \in \mathbb{C}^2$, for non-constant polynomials p, q, and reach a contradiction.

Take $w \in \mathbb{C} \setminus \{0\}$ and let $z_1 = w^k$, $z_2 = w^{-j}$. Then $p(w^k, w^{-j})q(w^k, w^{-j}) = 0$ and so, without loss of generality,

$$p(w^k, w^{-j}) = 0, \quad w \in \mathbb{C} \setminus \{0\}.$$

Let

$$p(z_1, z_2) = \sum_{r=0}^{j}\sum_{s=0}^{k} a_{rs} z_1^r z_2^s, \quad z_1, z_2 \in \mathbb{C},$$

where $a_{jk} = 0$. Then

$$\sum_{r=0}^{j}\sum_{s=0}^{k} a_{rs} w^{rk-sj} = 0, \quad w \in \mathbb{C} \setminus \{0\}.$$

Since j and k are coprime, distinct terms in the above summation give different powers of w, except for the terms a_{00} and a_{jk}. Thus $a_{rs} = 0$, $0 \leq r \leq j$, $0 \leq s \leq k$. So $p = 0$, which is a contradiction. □

Proof of Theorem 2: We shall first construct recursively for $m = -1, \ldots, n$, a matrix M_m in $L_{r\times r}$ satisfying

$$\det M_m(z) = \prod_{j=0}^{m+r}(1 - z^{v_j}), \tag{3.5}$$

and for $j = 0, \ldots, m + r$,

$$\tilde{u}M_m(e^{-iu}) = 0 \quad \text{when} \quad u \in \mathbb{R}^2, \quad uv_j = 0. \tag{3.6}$$

For $r=1$ this is satisfied by $M_m(z)=\prod_{j=0}^{m+1}(1-z^{v_j})$ and so we may assume $r\geq 2$.

First we define M_{-1}. For $j=0,\ldots,r-1$, define

$$P_j(u)=\prod_{k=0,k\neq j}^{r-1} iuv_k, \quad u\in \mathbb{R}^2. \tag{3.7}$$

Then $P_0,\ldots,P_{r-1}$ give a basis for the space H_{r-1} of all homogeneous bivariate polynomials of exact degree $r-1$. Since the polynomials $u_1^{r-1-j}u_2^j$, $j=0,\ldots,r-1$, also form a basis for H_{r-1}, there is a constant non-singular $r\times r$ matrix A with

$$\tilde{u}A=(P_0(u),\ldots,P_{r-1}(u)), \quad u\in \mathbb{R}^2. \tag{3.8}$$

Then define

$$M_{-1}(z):=(\det A)^{-1}A\begin{bmatrix}1-z^{v_0} & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & \cdots & 0 & 1-z^{v_{r-1}}\end{bmatrix}, \quad z\in(\mathbb{C}\setminus\{0\})^2. \tag{3.9}$$

Clearly M_{-1} satsifies (3.5). Take $0\leq j\leq r-1$ and $u\in\mathbb{R}^2$ with $uv_j=0$. Then by (3.9) and (3.8),

$$\tilde{u}M_{-1}(e^{-iu})=(\det A)^{-1}(P_0(u)(1-e^{-iuv_0}),\ldots,P_{r-1}(u)(1-e^{-iuv_{r-1}}))=0,$$

since $1-e^{-iuv_j}=0$ and $P_k(u)=0$, $k=0,\ldots,r-1$, except for $k=j$. Thus M_{-1} satisfies (3.6).

We may now suppose that for some m, $0\leq m\leq n$, M_{m-1} has been constructed satisfying (3.5) and (3.6). When $uv_{m+r}=0$, then $u=tv_{m+r}^{\perp}$ for some $t\in\mathbb{R}$ and

$$\begin{aligned}\tilde{u}M_{m-1}(e^{-iu})&=t^{r-1}(v_{m+r}^{\perp})\tilde{\ }M_{m-1}(e^{-itv_{m+r}^{\perp}})\\ &=t^{r-1}q(e^{-it})(p_1(e^{-it}),\ldots,p_r(e^{-it})),\end{aligned} \tag{3.10}$$

for univariate Laurent polynomials $q,p_1,\ldots,p_r$, where $p_1,\ldots,p_r$ have no common factor.

Now it is well-known (see Theorem 6.21 of [11]) that if $p_1,\ldots,p_r$ are univariate poynomials with no common factor, then there is an $r\times r$ matrix R of univariate polynomials with

$$\det R(z)=1, \quad z\in\mathbb{C}, \tag{3.11}$$

$$[p_1,\ldots,p_r]R=[1,0,\ldots,0]. \tag{3.12}$$

Clearly this can be extended to the case of Laurent polynomials $p_1,\ldots,p_r$, where R is now a matrix of Laurent polynomials.

Now choose $v \in \mathbb{Z}^2$ with $v_{m+r}^{\perp} v = 1$ and define

$$M_m(z) := M_{m-1}(z) R(z^v) \begin{bmatrix} 1 - z^{v_{m+r}} & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \quad z \in (\mathbb{C} \setminus \{0\})^2. \tag{3.13}$$

Clearly (3.5) is satisfied and (3.6) is satisfied for $j = 0, \ldots, m-1+r$. If $uv_{m+r} = 0$, then by (3.10),

$$\begin{aligned} \tilde{u} M_m(e^{-iu}) &= t^{r-1} q(e^{-it})(p_1(e^{-it}), \ldots, p_r(e^{-it})) R(e^{-it}) \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \\ &= t^{r-1} q(e^{-it})(1, 0, \ldots, 0) \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = 0, \end{aligned} \tag{3.14}$$

by (3.12). Thus M_m satisfies (3.5) and (3.6) and our recursive construction is complete.

Now take m, $-1 \le m \le n$. We shall prove that if U_m in $L_{r \times 1}$ satisfies

$$\tilde{u} U_m(e^{-iu}) = 0 \ \forall u \in \mathbb{R}^2 \text{ with } uv_j = 0 \text{ for some } j, \quad 0 \le j \le m + r, \tag{3.15}$$

then there is V in $L_{r \times 1}$ with

$$U_m = M_m V. \tag{3.16}$$

We prove this by induction on m. First take $m = -1$ and suppose U_{-1} satisfies (3.15). Take A as in (3.8) and for $j = 1, \ldots, r$, let $e_j \in \mathbb{R}^r$ be defined by $(e_j)_k = \delta_{jk}$, $k = 1, \ldots, r$. Then for $0 \le j \le r - 1$ and $uv_j = 0$, (3.15) gives

$$0 = \tilde{u} U_{-1}(e^{-iu}) = \tilde{u} A A^{-1} U_{-1}(e^{-iu}) = P_j(u) e_j A^{-1} U_{-1}(e^{-iu}).$$

Thus when $z^{v_j} = 1$, the jth row of $A^{-1} U_{-1}(z)$ equals zero. Since $1 - z^{v_j}$ is irreducible, by Lemma 4, it follows from the Hilbert Nullstellensatz that the jth row of $A^{-1} U_{-1}(z)$ must be divisible by $1 - z^{v_j}$, $j = 0, \ldots, r-1$ (see Chapter 1 of [5]). Hence for some V_{-1} in $L_{r \times 1}$,

$$A^{-1} U_{-1}(z) = \begin{bmatrix} 1 - z^{v_0} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 - z^{v_{r-1}} \end{bmatrix} V_{-1}(z), \quad z \in (\mathbb{C} \setminus \{0\})^2,$$

and hence by (3.9), $U_{-1} = (\det A) M_{-1} V_{-1}$, which proves the result.

Now suppose that the result holds for $m-1$, $0 \le m \le n$. Take U_m in $L_{r\times 1}$ satisfying (3.15). It follows from our inductive hypothesis that

$$U_m = M_{m-1}V_{m-1}$$

for some V_{m-1} in $L_{r\times 1}$. By (3.11) $\det R = 1$ and hence $R^{-1}(z^v)$ is in $L_{r\times r}$. So we may write

$$U_m(z) = M_{m-1}(z)R(z^v)W(z), \quad z \in (\mathbb{C}\setminus\{0\})^2, \tag{3.17}$$

for some W in $L_{r\times 1}$. Now take $u \in \mathbb{R}^2$ with $uv_{m+r} = 0$. Then from (3.15) and (3.17), with $u = tv_{m+r}^{\perp}$,

$$\begin{aligned} 0 &= \tilde{u}U_m(e^{-iu}) \\ &= \tilde{u}M_{m-1}(e^{-iu})R(e^{-it})W(e^{-iu}) \\ &= t^{r-1}q(e^{-it})(1,0,\dots,0)W(e^{-iu}) \end{aligned}$$

on recalling (3.10) and (3.14). Thus $W(e^{-iu})_{1,1} = 0$ when $uv_{m+r} = 0$ and hence $W(z)_{1,1}$ is divisible by $1 - z^{v_{m+r}}$. So

$$W(z) = \begin{bmatrix} 1 - z^{v_{m+r}} & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} V(z), \quad z \in (\mathbb{C}\setminus\{0\})^2, \tag{3.18}$$

for some V in $L_{r\times 1}$. From (3.17), (3.18) and (3.13) we have (3.16) and so the result holds for m. So the inductive proof is complete.

We now define $\phi = (\phi_1, \dots, \phi_r)$ by

$$\hat{\phi}(u) = \frac{\tilde{u}M_n(e^{-iu})}{(iuv_0)\dots(iuv_{n+r})}, \quad u \in \mathbb{R}^2. \tag{3.19}$$

By (3.6), $\hat{\phi}$ is continuous and so the components of ϕ lie in S_n, by the definition of S_n. We now show that ϕ is a local generator for S_n. Take f in S_n. Then

$$\hat{f}(u) = \frac{\tilde{u}U_n(e^{-iu})}{(iuv_0)\dots(iuv_{n+r})}, \quad u \in \mathbb{R}^2, \tag{3.20}$$

where $\hat{f}$ is continuous and so U_n in $L_{r\times 1}$ satisfies (3.15) for $m = n$. So by (3.16), $U_n = M_nV$ for some V in $L_{r\times 1}$ and so by (3.19) and (3.20),

$$\hat{f}(u) = \hat{\phi}(u)V(e^{-iu}), \quad u \in \mathbb{R}^2.$$

Thus f is a finite linear combination of the shifts of the components of ϕ. So ϕ is a local generator for S_n.

Now take any $\psi = (\psi_1, \ldots, \psi_r) \in S_n^r$. Then

$$\hat{\psi}(u) = \hat{\phi}(u) A(e^{-iu}), \quad u \in \mathbb{R}^2, \tag{3.21}$$

for some $A \in L_{r\times r}$, and so ψ satisfies (3.3) with

$$M = M_n A. \tag{3.22}$$

Suppose ψ is a local generator for S_n. Then

$$\hat{\phi}(u) = \hat{\psi}(u) B(e^{-iu}), \quad u \in \mathbb{R}^2,$$

for some $B \in L_{r\times r}$, and so from (3.22),

$$M_n = MB = M_n AB.$$

Thus $\det AB = 1$ and so A is invertible. Then (3.4) follows from (3.22) and (3.5).

Conversely suppose that ψ satisfies (3.4). From (3.22) we have $\det A = cz^k$ and so A is invertible. Then from (3.21),

$$\hat{\phi}(u) = \hat{\psi}(u) A^{-1}(e^{-iu}), \quad u \in \mathbb{R}^2.$$

So for any $f \in S_n$, there is $C \in L_{r\times 1}$ with

$$\hat{f}(u) = \hat{\phi}(u) C(e^{-iu}) = \hat{\psi}(u) A^{-1}(e^{-iu}) C(e^{-iu}), \quad u \in \mathbb{R}^2,$$

and hence ψ is a local generator for S_n. □

For the case $r = 1$, Theorem 2 states that the function ψ is a local generator for S_n if and only if

$$\hat{\psi}(u) = \frac{ce^{-iku} \prod_{j=0}^{n+1} (1 - e^{-iuv_j})}{(iuv_0) \ldots (iuv_{n+1})}, \quad u \in \mathbb{R}^2,$$

for some $k \in \mathbb{Z}^2$, $c \in \mathbb{R}$, $c \neq 0$, which is equivalent to

$$\psi = cB_n(. - k | v_0, \ldots, v_{n+1}),$$

i.e. ψ is a shift of a multiple of a box spline. For this reason we shall refer to any local generator ϕ for $S_n(v_0, \ldots, v_{n+r})$ as a multi-box spline, extending the terminology of [7] for the case $r = 2$.

§4. Stability

Suppose that $\phi = (\phi_1, \ldots, \phi_r)$ is a local generator for $S_n = S_n(v_0, \ldots, v_{n+r})$. Then any function f in S_n can be written in the form

$$f = \sum_{j \in \mathbb{Z}^2} \phi(. - j)c(j), \tag{4.1}$$

for a sequence c of $r \times 1$ matrices $c(j)$, $j \in \mathbb{Z}^2$, where c has finite support. We shall say that ϕ is **stable** if the shifts of ϕ form a Riesz basis, i.e. the representation (4.1) is stable in the sense that there are constants A, $B > 0$ so that for any f in S_n,

$$A \sum_{j \in \mathbb{Z}^2} |c(j)|^2 \leq ||f||_2^2 \leq B \sum_{j \in \mathbb{Z}^2} |c(j)|^2.$$

It follows from [10] that ϕ is stable if and only if no linear combination of $\phi_1, \ldots, \phi_r$ has a real 2π-periodic zero on the Fourier domain, i.e. there do not exist $u \in \mathbb{R}^2$ and non-zero $c \in \mathbb{Z}^r$ with

$$\hat{\phi}(u + 2\pi j)c^T = 0, \quad j \in \mathbb{Z}^2. \tag{4.2}$$

Theorem 3. *For the space $S_n = S_n(v_0, \ldots, v_{n+r})$, the following are equivalent.*

(a) *There is a stable local generator $\phi = (\phi_1, \ldots, \phi_r)$ of S_n.*

(b) *Every local generator $\phi = (\phi_1, \ldots, \phi_r)$ of S_n is stable.*

(c) *At most r lines in $M(v_0, \ldots, v_{n+r})$ intersect except at points in $\mathbb{Z}^2$.*

(d) *For each $u \in \mathbb{R}^2 \setminus 2\pi\mathbb{Z}^2$, there are at most r vectors v_j in $\{v_0, \ldots, v_{n+r}\}$ with $e^{iuv_j} = 1$.*

Proof: *(a)* $\Rightarrow$ *(b)*. Suppose $\phi = (\phi_1, \ldots, \phi_r)$ is a stable local generator of S_n. Let $\psi = (\psi_1, \ldots, \psi_r)$ be any local generator for S_n. Then

$$\hat{\phi}(u) = \hat{\psi}(u)A(e^{-iu}), \quad \hat{\psi}(u) = \hat{\phi}(u)B(e^{-iu}), \quad u \in \mathbb{R}^2,$$

for some $A, B \in L_{r \times r}$. Thus

$$\hat{\phi}(u) = \hat{\phi}(u)B(e^{-iu})A(e^{-iu}). \quad u \in \mathbb{R}^2,$$

So for any $u \in \mathbb{R}^2$,

$$\hat{\phi}(u + 2\pi j)(I - B(e^{-iu})A(e^{-iu})) = 0, \quad j \in \mathbb{Z}^2,$$

and since ϕ is stable, condition (4.2) shows that $I - B(e^{-iu})A(e^{-iu}) = 0$. Thus $B(e^{-iu})$ is non-singular for all $u \in \mathbb{R}^2$.

Now suppose that for $u \in \mathbb{R}^2$, $c \in \mathbb{Z}^r$, $\hat{\psi}(u + 2\pi j)c^T = 0$, for all $j \in \mathbb{Z}^2$. Then

$$\hat{\phi}(u + 2\pi j)B(e^{-iu})c^T = 0, \quad j \in \mathbb{Z}^2.$$

By stability of ϕ, $B(e^{-iu})c^T = 0$ and so $c = 0$. Thus ψ is stable. (Note that this proof does not depend on the particular structure of S_n and so holds for any shift-invariant space.)

$(c) \Leftrightarrow (d)$. Suppose that $x \in \mathbb{R}^2$ lies in a line in $M = M(v_0, \dots, v_{n+r})$ parallel to v_j, $0 \le j \le n+r$, i.e. $x = k + tv_j$, $k \in \mathbb{Z}^2$, $t \in \mathbb{R}$. Then $xv_j^\perp \in \mathbb{Z}$. Conversely suppose that $x \in \mathbb{R}^2$ satisfies $xv_j^\perp = l \in \mathbb{Z}$. Then we may choose $m \in \mathbb{Z}^2$ with $mv_j^\perp = l$ and so $(x-m)v_j^\perp = 0$. Thus $x - m = tv_j$, some $t \in \mathbb{R}$, and so x lies on a line in M parallel to v_j.

Now for $u \in \mathbb{R}^2$, $e^{iuv_j} = 1$ if and only if $uv_j \in 2\pi\mathbb{Z}$ if and only if $u^\perp v_j^\perp \in 2\pi\mathbb{Z}$ if and only if $(2\pi)^{-1}u^\perp$ lies on a line in M parallel to v_j. It follows that (c) and (d) are equivalent.

Now let $\phi = (\phi_1, \dots, \phi_r)$ be the generator of S_n constructed in the proof of Theorem 2. We shall complete the proof of Theorem 3 by showing that (d) holds if and only if ϕ is stable.

We recall that ϕ can be given by

$$\hat{\phi}(u) = \frac{P(u)D(e^{-iu})N(e^{-iu})}{(iuv_0)\dots(iuv_{n+r})}, \quad u \in \mathbb{R}^2, \tag{4.3}$$

where $P = (P_0, \dots, P_{r-1})$, given by (3.7), $D(z) = \mathrm{diag}(1 - z^{v_0}, \dots, 1 - z^{v_{r-1}})$ and $N \in L_{r\times r}$ satisfies

$$\det N(z) = \prod_{j=r}^{n+r} (1 - z^{v_j}), \quad z \in (\mathbb{C} \setminus \{0\})^2. \tag{4.4}$$

We recall further that $\hat{\phi}$ is defined recursively as $\hat{\phi} = f_n$, where

$$f_{-1}(u) = \frac{P(u)D(e^{-iu})}{(iuv_0)\dots(iuv_{r-1})} = \left(\frac{1 - e^{-iuv_0}}{iuv_0}, \dots, \frac{1 - e^{-iuv_{r-1}}}{iuv_{r-1}}\right), \quad u \in \mathbb{R}^2, \tag{4.5}$$

while for $m = 0, \dots, n$, for $uv_{m+r} \ne 0$,

$$f_m(u) = \frac{f_{m-1}(u)}{iuv_{m+r}} R_m(e^{-iu}) \begin{bmatrix} 1 - e^{-iuv_{m+r}} & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{4.6}$$

for $R_m \in L_{r\times r}$ satisfying $\det R_m(z) = 1$, $z \in (\mathbb{C} \setminus \{0\})^2$.

First we assume that ϕ is stable and show that (d) holds. Suppose to the contrary that (d) is false. Then, without loss of generality, we may choose $u \in \mathbb{R}^2 \setminus 2\pi\mathbb{Z}^2$ with $e^{-iuv_l} = 1$, $l = 0, \dots, r$. Then from (4.4), $\det N(e^{-iu}) = 0$ and so we may choose non-zero $c \in \mathbb{C}^r$ with $N(e^{-iu})c^T = 0$. Now for any $j \in \mathbb{Z}^2$, (4.3) gives

$$\hat{\phi}(y + 2\pi j)c^T = \frac{P(y + 2\pi j)D(e^{-iy})N(e^{-iy})c^T}{i(y + 2\pi j)v_0 \dots i(y + 2\pi j)v_{n+r}}, \quad y \in \mathbb{R}^2. \tag{4.7}$$

Note that $D(e^{-iu}) = 0$ and $N(e^{-iu})c^T = 0$, but at most one of the terms in the denominator of (4.7) can vanish when $y = u$. Thus (4.2) holds and so ϕ is not stable.

To finish we assume that (d) holds and prove that ϕ is stable, i.e. we show that for any $u \in \mathbb{R}^2$, there is no non-zero $c \in \mathbb{Z}^r$ satisfying (4.2).

First suppose that $u \notin 2\pi\mathbb{Z}^2$. We shall assume (4.2) holds for some non-zero $c \in \mathbb{Z}^r$ and reach a contradiction. By (d) we may assume, without loss of generality, that $e^{-iuv_j} \neq 1$, $j = r, \ldots, n+r$. So from (4.4), $\det N(e^{-iu}) \neq 0$. Then $d^T := N(e^{-iu})c^T \neq 0$ and by (4.2) and (4.3),

$$P(u + 2\pi j)D(e^{-iu})d^T = 0, \quad j \in \mathbb{Z}^2.$$

Thus the homogeneous polynomial $P(\cdot)D(e^{-iu})d^T$ must be identically zero and hence

$$D(e^{-iu})d^T = 0. \tag{4.8}$$

If for some k, $0 \le k \le r-1$, we have $e^{-iuv_k} \neq 1$, then from (4.8), $d_k = 0$. We shall now take k, $0 \le k \le r-1$, with $e^{-iuv_k} = 1$ and we shall also show $d_k = 0$, which will give the required contradiction.

Since $uv_k \in 2\pi\mathbb{Z}$, we may choose $j \in \mathbb{Z}^2$ with

$$(u + 2\pi j)v_k = 0. \tag{4.9}$$

Let $\mathcal{D}$ be a directional derivative with

$$\mathcal{D}(yv_k) = 1, \quad y \in \mathbb{R}^2. \tag{4.10}$$

Since the denominator of (4.3) vanishes at $u + 2\pi j$, (4.2) gives

$$\begin{aligned} 0 &= \mathcal{D}(P(y)D(e^{-iy})N(e^{-iy})c^T)(u + 2\pi j) \\ &= P(u + 2\pi j)\mathcal{D}(D(e^{-iy})N(e^{-iy})c^T)(u), \end{aligned} \tag{4.11}$$

by (4.8). By (4.9), $P_l(u + 2\pi j) = 0$, $l \neq k$, and

$$P_k(u + 2\pi j) = \prod_{l=0, l\neq k}^{r-1} i(u + 2\pi j)v_l \neq 0.$$

Then by (4.11), recalling that $e^{-iuv_k} = 1$,

$$0 = \mathcal{D}(1 - e^{-iyv_k})(u)d_k = id_k,$$

by (4.10). Thus $d_k = 0$, $k = 0, \ldots, r-1$, which contradicts $d \neq 0$.

Finally we take $u \in 2\pi\mathbb{Z}^2$ and show that there is no non-zero $c \in \mathbb{Z}^r$ satisfying (4.2) (which in this case holds independently of (d)). Without loss of generality we may suppose that $u = 0$.

First note that from (4.6), for $m = 0, \ldots, n$,

$$f_m(u)_1 = (f_{m-1}(u)R_m(e^{-iu}))_1 \frac{1 - e^{-iuv_{m+r}}}{iuv_{m+r}}, \quad u \in \mathbb{R}^2, \tag{4.12}$$

and so

$$f_m(0)_1 = (f_{m-1}(0)R_m(1))_1. \tag{4.13}$$

Also from (4.6) we see that for f_m to be continuous we require that for $m = 0, \ldots, n$,

$$(f_{m-1}(0)R_m(1))_j = 0, \quad j = 2, \ldots, r. \tag{4.14}$$

Now from (4.5), $f_{-1}(0) \neq 0$. Assume that for some m, $0 \leq m \leq n$, $f_{m-1}(0)_1 \neq 0$. Since $R_m(1)$ is non-singular, $f_{m-1}(0)R_m(1) \neq 0$. So from (4.13) and (4.14), $f_m(0)_1 \neq 0$. So by induction,

$$f_m(0)_1 \neq 0, \quad m = -1, \ldots, n. \tag{4.15}$$

Now for $0 \leq m \leq n$, let $\tilde{R}_m(1)$ denote the matrix gained by deleting the first row and column of $R_m(1)$. We claim $\tilde{R}_m(1)$ is non-singular. For suppose $(v_2, \ldots, v_r)\tilde{R}_m(1) = 0$, i.e. $(0, v_2, \ldots, v_r)R_m(1)_j = 0$, $j = 2, \ldots, r$. By (4.14), recalling that $R_m(1)$ is non-singular, $(0, v_2, \ldots, v_r) = \lambda f_{m-1}(0)$, some $\lambda \in \mathbb{C}$, and by (4.15), $\lambda = 0$. Thus $\tilde{R}_m(1)$ is non-singular.

Take $c \in \mathbb{Z}^r$ with $\tilde{c} := (c_2, \ldots, c_r) \neq 0$. Then for $u \in 2\pi\mathbb{Z}^2$, $uv_{n+r} \neq 0$, (4.6) gives

$$\hat{\phi}(u)c^T = f_n(u)c^T = \frac{f_{n-1}(u)}{iuv_{n+r}} R_n(1) \begin{bmatrix} 0 \\ \tilde{c}^T \end{bmatrix} = \frac{f_{n-1}(u)d^T}{iuv_{n+r}},$$

where $(d_2, \ldots, d_r)^T = \tilde{R}_n(1)\tilde{c}^T \neq 0$. Continuing in this way we obtain for $u \in 2\pi\mathbb{Z}^2$, $uv_j \neq 0$, $j = r, \ldots, n+r$,

$$\hat{\phi}(u)c^T = \frac{f_{-1}(u)e^T}{(iuv_n)\ldots(iuv_{n+r})}, \tag{4.16}$$

where $(e_2, \ldots e_r) \neq 0$. Choose k, $2 \leq k \leq r$, with $e_k \neq 0$. Then from (4.5) and (4.16), $\hat{\phi}(2\pi v_{k-1}^{\perp})c^T \neq 0$ and so (4.2) is not satisfied.

If $c = (1, 0, \ldots, 0)$, then $\hat{\phi}(0)c^T = f_n(0)_1 \neq 0$, by (4.15). Thus for any non-zero $c \in \mathbb{Z}^r$, (4.2) is not satisfied and the proof is complete. □

For the case $r = 1$, Theorem 3 states that the shifts of a box spline $B_n(.|v_0, \ldots, v_{n+1})$ are stable if and only if no lines in $M(v_0, \ldots, v_{n+1})$ intersect except at points in $\mathbb{Z}^2$. In fact under this condition the shifts of B_n satisfy the stronger property of being locally linearly independent [3,9]. For $r = 2$, Theorem 3 was proved in [7]. We shall now see that the maximum number of vectors $v_0, \ldots, v_{n+r}$ for which (c) of Theorem 3 can be satisfied is precisely $3r$. For $r = 1$ we may choose, for example, $\{(1,0), (0,1), (1,1)\}$. The case $r = 2$ is studied in some detail in [7] which gives, in particular, the choice $\{(1,0), (0,1), (1,1), (1,-1), (2,1), (1,-2)\}$.

Theorem 4. *For fixed $r \geq 1$, the maximum value of n for which the conditions of Theorem 3 can be satisfied is precisely $2r - 1$.*

Proof: Suppose that the conditions of Theorem 3 are satisfied. If the components of v_j are both odd, then $e^{iuv_j} = 1$ for $u = (\pi, \pi)$. If $(v_j)_1$ is even (respectively odd) and $(v_j)_2$ is odd (respectively even), then $e^{iuv_j} = 1$ for $u = (\frac{\pi}{2}, 0)$ (respectively $(0, \frac{\pi}{2})$). By condition (d), there are at most r vectors v_j of each of the above three types. Since the components of v_j cannot both be even, there are at most $3r$ vectors v_j, i.e. $n \leq 2r - 1$.

It remains to show that for $r \geq 1$, there is a choice of $3r$ vectors $v_0, \ldots, v_{3r-1}$ satisfying (d) of Theorem 3. For $r = 1$ and 2, we have already given such a choice. We now assume that we have such a choice for $r = s \geq 2$, and construct vectors v_{3s}, v_{3s+1}, v_{3s+2} so that $v_0, \ldots, v_{3s+2}$ satsify (d) for $r = s + 1$. We note that in (d) we may assume that u is in $[0, 2\pi)^2 \setminus \{0\}$ and $\frac{u}{2\pi}$ is rational.

Let $\{u_l : l = 1, \ldots, m\}$ denote all points in $[0, 2\pi)^2 \setminus \{0\}$ which satisfy $e^{iu_l v_j} = 1$ for at least two choices of j in $\{0, \ldots, 3s - 1\}$. for $l = 1, \ldots, m$, let $u_l = 2\pi(\frac{P_l}{R_l}, \frac{Q_l}{R_l})$, for integers P_l, Q_l, R_l with $0 \leq P_l, Q_l < R_l$. We let $v_{3s} = (N, N + 1)$, $v_{3s+1} = (N + 1, N)$, $v_{3s+2} = (N - 1, N + 1)$, where N is an even positive integer divisible by $R_1, \ldots, R_m$ and so that v_{3s+j} and v_k are linearly independent, $j = 0, 1, 2$, $k = 0, \ldots, 3s - 1$. Clearly the components of v_{3s+j} are coprime for $j = 0, 1, 2$.

Now take $1 \leq l \leq m$. Then

$$u_l v_{3s} = 2\pi \frac{P_l}{R_l} N + 2\pi \frac{Q_l}{R_l}(N + 1) = 2\pi \frac{Q_l}{R_l}, \quad (\text{mod } 2\pi\mathbb{Z}).$$

Thus $e^{iu_l v_{3s}} = 1$ only if $Q_l = 0$. Similarly $e^{iu_l v_{3s+1}} = 1$ only if $P_l = 0$ and $e^{iu_l v_{3s+2}} = 1$ only if $P_l = Q_l$. Since at most one of these can occur, $e^{iu_l v_j} = 1$ for at most $s + 1$ choices of j in $\{0, \ldots, 3s + 2\}$.

We now show that (d) holds for $r = s + 1$ and $u \notin \{u_1, \ldots, u_m\}$. Let $0 \neq u = 2\pi(\frac{P}{R}, \frac{Q}{R})$, for integers P, Q, R with $0 \leq P, Q < R$. Then $e^{-iuv_{3s+j}} = 1$, $j = 0, 1, 2$ if and only if

$$PN + Q(N + 1) \in R\mathbb{Z}, \tag{4.17}$$

$$P(N + 1) + QN \in R\mathbb{Z}, \tag{4.18}$$

$$P(N - 1) + Q(N + 1) \in R\mathbb{Z}. \tag{4.19}$$

Subtracting (4.19) from (4.17) shows that $P \in R\mathbb{Z}$ and hence $P = 0$. Then subtracting (4.18) from (4.17) shows similarly that $Q = 0$, which contradicts $u \neq 0$. Since $e^{-iuv_j} = 1$ for at most one choice of j in $\{0, \ldots, 3s - 1\}$, we see that $e^{-iuv_j} = 1$ for at most three choices of j in $\{0, \ldots, 3s + 2\}$. Since $s + 1 \geq 3$, (d) is satisfied for $r = s + 1$, which completes the proof by induction. □

While the construction in the above proof is useful for proving existence, it may not give a very 'good' choice of the vectors $\{v_0, \ldots, v_{3r-1}\}$ satisfying the conditions. For $r = 3$, a simple choice is

$$\{(1, 0), (0, 1), (1, 1), (1, -1), (2, 1), (1, -2), (1, 2), (2, -1), (1, 3)\},$$

which is not given by the construction in the above proof.

§5. Symmetry

We see from Theorem 2 that a generator for $S_n = S_n(v_0, \ldots, v_{n+r})$ is by no means unique. Indeed if $\phi = (\phi_1, \ldots, \phi_r)$ is a local generator for S_n, then $\psi = (\psi_1, \ldots, \psi_r)$ is also a local generator if and only if

$$\hat{\psi}(u) = \hat{\phi}(u) B(e^{-iu}), \quad u \in \mathbb{R}^2,$$

where B in $L_{r\times r}$ is invertible. For $r \geq 2$, there is a large choice of such B and hence a large choice of generators. We would therefore like to find generators which have 'good' properties, and one such property is symmetry.

We shall say $\phi = (\phi_1, \ldots, \phi_r)$ is symmetric if for $j = 1, \ldots, r$, there is $C_j \in \mathbb{Z}^2$ and $\sigma = 1$ or -1, such that

$$\phi_j(-x) = \sigma_j \phi_j(x + C_j), \quad x \in \mathbb{R}^2, \tag{5.1}$$

i.e. ϕ_j is even or odd about $\frac{1}{2}C_j$ as σ_j equals 1 or -1. We note that if $\tilde{\phi}_j = \phi_j(. - k)$ for some $k \in \mathbb{Z}^2$, then

$$\tilde{\phi}(-x) = \sigma_j \phi(x + k + C_j) = \sigma_j \tilde{\phi}_j(x + \tilde{C}_j), \quad x \in \mathbb{R}^2,$$

where $\tilde{C}_j = C_j + 2k$. Thus by replacing ϕ_j by one of its shifts, there is no loss of generality on assuming that $C_j \in \{0,1\}^2$. Note that taking Fourier transforms of (5.1) gives

$$\hat{\phi}_j(-u) = \sigma_j e^{iuC_j} \hat{\phi}_j(u), \quad u \in \mathbb{R}^2. \tag{5.2}$$

We conjecture that any space $S_n = S_n(v_0, \ldots, v_{n+r})$ has a symmetric local generator, but we have not proved this. What we now show is that any such generator has certain restrictions on the choice of σ_j, C_j in (5.1). This is very useful in restricting the choice when we are looking for symmetric local generators.

Theorem 5. *If $\phi = (\phi_1, \ldots, \phi_r)$ is a symmetric local generator of S_n satisfying (5.1) for $j = 1, \ldots, r$, then*

$$\sum_{j=1}^{r} C_j = \sum_{j=0}^{n+r} v_j, \quad \text{mod } 2\mathbb{Z}^2, \tag{5.3}$$

and

$$\sigma_1 \ldots \sigma_r = (-1)^{(n+1)(r+1)}. \tag{5.4}$$

Proof: By Theorem 2,

$$\hat{\phi}(u) = \frac{\tilde{u} M(e^{-iu})}{(iuv_0)\ldots(iuv_{n+r})}, \quad u \in \mathbb{R}^2, \tag{5.5}$$

for $M \in L_{r\times r}$ satisfying (3.4). Thus

$$\hat{\phi}(-u) = (-1)^n \frac{\tilde{u} M(e^{iu})}{(iuv_0)\dots(iuv_{n+r})}, \quad u \in \mathbb{R}^2. \tag{5.6}$$

From (5.2), (5.5) and (5.6) we have for $u \in \mathbb{R}^2$,

$$\sigma_j e^{iuC_j} M(e^{-iu})_j = (-1)^n M(e^{iu})_j, \quad j = 1, \dots, r. \tag{5.7}$$

Putting $V = \sum_{j=0}^{n+r} v_j$, (3.4) gives

$$\det M(z^{-1}) = (-1)^{n+r+1} z^{-2k-V} \det M(z), \quad z \in (\mathbb{C} \setminus \{0\})^2,$$

and comparing with (5.7) gives, for $C = \sum_{j=1}^{r} C_j$,

$$\sigma_1 \dots \sigma_r e^{iuC} = (-1)^{nr} (-1)^{n+r+1} e^{iu(2k+V)}, \quad u \in \mathbb{R}^2,$$

from which (5.3) and (5.4) follow. □

For $r = 1$, Theorem 5.1 shows that a box-spline $B_n(.|v_0, \dots, v_{n+1})$ is even about $\frac{1}{2}\sum_{j=0}^{n+1} v_j$, as is well-known. We note that if n and r are even, then from (5.4) we must have $\sigma_j = -1$ for some j, $1 \le j \le r$, and so, in particular, ϕ_j cannot be a positive function.

We now consider other types of symmetries. Let T be a unitary 2×2 matrix of real numbers, and suppose that

$$\{v_0 T, \dots, v_{n+r} T\} = \{t_0 v_0, \dots, t_{n+r} v_{n+r}\}, \tag{5.8}$$

where $t_j = 1$ or -1, $j = 0, \dots, n+r$. Then from the definition (2.1), S_n is invariant under T, i.e. $\{f(\cdot T) : f \in S_n\} = S_n$, where $f(\cdot T)(x) = f(xT)$, $x \in \mathbb{R}^2$. Moreover from Thorem 2, if ϕ is a local generator of S_n, then so is $\phi(\cdot T)$. We shall suppose that ϕ is symmetric under T in the sense that for a permutation P of $\{1, \dots, r\}$,

$$\phi_j(xT) = \tau_j \phi_{P(j)}(x + a_j), \quad x \in \mathbb{R}^2, \quad j = 1, \dots, r, \tag{5.9}$$

for $a_j \in \mathbb{Z}^2$, $\tau_j = 1$ or -1, $j = 1 \dots, r$. If $P(j) = j$, then we may assume, as before, that $a_j \in \{0,1\}^2$. We note that the symmetry considered in Theorem 5 is a special case of that above where $T = -I$ and $P = I$. Now $(\widetilde{uT})$ gives a basis for H_{r-1} and so, as in the proof of Theorem 2, there is a non-singular constant $r \times r$ matrix A with $(\widetilde{uT}) = \tilde{u} A$.

Theorem 6. *If $\phi = (\phi_1, \dots, \phi_r)$ is a local generator of S_n satisfying (5.9), then*

$$\tau_1 \dots \tau_r = (t_0 \dots t_{n+r})^{r-1} \sigma(P) \det A, \tag{5.10}$$

where $\sigma(P) = 1$ or -1 as P is even or odd.

Proof: From (5.5) and (5.8),

$$\hat{\phi}(\cdot T) = t_0 \dots t_{n+r} \frac{\tilde{u} A M(e^{-iuT})}{(iuv_0)\dots(iuv_{n+r})}, \quad u \in \mathbb{R}^2. \tag{5.11}$$

From (5.9), (5.5) and (5.11) we have for $u \in \mathbb{R}^2$,

$$\tau_j e^{iua_j} M(e^{-iu})_{k,P(j)} = t_0 \dots t_{n+r} AM(e^{-iuT})_{k,j}, \quad j,k = 1,\dots,r. \tag{5.12}$$

Now from (3.4),

$$\det M(e^{-iuT}) = t_0 \dots t_{n+r} e^{-iut} \det M(e^{-iu}),$$

for some $l \in \mathbb{Z}^2$, and comparing with (5.12) gives

$$\tau_1 \dots \tau_r \sigma(P) = (t_0 \dots t_{n+r})^{r+1} \det A e^{-ium},$$

for some $m \in \mathbb{Z}^2$, which gives (5.10). □

Henceforward we assume that T is symmetric, i.e. $T^2 = I$. Then $P^2 = I$ and if $P(j) = k \neq j$ we may, without loss of generality, choose ϕ_k to equal $\phi_j(\cdot T)$, and thus $\tau_j = \tau_k = 1$, $a_j = a_k = 0$.

Now suppose that ϕ is symmetric under T, and ϕ is also symmetric (as in (5.1)). Then if $P(j) = k \neq j$, $x \in \mathbb{R}^2$,

$$\phi_k(-x) = \phi_j(-xT) = \sigma_j \phi_j(xT + C_j) \tag{5.13}$$

and

$$\phi_k(-x) = \sigma_k \phi_k(x + C_k) = \sigma_k \phi_j(xT + C_k T). \tag{5.14}$$

Since (5.13) and (5.14) hold for all $x \in \mathbb{R}^2$, and ϕ_j has compact support,

$$\sigma_j = \sigma_k, \quad C_j = C_k T. \tag{5.15}$$

Next suppose that $P(j) = j$. Then for any $x \in \mathbb{R}^2$,

$$\begin{aligned} \phi_j(-xT) &= \sigma_j \phi_j(xT + C_j) \\ &= \sigma_j \phi_j((x + C_j T)T) \\ &= \sigma_j \tau_j \phi_j(x + C_j T + a_j), \end{aligned} \tag{5.16}$$

and

$$\phi_j((-x)T) = \tau_j \phi_j(-x + a_j) = \sigma_j \tau_j \phi_j(x - a_j + C_j). \tag{5.17}$$

Thus

$$C_j T + 2a_j = C_j. \tag{5.18}$$

We now consider the case

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Here $A_{ij} = \delta_{i,r-j+1}$, $i,j = 1,\dots,r$, and so (5.10) becomes

$$\tau_1 \dots \tau_r = (-1)^{\frac{1}{2}r(r-1)} \sigma(P)(t_0 \dots t_{n+r})^{r-1}. \tag{5.19}$$

If $P(j) = k \neq j$, then (5.15) holds. If $P(j) = j$, then from (5.18), $2a_j = ((C_j)_1 - (C_j)_2)(1,-1)$ and hence $a_j = 0$ and $C_j = 0$ or $(1,1)$. Thus we have $a_j = 0$, $j = 1,\dots,r$.

Finally consider the case

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Here $A_{ij} = (-1)^{r-i}\delta_{i,j}$, $i,j = 1,\dots,r$, and so (5.10) again gives (5.19). Again (5.15) holds if $P(j) = k \neq j$. If $P(j) = j$, then from (5.18), $a_j = ((C_j)_1, 0)$.

§6. Examples

We shall illustrate the preceding work with three examples. Throughout we write $z = e^{-iu}$, $w = e^{-iv}$.

Example 1. Take $n = 0$, $r = 3$ and

$$\{v_0, \ldots, v_3\} = \{(1,0), (0,1), (1,1), (1,-1)\}.$$

The mesh $M = M(v_0, \ldots, v_3)$ is called a type-2 triangulation. From Theorem 1, $S_0 = S_0(v_0, \ldots, v_3)$ comprises all piecewise constant functions f over M with compact support such that at any point $x = (j + \frac{1}{2}, k + \frac{1}{2})$, $j, k \in \mathbb{Z}$, the jump in f across the mesh lines $\{(j+t, k+t) : t \in \mathbb{R}\}$ and $\{(j+t, k+1-t) : t \in \mathbb{R}\}$ remains constant across x.

A symmetric local generator $\phi = (\phi_1, \phi_2, \phi_3)$ of S_0 is defined as follows. The function ϕ_1 is the characteristic function of $[0,1]^2$, i.e. the box-spline on $M(v_0, v_1)$ given by

$$\hat{\phi}_1(u,v) = \frac{(1-z)(1-w)}{(iu)(iv)}, \quad u, v \in \mathbb{R}.$$

The other functions are given by

$$\hat{\phi}_2(u,v) = \frac{v(1-z)(1+w) - u(1-w)(1+z)}{uv(u-v)} \quad u, v \in \mathbb{R},$$

$$\hat{\phi}_3(u,v) = z\hat{\phi}_2(-u,v) \quad u, v \in \mathbb{R}.$$

Note that there is a local generator with smaller support, $\psi = (\phi_1, \psi_2, \psi_3)$, where $\psi_2 = \frac{1}{2}(\phi_1 + \phi_2)$ is the characteristic function of the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $\psi_3 = \frac{1}{2}(\phi_1 + \phi_3)$ is the characteristic function of the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$. However the generator ψ is not symmetric.

Since only two lines in M intersect other than in $\mathbb{Z}^2$, Theorem 3 shows that ϕ is stable. Equation (5.1) is satisfied with $\sigma_1 = 1$, $\sigma_2 = \sigma_3 = -1$, $C_1 = C_2 = C_3 = (1,1)$. For $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we have

$$\phi_j(\cdot T) = \phi_j, \quad j = 1, 2, \quad \phi_3(\cdot T) = -\phi_3.$$

For $T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, we have

$$\phi_1(\cdot T) = \phi_1(. + (1,0)), \quad \phi_2(\cdot T) = \phi_3(. + (1,0)),$$

and thus (5.19) is satisfied with $\tau_j = 1$, $j = 1, 2, 3$, $\sigma(P) = -1$.

Example 2. Take $n = 2$, $r = 2$ and

$$\{v_0, \ldots, v_4\} = \{(1,0), (0,1), (1,1), (2,1), (1,2)\}.$$

Since only two lines in $M = M(v_0, \ldots, v_4)$ intersect other than in $\mathbb{Z}^2$, we see from Corollary 1 that $S_2 = S_2(v_0, \ldots, v_4)$ comprises all C^1 quadratic spline functions over M with compact support.

A symmetric local generator of S_2 is $\phi = (\phi_1, \phi_2)$, where for $u, v \in \mathbb{R}$,

$$\hat{\phi}_1(u,v) = \frac{(2u+v)(w^{-1} - zw)(1+z^{-1}) - (u+2v)(z^{-1} - zw)(1+w^{-1})}{iuv(u+v)(2u+v)(u+2v)},$$

$$\begin{aligned}\hat{\phi}_2(u,v) =& \frac{(w^{-1} - zw)(zw - 3w + 3 - z^{-1})}{iuv(u+v)(u+2v)} \\ &+ \frac{(z^{-1} - zw)(zw - 3z + 3 - w^{-1})}{iuv(u+v)(2u+v)}.\end{aligned}$$

By Theorem 3, ϕ is stable. Equation (5.1) is satisfied with $\sigma_1 = -1$, $\sigma_2 = 1$, $C_1 = 0$, $C_2 = (1,1)$. For $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\phi_1(\cdot T) = -\phi_1$, $\phi_2(\cdot T) = \phi_2$.

Example 3. Take $n = 3$, $r = 4$ and

$$\{v_0, \ldots, v_7\} = \{(1,0), (0,1), (1,1), (1,-1), (2,1), (1,2), (2,-1), (1,-2)\}.$$

Since at most three lines in $M = M(v_0, \ldots, v_7)$ intersect other than in $\mathbb{Z}^2$, we see from Corollary 1 that $S_3 = S_3(v_0, \ldots, v_7)$ comprises all C^2 cubic spline functions over M with compact support.

A symmetric local generator of S_3 is $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$, defined as follows for $u, v \in \mathbb{R}$. Letting f denote ϕ_1 as in Example 2,

$$\hat{\phi}_1(u,v) = \frac{f(u,v)}{i(u-v)}, \quad \hat{\phi}_2(u,v) = \hat{\phi}_1(-u,v),$$

$$\begin{aligned}\hat{\phi}_3(u,v) =& \frac{(zw^2 - 1)(z^{-1} + w^{-1})}{uv(u+v)(u+2v)} + \frac{(zw - z^{-1})}{uv(u+v)(2u+v)} \\ &+ \frac{(z^{-1}w^2 - 1)(z + w^{-1})}{uv(u+v)(u-2v)} + \frac{(z^{-1}w - z)}{uv(u+v)(2u-v)},\end{aligned}$$

$$\hat{\phi}_4(u,v) = \hat{\phi}_3(v,u).$$

By Theorem 3, ϕ is stable. Equation (5.1) is satisfied with $\sigma_j = 1$, $j = 1, \ldots, 4$, $C_1 = C_2 = 0$, $C_3 = (0,1)$, $C_4 = (1,0)$.

For $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we have

$$\phi_j(\cdot T) = \phi_j, \quad j = 1, 2, \quad \phi_4(\cdot T) = \phi_3(\cdot T),$$

while for $T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, we have

$$\phi_2 = \phi_1(\cdot T), \quad \phi_3(\cdot T) = \phi_3, \quad \phi_4(\cdot T) = \phi_3(. + (1,0)).$$

In both cases (5.19) is satisfied with

$$\tau_j = 1, \quad j = 1, \ldots, 4, \quad \sigma(P) = -1, \quad t_0 \ldots t_7 = -1.$$

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