

**UNCERTAINTY PRINCIPLES IN BANACH SPACES
AND SIGNAL RECOVERY**

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Abstract

A very general uncertainty principle is given for operators on Banach spaces. Many consequences are derived, including uncertainty principles for Bessel sequences in Hilbert spaces and for integral operators between measure spaces. In particular it implies an uncertainty principle for $\mathcal{L}^p(G)$, $1 \leq p \leq \infty$, for a locally compact Abelian group G , concerning simultaneous approximation of $f \in \mathcal{L}^p(G)$ by gf and $H * f$ for suitable g and H . Taking g and \widehat{H} to be characteristic functions then gives an uncertainty principle about ϵ -concentration of f and \widehat{f} , which generalizes a result of Smith, which in turn generalizes a well-known result of Donoho and Stark. The paper also generalizes to the setting of Banach spaces a related result of Donoho and Stark on stable recovery of a signal which has been truncated and corrupted by noise. In particular this can be applied to the recovery of missing coefficients in a series expansion.

Keywords. Uncertainty principles, time- or frequency-concentrated functions, signal recovery

1. INTRODUCTION

The classical Heisenberg uncertainty principle gives a lower bound for the product of the variances of $|f|^2$ and $|\widehat{f}|^2$, for $f \in \mathcal{L}^2(\mathbb{R})$, $\|f\|_2 = 1$, and its Fourier transform \widehat{f} . As well as its well-known original interpretation in quantum theory [5], it also has relevance to signal processing, as it gives a restriction on how well the ‘instantaneous frequency’ of a signal can be measured [2]. The uncertainty principle has been generalized to operators on Hilbert spaces, see for instance [4]. This leads to uncertainty principles for functions in various spaces, for example functions on spheres in Euclidean space of any dimension, see [3].

In [1], Donoho and Stark gave a new type of uncertainty principle. A function f in $\mathcal{L}^2(\mathbb{R})$ is said to be ϵ -concentrated on a measurable set $T \subset \mathbb{R}$ if $\|f - f|_T\|_2 \leq \epsilon\|f\|_2$. It is shown in [1] that if f is η -concentrated on T and \widehat{f} is δ -concentrated on W , then $|T|^{1/2}|W|^{1/2} > 1 - \eta - \delta$. A corresponding result is shown for functions on a finite set (with the finite Fourier transform), and also for functions in $\mathcal{L}^1(\mathbb{R})$, though in this latter case the definition of δ -concentrated of \widehat{f} on W needs to be modified (see below).

In [8], Smith unified and generalized the above results to functions in $\mathcal{L}^p(G)$, $1 \leq p \leq 2$, for a locally compact Abelian group G (with the Fourier transform on its character group \widehat{G}). Following [1] for the case $p = 1$, the definition of δ -concentration of \widehat{f} on W is modified for $1 \leq p < 2$ to require $\|f - F\|_p \leq \delta\|f\|_p$ for some $F \in \mathcal{L}^p(G)$ with $\widehat{F} = \widehat{F}|_W$.

We also note that the uncertainty principle of Donoho and Stark has been further extended to the settings of Gelfand pairs [10], commutative hypergroups [9] and integral operators with bounded kernels [6]. In addition, analogous inequalities have been obtained for Hopf algebras [7].

In Corollary 2.1 of this paper we give an uncertainty principle which generalizes Smith’s result in a number of ways. Firstly it holds for all p , $1 \leq p \leq \infty$. More interestingly, in the definition of η -concentration, $\|f - f|_T\|_p$ is replaced by $\|f - gf\|_p$ and $\widehat{F} = \widehat{F}|_W$ is replaced by $\|f - H * f\|_p \leq \epsilon$, for suitable functions g and H , where $\|g\|_\infty = 1$, while in the uncertainty principle the measures of T and W are replaced by $\|g\|_p^p$ and $\|H\|_q^q$ respectively (where $\frac{1}{p} + \frac{1}{q} = 1$). Thus the result can be viewed as giving restrictions on how well f can be simultaneously approximated by multiplication by a function and by convolution with a function. By putting $g = \chi_T$ (the characteristic function of T), $\widehat{H} = \chi_W$, and $\epsilon = \delta$, we recover Smith’s result, but even in this case

our inequality may be stronger since $\|H\|_q^q$ could be strictly less than the measure of W .

The result given by Corollary 2.1, discussed above, is deduced from a very general uncertainty principle for operators on Banach spaces, Theorem 2.1, which can be simply proved using essentially only the triangle inequality. Another consequence of Theorem 2.1 is an uncertainty principle involving a Hilbert space and \mathcal{L}^2 -functions on a general measure space, which is given in Theorem 2.2. This in turn implies a result on Bessel sequences in Hilbert space (Corollary 2.2), and a result on integral operators between measure spaces (Corollary 2.4).

In [1], the authors applied the ideas of their uncertainty principles to show, paradoxically, that uncertain signals may be recovered with certainty! To be more precise, a signal with Fourier transform supported on a set W is corrupted by noise and also has all information lost on a set T . Then provided that $|T||W| < 1$, the signal can be stably reconstructed, i.e. the norm of the error in reconstruction is bounded by a fixed constant times the norm of the noise. In Section 3 we extend this result to operators on Banach spaces, where the condition $|T||W| < 1$ is replaced by the requirement that the uncertainty product appearing in the uncertainty principle in Theorem 2.1 is strictly less than one. Recalling Corollary 2.2 on Bessel sequences, this can be applied to the recovery of missing coefficients in a series expansion, for example in a Fourier series.

2. UNCERTAINTY PRINCIPLES

We first give a very general uncertainty principle for operators on Banach spaces. Although the statement of this result may appear ungainly, its proof is very simple and all the other uncertainty principles in this paper follow from it. We shall denote by \mathcal{X}' the dual of a Banach space \mathcal{X} and by $P^* : \mathcal{Y}' \rightarrow \mathcal{X}'$ the adjoint of a linear operator $P : \mathcal{X} \rightarrow \mathcal{Y}$.

Theorem 2.1. *Let \mathcal{X} and \mathcal{Y} be Banach spaces, and $P : \mathcal{X} \rightarrow \mathcal{Y}$, $Q : \mathcal{Y} \rightarrow \mathcal{X}$, $R : \mathcal{X} \rightarrow \mathcal{X}$ bounded linear operators, where $Rx = Qx$ for $x \in \mathcal{X} \cap \mathcal{Y}$. Suppose $\mathcal{S} \subset \mathcal{Y}'$ and for constants $A, B > 0$,*

$$\|P^*\alpha\|_{\mathcal{X}'} \leq A, \quad \alpha \in \mathcal{S}, \quad (2.1)$$

$$\|Qy\|_{\mathcal{X}} \leq B \sup\{|\alpha(y)| : \alpha \in \mathcal{S}\}, \quad y \in P\mathcal{X}. \quad (2.2)$$

If $x, y \in \mathcal{X}$, $Py \in \mathcal{X}$ and $\|x\|_{\mathcal{X}} = 1$,

$$\|x - y\|_{\mathcal{X}} \leq \delta, \quad \|x - Py\|_{\mathcal{X}} \leq \epsilon, \quad \|x - Rx\|_{\mathcal{X}} \leq \eta, \quad (2.3)$$

then

$$AB(1 + \delta) \geq 1 - \eta - \|R\|\epsilon. \quad (2.4)$$

Proof: For any $t \in \mathcal{X}$, (2.2) gives

$$\begin{aligned} \|QPt\|_{\mathcal{X}} &\leq B \sup\{|\alpha(Pt)| : \alpha \in \mathcal{S}\} \\ &= B \sup\{|P^*\alpha(t)| : \alpha \in \mathcal{S}\} \\ &\leq B \sup\{\|P^*\alpha\|_{\mathcal{X}'} \|t\|_{\mathcal{X}} : \alpha \in \mathcal{S}\} \\ &\leq AB\|t\|_{\mathcal{X}}, \end{aligned} \quad (2.5)$$

by (2.1). For $x \in \mathcal{X}$, $\|x\|_{\mathcal{X}} = 1$,

$$\begin{aligned} \|x\|_{\mathcal{X}} - \|QPx\|_{\mathcal{X}} &\leq \|x - QPx\|_{\mathcal{X}} \\ &\leq \|x - Rx\|_{\mathcal{X}} + \|Rx - QPy\|_{\mathcal{X}} + \|QPy - QPx\|_{\mathcal{X}} \\ &\leq \eta + \|Rx - RPy\|_{\mathcal{X}} + \|QP(y - x)\|_{\mathcal{X}} \\ &\leq \eta + \|R\|\epsilon + AB\delta, \end{aligned} \quad (2.6)$$

by (2.3) and (2.5). So again applying (2.5),

$$\begin{aligned} AB(1 + \delta) &= AB\|x\|_{\mathcal{X}} + AB\delta \\ &\geq \|QPx\|_{\mathcal{X}} + AB\delta \\ &\geq 1 - \eta - \|R\|\epsilon, \end{aligned}$$

by (2.6). ■

As a special case we may take $\mathcal{S} = \{\alpha \in \mathcal{Y}' : \|\alpha\|_{\mathcal{Y}'} \leq 1\}$ and then in (2.1) and (2.2) we can take $A = \|P\|$, $B = \|Q|_{P\mathcal{X}}\|$. In this case the proof of (2.5) is simply $\|QP\| \leq \|Q|_{P\mathcal{X}}\| \|P\|$. In particular, this choice gives the following.

Corollary 2.1. *Let G be a locally compact Abelian group with Haar measure. Take $1 \leq p \leq \infty$, $g \in \mathcal{L}^\infty(G) \cap \mathcal{L}^p(G)$ with $\|g\|_\infty = 1$, and $H \in \mathcal{L}^q(G)$, where $\frac{1}{p} + \frac{1}{q} = 1$. For $f \in \mathcal{L}^p(G)$ with $\|f\|_p = 1$, suppose there is F in $\mathcal{L}^p(G)$ with $H * F \in \mathcal{L}^p(G)$ and*

$$\|f - F\|_p \leq \delta, \quad \|f - H * F\|_p \leq \epsilon, \quad \|f - gf\|_p \leq \eta. \quad (2.7)$$

Then

$$\|g\|_p \|H\|_q (1 + \delta) \geq 1 - \eta - \epsilon. \quad (2.8)$$

Proof: In Theorem 2.1, take $\mathcal{X} = \mathcal{L}^p(G)$, $\mathcal{Y} = \mathcal{L}^\infty(G)$. Define $Qy = gy$, $y \in \mathcal{Y}$, $Rx = gx$, $x \in \mathcal{X}$, $Px = H * x$. Take $\mathcal{S} = \{\alpha \in \mathcal{Y}' : \|\alpha\|_{\mathcal{Y}'} \leq 1\}$, as discussed above. Since for $x \in \mathcal{L}^p(G)$,

$$\|Px\|_\infty \leq \|x\|_p \|H\|_q,$$

we have $\|P\| \leq \|H\|_q$ and so we may take $A = \|H\|_q$ in (2.1). In (2.2) we may take $B = \|Q\| \leq \|g\|_p$. Since $\|R\| \leq \|g\|_\infty = 1$, (2.8) follows. ■

As a special case of Corollary 2.1, we may take measurable sets $T \subset G$ and $W \subset \widehat{G}$, with finite measure, and let $g = \chi_T$, $\widehat{H} = \chi_W$, provided that $H \in \mathcal{L}^q(G)$. Since $\|g\|_p = \mu(T)^{1/p}$, where μ denotes Haar measure on G , (2.8) becomes

$$\mu(T)^{1/p} \|H\|_q (1 + \delta) \geq 1 - \eta - \epsilon. \quad (2.9)$$

First suppose $p = 2$. Then $\widehat{H * f} = \widehat{H} \widehat{f} = \chi_W \widehat{f} \in \mathcal{L}^2(\widehat{G})$ and so $H * f \in \mathcal{L}^2(G)$. Thus we may take $F = f$ and $\delta = 0$, and the condition $\|f - H * F\|_2 \leq \epsilon$ in (2.7) is equivalent to $\|\widehat{f} - \widehat{f}|_W\|_2 \leq \epsilon$, i.e. \widehat{f} is ϵ -concentrated on W . Also $\|H\|_2 = \|\chi_W\|_2 = \widehat{\mu}(W)^{1/2}$, where $\widehat{\mu}$ denotes Haar measure on \widehat{G} . Thus (2.8) becomes

$$\mu(T)^{1/2} \widehat{\mu}(W)^{1/2} \geq 1 - \eta - \epsilon,$$

which is the case $p = 2$ of the uncertainty principle of Smith in [8].

For the case $1 \leq p \leq 2$, $\|H\|_q \leq \|\widehat{H}\|_p = \widehat{\mu}(W)^{1/p}$ and so (2.8) gives

$$\mu(T)^{1/p} \widehat{\mu}(W)^{1/p} (1 + \delta) \geq 1 - \eta - \epsilon. \quad (2.10)$$

If $\widehat{F} = \widehat{F}|_W$, then $\widehat{F} = \widehat{H} \widehat{F}$ and thus $f - F = f - H * F$. So we may take $\delta = \epsilon$ in (2.7) and the result again reduces to the uncertainty principle in [8]. However we may have $\|H\|_q < \widehat{\mu}(W)^{1/p}$, so that (2.9) is stronger than (2.10). To illustrate this, consider the case $G = \mathbb{R}$ (and hence $\widehat{G} = \mathbb{R}$), and W is an interval. Without loss of generality

we may suppose $W = [-a, a]$, $a > 0$. Then $H(t) = 2t^{-1} \sin(at)$, $t \in \mathbb{R} \setminus \{0\}$, and for $1 < q < \infty$,

$$\begin{aligned} \|H\|_q^q &= (2a)^q \int_{-\infty}^{\infty} \left| \frac{\sin(at)}{at} \right|^q dt \\ &= \frac{(2a)^q}{a} \int_{-\infty}^{\infty} \left| \frac{\sin v}{v} \right|^q dv = 2\widehat{\mu}(W)^{q-1} \int_{-\infty}^{\infty} \left| \frac{\sin v}{v} \right|^q dv. \end{aligned}$$

Putting $J_\infty = 1$ and

$$J_q = \left(2 \int_{-\infty}^{\infty} \left| \frac{\sin v}{v} \right|^q dv \right)^{1/q}, \quad 1 < q < \infty,$$

we have $\|H\|_q = J_q \widehat{\mu}(W)^{1/p}$ and so for $1 \leq p < \infty$, (2.9) becomes

$$\mu(T)^{1/p} \widehat{\mu}(W)^{1/p} (1 + \delta) J_q \geq 1 - \eta - \epsilon,$$

which coincides with (2.10) for $p = 1$ and 2 , and is stronger than (2.10) for $1 < p < 2$.

We now return to Theorem 2.1 and note that if we put $\mathcal{X} = \mathcal{Y}$, $R = Q$, $x = y$, then (2.3) and (2.4) can be expressed as the simpler forms

$$\|x - Px\| \leq \epsilon, \quad \|x - Qx\| \leq \eta, \quad (2.11)$$

$$AB \geq 1 - \eta - \|Q\|\epsilon. \quad (2.12)$$

Henceforward in this section, we shall consider this case when $\mathcal{X} = \mathcal{Y} = \mathcal{H}$ for a Hilbert space \mathcal{H} . Our next result is a special case of Theorem 2.1, but as it will generate all the remaining results, we give it the rank of a theorem.

Theorem 2.2. *Let \mathcal{H} be a Hilbert space and (Y, ν) a measure space. Take $h \in \mathcal{L}^2(Y, \nu)$ with $\|h\|_\infty = 1$. Let $P : \mathcal{H} \rightarrow \mathcal{H}$, $U : \mathcal{H} \rightarrow \mathcal{L}^2(Y, \nu)$ be bounded linear operators such that for all $x \in P\mathcal{H}$,*

$$(Ux)(u) = \langle x, K_u \rangle, \quad u \in W,$$

where $h = 0$ a.e. outside $W \subset Y$ and $K_u \in \mathcal{H}$ with

$$\|P^* K_u\| \leq A, \quad u \in W.$$

If $x \in \mathcal{H}$, $\|x\| = 1$,

$$\|x - Px\| \leq \epsilon, \quad \|x - U^* h U x\| \leq \eta,$$

then

$$A\|U\| \|h\|_2 \geq 1 - \eta - \|U\|^2 \epsilon.$$

Proof: In Theorem 2.1, under the situation leading to (2.11) and (2.12), we put $\mathcal{S} = \{K_u : u \in W\}$ and $Q = U^*MU$, where $Mw = hw$, $w \in \mathcal{L}^2(Y, \nu)$. Then $\|P^*\alpha\| \leq A$, $\alpha \in \mathcal{S}$, and Q is bounded with $\|Q\| \leq \|U^*\| \|M\| \|U\| \leq \|U\|^2$. In addition, for $x \in P\mathcal{H}$,

$$\begin{aligned} \|Qx\|^2 = \|U^*MUx\|^2 &\leq \|U^*\|^2 \int_W |h|^2 |Ux|^2 \\ &= \|U\|^2 \int_W |h(u)|^2 |\langle x, K_u \rangle|^2 d\nu(u) \\ &\leq \|U\|^2 \|h\|_2^2 \sup\{|\langle x, K_u \rangle|^2 : u \in W\}, \end{aligned}$$

and so

$$\|Qx\| \leq \|U\| \|h\|_2 \sup\{|\langle x, \alpha \rangle| : \alpha \in \mathcal{S}\}.$$

Hence from Theorem 2.1,

$$A\|U\| \|h\|_2 \geq 1 - \eta - \|Q\|\epsilon \geq 1 - \eta - \|U\|^2\epsilon,$$

as desired. ■

In particular, we can take h as the characteristic function of W with $\nu(W) < \infty$. Then $\|h\|_2 = \nu(W)^{1/2}$.

Theorem 2.2 can be used to obtain an uncertainty principle on Bessel sequences of a Hilbert space \mathcal{H} . Recall that a sequence $\{\phi_n\}_{n \in I}$, where I is a countable index set, is a *Bessel sequence* in \mathcal{H} with bound C if

$$\sum_{n \in I} |\langle f, \phi_n \rangle|^2 \leq C \|f\|^2, \quad f \in \mathcal{H}. \quad (2.13)$$

Orthonormal bases, Riesz bases and frames are examples of Bessel sequences.

Corollary 2.2. *Let \mathcal{H} be a Hilbert space and $\{\phi_n\}_{n \in I}$ a Bessel sequence in \mathcal{H} with bound C . Take a finite subset W of I and consider $Q : \mathcal{H} \rightarrow \mathcal{H}$ given by*

$$Qf = \sum_{n \in W} \langle f, \phi_n \rangle \phi_n.$$

Let $P : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator with $\|P^\phi_n\| \leq A$, $n \in W$. If $f \in \mathcal{H}$, $\|f\| = 1$, $\|f - Pf\| \leq \epsilon$, $\|f - Qf\| \leq \eta$, then*

$$AC^{1/2}|W|^{1/2} \geq 1 - \eta - C\epsilon.$$

Proof: In Theorem 2.2, take $Y = I$ with the counting measure, and define U by

$$(Uf)_n = \langle f, \phi_n \rangle, \quad f \in \mathcal{H}, \quad n \in I.$$

By (2.13), we have $\|U\| \leq C^{1/2}$. Now for $f \in \mathcal{H}$, $c = \{c_n\}_{n \in I} \in \ell^2(I)$,

$$\langle Uf, c \rangle = \sum_{n \in I} \langle f, \phi_n \rangle \overline{c_n} = \langle f, \sum_{n \in I} c_n \phi_n \rangle,$$

and so

$$U^*c = \sum_{n \in I} c_n \phi_n, \quad c = \{c_n\}_{n \in I} \in \ell^2(I).$$

Let h in Theorem 2.2 be the characteristic function of W . Then for $f \in \mathcal{H}$,

$$U^*hUf = \sum_{n \in W} \langle f, \phi_n \rangle \phi_n = Qf.$$

Thus the result follows from Theorem 2.2 and the inequality $\|U\| \leq C^{1/2}$. ■

A special case of Corollary 2.2 that is of interest is the following.

Corollary 2.3. *For a countable index set I , let $\{\phi_n\}_{n \in I}$ be an orthonormal set in $\mathcal{L}^2([0, 2\pi])$ with $\|\phi_n\|_\infty \leq 1$, $n \in I$. Let T be a measurable subset of $[0, 2\pi]$ and $W \subset I$. Then for $f = \sum_{n \in I} c_n \phi_n \in \mathcal{L}^2([0, 2\pi])$ with $\|f\|_2 = 1$, $\int_{[0, 2\pi] \setminus T} |f|^2 \leq \epsilon^2$, $\sum_{I \setminus W} |c_n|^2 \leq \eta^2$, $\epsilon + \eta \leq 1$,*

$$|T||W| \geq (1 - \epsilon - \eta)^2.$$

Proof: In Corollary 2.2, take \mathcal{H} to be the \mathcal{L}^2 -span of $\{\phi_n\}_{n \in I}$, and define $Pf = gf$, where g is the characteristic function of T . Then $C = 1$, $A = |T|^{1/2}$ and the result follows. ■

We remark that [1] also gave an uncertainty principle for orthonormal sets but in that case f is ϵ -concentrated on T in the \mathcal{L}^1 -norm, although the sequence $\{c_n\}_{n \in I}$ is, as above, η -concentrated in the ℓ^2 -norm. In Corollary 2.2 we may choose, in particular, $\phi_n(t) = e^{int}$, $n \in \mathbb{Z}$. In this case the condition $\sum_{I \setminus W} |c_n|^2 \leq \eta^2$ states that the Fourier series of f is η -concentrated on W , and so Corollary 2.3 reduces to the uncertainty principle of Smith in [8] for $G = \mathbb{R}/2\pi\mathbb{Z}$ and $p = 2$.

Another consequence of Theorem 2.2 is as follows.

Corollary 2.4. *Let (X, μ) and (Y, ν) be measure spaces. Take $g \in \mathcal{L}^2(X, \mu)$, $h \in \mathcal{L}^2(Y, \nu)$ with $\|g\|_\infty < \infty$, $\|h\|_\infty = 1$ and choose $T \subset X$, $W \subset Y$ so that $g = 0$ a.e. outside T and $h = 0$ a.e. outside W . Let $U : \mathcal{L}^2(X, \mu) \rightarrow \mathcal{L}^2(Y, \nu)$ be a bounded linear operator such that for $f \in \mathcal{L}^2(X, \mu)$, $\text{supp } f \subset T$, $u \in W$,*

$$Uf(u) = \int_X K(t, u) f(t) d\mu(t)$$

for some $K(\cdot, u) \in \mathcal{L}^2(X, \mu)$ with

$$|K(t, u)| \leq \kappa, \quad t \in T, u \in W.$$

If $f \in \mathcal{L}^2(X, \mu)$, $\|f\|_2 = 1$, $\|f - gf\|_2 \leq \epsilon$, $\|f - U^*hUf\|_2 \leq \eta$, then

$$\kappa \|U\| \|g\|_2 \|h\|_2 \geq 1 - \eta - \|U\|^2 \epsilon.$$

Proof: In Theorem 2.2, take $\mathcal{H} = \mathcal{L}^2(X, \mu)$, $Pf = gf$ and $K_u = \overline{K(\cdot, u)}$, $u \in W$. Now for $u \in W$,

$$\|P^*K_u\|_2^2 = \int_X |\overline{g(t)}|^2 |K_u(t)|^2 d\mu(t) \leq \kappa^2 \|g\|_2^2.$$

So in Theorem 2.2, A is replaced by $\kappa \|g\|_2$ and the result follows. \blacksquare

Taking $X = G$, $Y = \widehat{G}$, μ, ν the respective Haar measures, U the Fourier transform, and g, h the characteristic functions of T, W , we again recover the uncertainty principle of Smith in [8] for $p = 2$.

3. SIGNAL RECOVERY

Now recall the result on signal recovery in [1] as discussed in Section 1. We shall extend it to the setting of Theorem 2.1 with $\mathcal{X} = \mathcal{Y}$, $R = Q$, $x = y$. It follows from (2.5) in the proof of Theorem 2.1 that

$$\|QP\| \leq AB, \tag{3.1}$$

where A and B are the positive constants in (2.1) and (2.2). For any signal $x \in \mathcal{X}$ satisfying $\|x - Px\| \leq \epsilon$, our goal is to recover it from the truncated signal $x - Qx$ which is corrupted by noise m . We also consider the analogous problem of recovering $y \in \mathcal{X}$ for which $\|y - Qy\| \leq \eta$ from $y - Py$ under the presence of noise n .

Theorem 3.1. *Let \mathcal{X} be a Banach space, P and Q bounded linear operators on \mathcal{X} , and $\mathcal{S} \subset \mathcal{X}'$. Let A and B be positive constants satisfying (2.1) and (2.2). If $AB < 1$, then there exists a linear operator L_{QP} on \mathcal{X} such that for any $x \in \mathcal{X}$ with $\|x - Px\| \leq \epsilon$, and*

$$r = (x - Qx) + m, \quad (3.2)$$

where $m \in \mathcal{X}$, there holds

$$\|x - L_{QP}r\| \leq \frac{1}{1 - AB}(\|m\| + \|Q\|\epsilon). \quad (3.3)$$

In addition, if $\|PQ\| \leq \|QP\|$, then there exists a linear operator L_{PQ} on \mathcal{X} such that for any $y \in \mathcal{X}$ with $\|y - Qy\| \leq \eta$, and

$$s = (y - Py) + n, \quad (3.4)$$

where $n \in \mathcal{X}$, there holds

$$\|y - L_{PQ}s\| \leq \frac{1}{1 - AB}(\|n\| + \|P\|\eta). \quad (3.5)$$

Proof: First we observe that for any $x \in \mathcal{X}$,

$$(I - QP)x = (I - Q)x - Q(Px - x). \quad (3.6)$$

By (3.1), $\|QP\| \leq AB < 1$ and so the linear operator

$$L_{QP} = (I - QP)^{-1} \quad (3.7)$$

exists. Then it follows from (3.6) that

$$x = L_{QP}(I - Q)x - L_{QP}Q(Px - x).$$

Applying (3.2) and the inequality $\|x - Px\| \leq \epsilon$ gives

$$\|x - L_{QP}r\| \leq \|L_{QP}m\| + \|L_{QP}Q(Px - x)\| \leq \|L_{QP}\|(\|m\| + \|Q\|\epsilon).$$

Now by (3.7) and (3.1),

$$\|L_{QP}\| \leq \frac{1}{1 - \|QP\|} \leq \frac{1}{1 - AB}$$

which leads to (3.3).

Next, if $\|PQ\| \leq \|QP\|$, we have $\|PQ\| \leq AB < 1$. The same arguments can be used to deduce the second half of the theorem. ■

We note that in order to interpret the conditions $\|x - Px\| \leq \epsilon$ and $\|y - Qy\| \leq \eta$ in Theorem 3.1 as analogs of ϵ -concentration and η -concentration, we would need to assume $\|x\| = \|y\| = 1$, though this condition is not needed in the proof. If the operator L_{QP} exists, then analogous to the situation considered in [1], we say that any $x \in \mathcal{X}$ with $\|x\| = 1$, $\|x - Px\| \leq \epsilon$ can be *stably reconstructed* from r in (3.2). Similarly, the existence of the operator L_{PQ} means that any $y \in \mathcal{X}$ with $\|y\| = 1$, $\|y - Qy\| \leq \eta$ can be stably reconstructed from s in (3.4).

As noted above, the recovery of $x \in \mathcal{X}$ for which $\|x\| = 1$, $\|x - Px\| \leq \epsilon$ lies in the computation of $L_{QP}r$, where r is as in (3.2). In this connection, recall that

$$L_{QP} = (I - QP)^{-1} = \sum_{j=0}^{\infty} (QP)^j.$$

For $k \geq 0$, define

$$x^{(k)} = \sum_{j=0}^k (QP)^j r.$$

Then $x^{(k)} \rightarrow L_{QP}r$ as $k \rightarrow \infty$. In general, for $k \geq 1$,

$$x^{(k)} = r + QP x^{(k-1)}. \quad (3.8)$$

It is well known that the iterations in (3.8) converge in a geometric rate to the fixed point x^* given by

$$x^* = r + QP x^*,$$

i.e. $x^* = L_{QP}r$. Hence we may approximate x by x^* , up to the accuracy in (3.3). If $\epsilon = 0$ (which corresponds to x being a fixed point of P) and $m = 0$, then the iteration (3.8) enables x to be recovered perfectly.

Similarly, for the case when $\|PQ\| \leq \|QP\|$, any $y \in \mathcal{X}$ with $\|y\| = 1$, $\|y - Qy\| \leq \eta$ can be recovered from the iteration

$$y^{(k)} = s + PQ y^{(k-1)},$$

where s is as in (3.4), up to the accuracy in (3.5). Perfect recovery is possible when $\eta = 0$ and $n = 0$.

Let us comment on the special Hilbert space setting of Theorem 2.2 when the function h is real-valued. In this case, the operator $Q = U^*MU$, where $Mw = hw$, $w \in \mathcal{L}^2(Y, \nu)$, is self-adjoint. If the operator P is also self-adjoint, then $\|PQ\| = \|(QP)^*\| = \|QP\|$ which means signal recovery in the sense of both (3.3) and (3.5) is possible for all such settings covered by Theorem 2.2, whenever the constants A and B satisfy $AB < 1$.

We further specialize our discussion to the situation in Corollary 2.4. Take $X = G$, $Y = \widehat{G}$, μ, ν the respective Haar measures, U the Fourier transform, and g, h the characteristic functions of T, W . Then Theorem 3.1 gives various signal recovery results for signals with missing components in time or frequency, for instance missing Fourier coefficients. The case $G = \mathbb{R}$ (and so $\widehat{G} = \mathbb{R}$) with $\eta = 0$ leads to the result of Donoho and Stark in [1] on recovering missing segments of a bandlimited signal.

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