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Construction of orthogonal bases for polynomials in Bernstein form on triangular and simplex domains

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Abstract

A scheme for constructing orthogonal systems of bivariate polynomials in the Bernstein–Bézier form over triangular domains is formulated. The orthogonal basis functions have a hierarchical ordering by degree, facilitating computation of least-squares approximations of increasing degree (with permanence of coefficients) until the approximation error is subdued below a prescribed tolerance. The orthogonal polynomials reduce to the usual Legendre polynomials along one edge of the domain triangle, and within each fixed degree are characterized by vanishing Bernstein coefficients on successive rows parallel to that edge. Closed-form expressions and recursive algorithms for computing the Bernstein coefficients of these orthogonal bivariate polynomials are derived, and their application to surface smoothing problems is sketched. Finally, an extension of the scheme to the construction of orthogonal bases for polynomials over higher-dimensional simplexes is also presented.

Keywords: Orthogonal polynomials; Barycentric coordinates; Triangular domains; Legendre polynomials; Bernstein representation; Higher-dimensional simplexes

1. Introduction

Orthogonal polynomials play a fundamental role in problems of least-squares approximation of functions over finite domains (Davis, 1975). In the univariate case, the Legendre polynomials are easily constructed by a simple recurrence relation. They also possess elegant Bernstein representations,

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and the transformation between Legendre and Bernstein forms is comparatively well-conditioned (Farouki, 2000b). The Bernstein form of a polynomial over a finite domain is advantageous for practical computations, on account of its intrinsic numerical stability (Farouki and Goodman, 1996; Farouki and Rajan, 1987) and the geometrical insight and elegant algorithms it entails (Farin, 1986, 1993).

In the bivariate (or multivariate) context, the construction of orthogonal polynomials in Bernstein form over fundamental (i.e., simplex) domains is more challenging. For example, to facilitate approximation by polynomials of successively higher degree, until a desired tolerance is attained, a systematic degree ordering must be imposed on the orthogonal basis functions. This is, in general, incompatible with the symmetry properties and simple recursive definitions that characterize the univariate Legendre polynomials.

Although multivariate orthogonal polynomials were introduced quite early, most of the developments in this field have occurred recently. Standard texts on orthogonal polynomials focus on the univariate case (Askey, 1975; Szegö, 1975). Orthogonal polynomials on the simplex (namely, Rodrigues' formula and orthogonalized monomials as a biorthogonal system) were given by Appell and Kampé de Fériet (1926). Jackson (1936) discussed formal properties of orthogonal systems in two variables, and Koornwinder (1975, 1976) gave bivariate analogs of the Jacobi polynomials; see also (Bertran, 1975; Krall and Sheffer, 1967). More recently, Kowalski (1982a, 1982b) and Xu (1993a, 1993b, 1994a, 1994b, 1994c) discuss some fundamental properties of multivariate orthogonal systems, including recurrence relations. Detailed constructions of orthogonal systems for particular domain geometries have received less attention—a system for hexagonal domains has been described by Dunkl (1987), while Sauer (1994) gives a biorthogonal system (for a singular weight function) over triangles.

In this paper, we propose a particular approach to constructing systems of bivariate orthogonal polynomials in Bernstein form over triangular domains. The distinguishing properties of these polynomials are: (i) they coincide with the univariate Legendre polynomials along one edge of the domain triangle; and (ii) their "Bernstein–Bézier control nets" exhibit successive sequences of vanishing coefficients on rows parallel to this edge. Closed-form expressions for the Bernstein coefficients of these orthogonal bivariate polynomials are derived, together with a recursive scheme for their generation. Moreover, the method may be generalized in a natural manner to allow the construction of systems of orthogonal d-variate polynomials over d-simplex domains.

It should be emphasized that the orthogonal bases described herein are not new (see references cited above). The contribution of this paper, however, is to give explicit constructions of these bases in Bernstein form, so as to take full advantage of the numerical stability, simple recursive algorithms, degree ordering, and geometrical insights associated with the Bernstein form.

Our plan for this paper is as follows. In Section 2 we outline some basic features of the univariate Legendre polynomials, and their Bernstein representations, that are required subsequently. After reviewing barycentric coordinates, the Bernstein basis, and integration over triangular domains in Section 3, we describe the construction of a degree-ordered orthogonal system of polynomials over such domains, and derive formulae and recursive algorithms for computing their Bernstein coefficients. In Section 4 we sketch an application of the orthogonal basis to surface smoothing problems, under linear interpolation constraints. The scheme is then generalized in Section 5 to furnish constructions for orthogonal polynomial bases over d-dimensional simplexes. Finally, we briefly discuss further possible extensions and generalizations of the orthogonal basis scheme in Section 6, and offer some concluding remarks in Section 7.

2. The univariate case

To emphasize symmetry properties, the univariate Legendre polynomials are traditionally defined on the interval $[-1,+1]$. For our purposes, however, it is more convenient to use $[0,1]$. The Legendre polynomials $L_0(u), L_1(u), \ldots$ on $u \in [0,1]$ can be generated through the recurrence

$$(r+1)L_{r+1}(u) = (2r+1)(2u-1)L_r(u) - rL_{r-1}(u) \tag{1}$$

for $r = 1, 2, \ldots$, commencing with $L_0(u) = 1$ and $L_1(u) = 2u - 1$. Successive polynomials are then $L_2(u) = 6u^2 - 6u + 1$, $L_3(u) = 20u^3 - 30u^2 + 12u - 1$, etc. The orthogonality of these polynomials is expressed by the relation

$$\int_0^1 L_r(u)L_s(u)\,\mathrm{d}u = \begin{cases} \frac{1}{2r+1} & \text{if } r = s, \\ 0 & \text{if } r \neq s, \end{cases} \tag{2}$$

which allows us to sequentially compute the Legendre coefficients

$$\ell_r = (2r+1)\int_0^1 L_r(u)f(u)\,\mathrm{d}u \quad \text{for } r = 0, \ldots, n$$

of the degree n least-squares polynomial approximant

$$P_n(u) = \sum_{r=0}^{n} \ell_r L_r(u), \tag{3}$$

to a given integrable function $f(u)$ on the interval $u \in [0,1]$.

As an alternative to the recurrence relation (1), the Legendre polynomials may be defined through Rodrigues' formula

$$L_r(u) = \frac{1}{r!}\frac{\mathrm{d}^r}{\mathrm{d}u^r}\left(u^2 - u\right)^r \quad \text{for } r = 0, 1, \ldots. \tag{4}$$

This can be used to express $L_r(u)$ in terms of the univariate Bernstein basis

$$b_i^r(u) = \binom{r}{i}(1-u)^{r-i}u^i \quad \text{for } i = 0, \ldots, r.$$

Lemma 1. *The Legendre polynomial $L_r(u)$ has the Bernstein representation*

$$L_r(u) = \sum_{i=0}^{r}(-1)^{r+i}\binom{r}{i}b_i^r(u). \tag{5}$$

Proof. We write $(u^2-u)^r = (-1)^r(1-u)^r u^r$ and invoke Leibniz's rule:

$$\frac{(-1)^r}{r!}\frac{\mathrm{d}^r}{\mathrm{d}u^r}(1-u)^r u^r = \frac{(-1)^r}{r!}\sum_{i=0}^{r}\binom{r}{i}\frac{\mathrm{d}^i}{\mathrm{d}u^i}(1-u)^r\frac{\mathrm{d}^{r-i}}{\mathrm{d}u^{r-i}}u^r$$

$$= \frac{(-1)^r}{r!}\sum_{i=0}^{r}\binom{r}{i}\frac{r!}{(r-i)!}(-1)^i(1-u)^{r-i}\frac{r!}{i!}u^i = \sum_{i=0}^{r}(-1)^{r+i}\binom{r}{i}b_i^r(u). \qquad \square$$

Note the elegant simplicity of (5)—the Bernstein coefficients of $L_r(u)$ are simply the sequence of binomial coefficients of order r, taken with alternating signs (beginning with a "+" or "−" according to whether r is even or odd). We may also wish to express all the Legendre polynomials $L_0(u), \ldots, L_n(u)$ of degree $\leqslant n$ in the Bernstein basis of fixed degree n. This can be accomplished by $(n-r)$-fold "degree elevation" (Farouki and Rajan, 1988) of $L_r(u)$, giving

$$L_r(u) = \sum_{i=0}^{n} \gamma_i^{r,n} b_i^n(u) \quad \text{for } r = 0, \ldots, n-1, \tag{6}$$

where

$$\gamma_i^{r,n} = \binom{n}{i}^{-1} \sum_{k=\max(0,i+r-n)}^{\min(i,r)} (-1)^{r+k} \binom{r}{k}\binom{r}{k}\binom{n-r}{i-k} \quad \text{for } i = 0, \ldots, n. \tag{7}$$

The Legendre polynomials (5) may be considered to arise from application of the Gram–Schmidt orthogonalization process (Hoffman and Kunze, 1971) to the set of monomials $1, u, u^2, \ldots$. Starting with $L_0(u) = 1$, we define[1]

$$L_r(u) = u^r - \sum_{i=0}^{r-1} \frac{\int_0^1 u^r L_i(u)\,\mathrm{d}u}{\int_0^1 L_i^2(u)\,\mathrm{d}u} L_i(u) \quad \text{for } r = 1, 2, \ldots.$$

In other words, we obtain $L_r(u)$ by subtracting from u^r its "projections" onto each of the lower-order Legendre polynomials, $L_0(u), \ldots, L_{r-1}(u)$.

3. Orthogonal polynomials on triangles

One way of constructing a set of bivariate orthogonal polynomials is to form *tensor products* of the Legendre polynomials, $L_{rs}(u, v) = L_r(u)L_s(v)$. This is appropriate whenever the unit square $(u, v) \in [0, 1] \times [0, 1]$ is the domain of interest. However, the construction of an orthogonal basis over a triangular (simplicial) domain is a more challenging and fundamental problem. Before embarking upon this, we need to review basic concepts concerning barycentric coordinates and the Bernstein polynomial basis over triangular domains.

3.1. Barycentric coordinates

Let T be a *reference triangle* in the plane, defined by vertices $\mathbf{p}_k = (x_k, y_k)$ for $k = 1, 2, 3$. If these vertices are not collinear, the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \tag{8}$$

is positive if the vertices are labelled in a counter-clockwise sense, and T has area $A = \frac{1}{2}\Delta$. If T_1, T_2, T_3 are the triangles subtended at $\mathbf{p} = (x, y)$ by the sides of T, their (signed) areas are $A_1 = \frac{1}{2}\Delta_1$, $A_2 = \frac{1}{2}\Delta_2$,

[1] Note that this gives a different normalization for $L_r(u)$ than the formulae (4) and (5).

$A_3 = \frac{1}{2}\Delta_3$, where Δ_k is obtained from Δ by replacing the elements $1, x_k, y_k$ by $1, x, y$. We may then write $\mathbf{p} = u\mathbf{p}_1 + v\mathbf{p}_2 + w\mathbf{p}_3$, the barycentric coordinates (u, v, w) of $\mathbf{p}$ with respect to T being defined by the area-ratios

$$u = \frac{\Delta_1}{\Delta}, \qquad v = \frac{\Delta_2}{\Delta}, \qquad w = \frac{\Delta_3}{\Delta}, \tag{9}$$

which evidently satisfy the normalization condition

$$u + v + w = 1. \tag{10}$$

3.2. Bernstein bases over triangles

To construct a basis for bivariate polynomials of degree n over T, we raise the left-hand side of (10) to the nth power and perform a trinomial expansion:

$$(u + v + w)^n = \sum_{\substack{0\leqslant i,j,k\leqslant n\\ i+j+k=n}} \frac{n!}{i!j!k!} u^i v^j w^k = 1. \tag{11}$$

The sum contains $\frac{1}{2}(n+1)(n+2)$ linearly-independent polynomials, which define the Bernstein basis for the space Π_n of degree-n polynomials over the reference triangle T. It is convenient to introduce the compact notation $\alpha = (i, j, k)$ to denote triples of non-negative integers, and we write $|\alpha| = i + j + k$. The Bernstein basis of degree n over T is thus denoted by

$$b^n_\alpha(u, v, w) = \frac{n!}{i!j!k!} u^i v^j w^k, \quad |\alpha| = n. \tag{12}$$

Since these functions are *non-negative* over T, and form a *partition of unity*, any degree-n polynomial

$$P(u, v, w) = \sum_{|\alpha|=n} c_\alpha b^n_\alpha(u, v, w) \tag{13}$$

with Bernstein coefficients c_α satisfies the *convex hull property* over T, namely $\min_{|\alpha|=n} c_\alpha \leqslant P(u, v, w) \leqslant \max_{|\alpha|=n} c_\alpha$. We will also have occasion to use the *degree elevation* algorithm for the Bernstein representation (13). Multiplying both sides by $1 = u + v + w$, we may write

$$P(u, v, w) = \sum_{|\alpha|=n+1} c'_\alpha b^{n+1}_\alpha(u, v, w),$$

where the degree-elevated coefficients c'_α are given by

$$c'_{ijk} = \frac{ic_{i-1,j,k} + jc_{i,j-1,k} + kc_{i,j,k-1}}{n+1} \quad \text{for } i + j + k = n + 1.$$

Of course, this can be repeated to raise the degree to any value $> n$. Further details on the Bernstein form over triangular domains may be found in (Farin, 1986).

3.3. Integration over triangular domains

Consider two pairs of neighboring lines, parallel to the sides $u = 0$ and $v = 0$ of T. If these lines correspond to fixed values $u, u + \mathrm{d}u$ and $v, v + \mathrm{d}v$ of the barycentric coordinates, they delineate a parallelogram of area $\mathrm{d}A = \Delta\,\mathrm{d}u\,\mathrm{d}v$. Thus, the integral of a function $f(u,v,w)$ of the barycentric coordinates over T can be expressed as

$$\iint_T f(u,v,w)\,\mathrm{d}A = \Delta \int_{v=0}^{v=1} \int_{u=0}^{u=1-v} f(u,v,1-u-v)\,\mathrm{d}u\,\mathrm{d}v. \tag{14}$$

Of course, we could also formulate (14) as a double integral over v and w, or w and u, by using $u+v+w=1$.

Lemma 2. *For all $|\alpha| = n$, the Bernstein basis functions on T satisfy*

$$\iint_T b^n_\alpha(u,v,w)\,\mathrm{d}A = \frac{\Delta}{(n+1)(n+2)}. \tag{15}$$

Proof. Substituting the expansion of $w^k = (1-u-v)^k$ into (12) allows the integral to be evaluated; the result (15) then follows from some well-known binomial-coefficient identities. □

Given two bivariate polynomials $p(u,v,w)$ and $q(u,v,w)$, we now define their *inner product* $\langle p,q\rangle$ over T by

$$\langle p,q\rangle = \frac{1}{\Delta} \iint_T pq\,\mathrm{d}A,$$

and when $\langle p,q\rangle = 0$ we say that p and q are *orthogonal* (and we write $p \perp q$). The *norm* of a single polynomial $p(u,v,w)$ is defined by

$$\|p\| = \sqrt{\langle p,p\rangle}.$$

3.4. Orthogonal bases on triangular domains

We wish to construct a basis of linearly independent and mutually orthogonal polynomials in the barycentric coordinates (u,v,w) over T. For polynomials of maximum degree n, there must be $\frac{1}{2}(n+1)(n+2)$ basis functions.

For convenience in computing least-squares approximants of successively higher degree, we want these basis functions to exhibit a hierarchical ordering:

$$\begin{array}{cl} 1 & \text{constant basis function } L_{0,0}, \\ 2 & \text{linear basis functions } L_{1,0}, L_{1,1}, \\ 3 & \text{quadratic basis functions } L_{2,0}, L_{2,1}, L_{2,2}, \\ \cdots & \qquad \cdots \\ n+1 & \text{degree } n \text{ basis functions } L_{n,0}, \ldots, L_{n,n}. \end{array} \tag{16}$$

The least-squares degree-n polynomial approximation $P_n(u,v,w)$ to a given function $f(u,v,w)$ over the domain T, that minimizes

$$\varepsilon^2 = \iint_T \big[f(u,v,w) - F(u,v,w)\big]^2\,\mathrm{d}A$$

over degree-n polynomials $F(u,v,w)$, can then be simply "written down" as

$$P_n(u,v,w)=\sum_{r=0}^{n}\sum_{i=0}^{r}\ell_{r,i}L_{r,i}(u,v,w),\quad \text{with } \ell_{r,i}=\frac{\langle L_{r,i},f\rangle}{\langle L_{r,i},L_{r,i}\rangle}.$$

We note that an orthogonal basis of the form (16) cannot be unique—if $L_m=[L_{m,0},\dots,L_{m,m}]^{\mathrm{T}}$ is a vector containing the degree m basis functions, and M is any orthogonal $(m+1)\times(m+1)$ matrix, we can always replace those functions in (16) by the polynomials $\tilde{L}_m=ML_m=[\tilde{L}_{m,0},\dots,\tilde{L}_{m,m}]$ without compromising orthogonality of the system (Jackson, 1936; Kowalski, 1982a).

We can, in principle, employ the Gram–Schmidt process to construct the orthogonal basis (16). However, Gram–Schmidt orthogonalization is a very cumbersome approach, since the introduction of each basis function requires the evaluation of many integrals. Instead, we desire either a recursion relation analogous to (1) for the univariate case, or a closed-form expression for the basis functions, similar to the Bernstein form (5).

Appell and Kampé de Fériet (1926) and Sauer (1994) have discussed *biorthogonal* systems (for Jacobi weight functions): these comprise two distinct bases such that, within each basis, members of different degree are orthogonal, but those of the same degree are not. Regardless of degree, the members of each basis are orthogonal to all but the unique corresponding member of the other basis. Koornwinder (1975) described a method for generating bivariate analogs of the Jacobi polynomials that are orthogonal on a triangular domain; see also Xu (1994a). Since, on the triangle $S=\{(x,y)\mid x,y\geqslant 0,x+y\leqslant 1\}$, the integral of a function $f(x,y)$ can be re-written as

$$\iint\limits_{S} f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_0^1\!\!\int_0^1 f(1-\xi,\xi\eta)\xi\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

a set of orthogonal bivariate polynomials on S is defined by the products

$$P_{r-k}^{(2k+1,0)}(2x-1)\times(1-x)^k L_k\left(\frac{y}{1-x}\right) \tag{17}$$

for $k=0,\dots,r$ where $P_k^{(\alpha,\beta)}$ are the Jacobi polynomials (Szegö, 1975), orthogonal with respect to the weight $w(x)=(1-x)^\alpha(1+x)^\beta$ on $x\in[-1,+1]$, and L_k are the Legendre polynomials on $x\in[0,1]$. This approach to constructing a set of bivariate orthogonal polynomials goes back to Proriol (1957)—see (Koornwinder, 1975). Szegö (1975) gives the following expression for the Jacobi polynomials

$$P_k^{(\alpha,\beta)}(2x-1)=\sum_{i=0}^{k}\binom{k+\alpha}{k-i}\binom{k+\beta}{i}(x-1)^i x^{k-i},$$

which can be written in terms of the Bernstein basis on $x\in[0,1]$ as

$$P_k^{(\alpha,\beta)}(2x-1)=\sum_{i=0}^{k}(-1)^{k+i}\frac{\binom{k+\alpha}{i}\binom{k+\beta}{k-i}}{\binom{k}{i}}b_i^k(x). \tag{18}$$

The result of Lemma 1, for example, corresponds to taking $\alpha=\beta=0$ above. The system of orthogonal bivariate polynomials that we shall construct below is equivalent to substituting (18) with $\alpha=2k+1$ and $\beta=0$ into (17), and choosing $r=\frac{1}{2}(n+1)(n+2)$ for a basis of given total degree n on S. However, our approach is based on direct *ab initio* construction in the Bernstein basis: it yields simple (and numerically stable) recursive algorithms, and explicitly incorporates the desired degree-ordering (16) of the basis functions.

Even degree-ordered orthogonal (and orthonormal) polynomials are far from being unique—if we arrange all orthogonal polynomials of the same total degree (n, say) in a degree-ordered basis into a vector (of length $n+1$), and multiply this vector by any orthogonal $(n+1) \times (n+1)$ matrix, we obtain another vector of orthogonal polynomials of the same total degree. These freedoms enable us to combine orthogonality (and even orthormality) with additional constraints on the polynomials, to obtain a particular set of orthogonal polynomials that allows a stable and relatively simple recursive construction of one specific orthogonal basis. However, we remark that there are several other ways of choosing an orthogonal basis: for example, it is very common to choose those polynomials which are orthogonal to all polynomials of lower degree, and have monomials as leading terms; see (Xu, 1993a, 1993b, 1994a, 1994b, 1994c).

Our aim here is to give an explicit construction for the Bernstein form of an orthogonal basis for bivariate polynomials over triangles, whose members are mutually orthogonal (with respect to a unit weight function)—whether they are of the same or different degrees. Moreover, we would like this basis to exhibit a degree-ordering of the form (16), and to be as close an analog of the univariate Legendre basis as possible, so that the Bernstein coefficients of the basis functions exhibit a structure making them amenable to computation by recursive algorithms. It is not possible to incorporate all these attributes without some sacrifice, however. Specifically, the orthogonal basis functions are asymmetric with respect to the domain triangle, which may be of concern in applications concerned with data sets on large triangulations (e.g., fitting smooth functions to scattered data or finite-element analysis).

3.5. Degree-ordered orthogonal polynomials

To construct a simple closed-form representation of a degree-ordered system of orthogonal polynomials on a given domain triangle T, it is advantageous to express these polynomials in Bernstein form. We shall begin by showing that this form admits a useful characterization of the condition for any polynomial of degree n to be orthogonal to all polynomials of degree $< n$.

Now for $m \geqslant 1$, consider the space $\mathcal{L}_m$ of degree-m polynomials that are orthogonal to all polynomials of degree $< m$ over T:

$$\mathcal{L}_m = \{p \in \Pi_m \mid p \perp \Pi_{m-1}\}.$$

For any integrable function $f(u, v, w)$ over T, we also define the operator

$$S_n(f) = (n+1)(n+2) \sum_{|\alpha|=n} \langle f, b_\alpha^n \rangle b_\alpha^n$$

(note that $S_n(1) = 1$). It is shown in (Derriennic, 1985) that, for $n \geqslant m$, $\mathcal{L}_m$ is an eigenspace of the operator S_n corresponding to the single eigenvalue

$$\lambda_{m,n} = \frac{(n+2)!n!}{(n+m+2)!(n-m)!}.$$

Lemma 3. *Let* $p=\sum_{|\alpha|=n} c_\alpha b_\alpha^n \in \mathcal{L}_m$ *and* $q=\sum_{|\alpha|=n} d_\alpha b_\alpha^n \in \Pi_n$ *with* $m \leqslant n$. *Then we have*

$$\langle p,q\rangle = \frac{(n!)^2}{(n+m+2)!(n-m)!}\sum_{|\alpha|=n} c_\alpha d_\alpha .$$

Proof. Since p is an eigenfunction of S_n with eigenvalue $\lambda_{m,n}$, we have

$$S_n(p)=(n+1)(n+2)\sum_{|\alpha|=n}\langle p,b_\alpha^n\rangle b_\alpha^n=\lambda_{m,n}p=\lambda_{m,n}\sum_{|\alpha|=n}c_\alpha b_\alpha^n .$$

Identifying coefficients of the basis functions b_α^n in the above sums, we obtain

$$\langle p,b_\alpha^n\rangle=\frac{\lambda_{m,n}c_\alpha}{(n+1)(n+2)}=\frac{(n!)^2 c_\alpha}{(n+m+2)!(n-m)!}$$

for all $|\alpha|=n$. Hence, we have

$$\langle p,q\rangle=\sum_{|\alpha|=n} d_\alpha\langle p,b_\alpha^n\rangle=\frac{(n!)^2}{(n+m+2)!(n-m)!}\sum_{|\alpha|=n}c_\alpha d_\alpha . \qquad \square$$

Lemma 4. *Let* $p=\sum_{|\alpha|=n} c_\alpha b_\alpha^n \in \Pi_n$. *Then we have*

$$p\in\mathcal{L}_n \quad\Leftrightarrow\quad \sum_{|\alpha|=n}c_\alpha d_\alpha=0 \quad \text{for all } q=\sum_{|\alpha|=n}d_\alpha b_\alpha^n\in\Pi_{n-1}.$$

Proof. Let $p=\sum_{|\alpha|=n} c_\alpha b_\alpha^n\in\mathcal{L}_n$. Then $\langle p,q\rangle=0$ for any $q=\sum_{|\alpha|=n}d_\alpha b_\alpha^n\in\Pi_{n-1}$, and hence we have $\sum_{|\alpha|=n}c_\alpha d_\alpha=0$ by Lemma 3.

Conversely, suppose $p=\sum_{|\alpha|=n}c_\alpha b_\alpha^n\in\Pi_n$ satisfies $\sum_{|\alpha|=n}c_\alpha d_\alpha=0$ for every $q=\sum_{|\alpha|=n}d_\alpha b_\alpha^n\in\Pi_{n-1}$. Let $r=\sum_{|\alpha|=n}k_\alpha b_\alpha^n\in\Pi_{n-1}$ be the projection of p onto Π_{n-1} so that $p-r\perp\Pi_{n-1}$, i.e., $p-r\in\mathcal{L}_n$. Then, by the first part of the proof, we have $\sum_{|\alpha|=n}(c_\alpha-k_\alpha)k_\alpha=0$. Since, by supposition, we also have $\sum_{|\alpha|=n}c_\alpha k_\alpha=0$, we deduce that $\sum_{|\alpha|=n}k_\alpha^2=0$. Hence, $k_\alpha=0$ for $|\alpha|=n$, which implies that $r\equiv 0$ and $p\in\mathcal{L}_n$. $\square$

3.6. Construction of the orthogonal basis

With $n\geqslant 1$, we now construct polynomials $P_{n,r}(u,v,w)\in\mathcal{L}_n$, $r=0,\dots,n$, such that $P_{n,r}\perp P_{n,s}$ for $r\neq s$. Thus, choosing $P_{0,0}=1$, the polynomials

$$P_{n,r}(u,v,w) \quad \text{for } 0\leqslant r\leqslant n \text{ and } n=0,1,2,\dots$$

constitute a degree-ordered orthogonal sequence on T. The basic idea in this construction is to make $P_{n,r}$ coincide with the univariate Legendre polynomial (5) of degree r along one edge of T (conventionally, the edge $w=0$), and to make its variation along each chord parallel to that edge a scaled version of this Legendre polynomial. The variation of $P_{n,r}$ with w can then be arranged so as to ensure its orthogonality on T with *every* polynomial of degree $<n$, and with the other basis polynomials $P_{n,s}$ of degree n for $s\neq r$.

Consider the bivariate polynomials defined by

$$P_{n,r}(u,v,w)=\sum_{i=0}^{r}(-1)^i\binom{r}{i}b_i^r(u,v)\sum_{j=0}^{n-r}(-1)^j\binom{n+r+1}{j}b_j^{n-r}(w,u+v) \tag{19}$$

for $r=0,\ldots,n$ and $n=0,1,2,\ldots$, where we write

$$b_i^r(s,t):=\binom{r}{i}s^i t^{r-i} \quad \text{for } i=0,\ldots,r.$$

Noting that

$$\frac{b_i^r(u,v)}{(u+v)^r}=\binom{r}{i}\left(\frac{u}{u+v}\right)^i\left(\frac{v}{u+v}\right)^{r-i}=b_i^r\left(\frac{u}{u+v}\right)=b_i^r\left(\frac{u}{1-w}\right), \tag{20}$$

we may write

$$P_{n,r}(u,v,w)=L_r\left(\frac{u}{1-w}\right)(1-w)^r q_{n,r}(w) \tag{21}$$

for $r=0,\ldots,n$, where

$$q_{n,r}(w)=\sum_{j=0}^{n-r}(-1)^j\binom{n+r+1}{j}b_j^{n-r}(w)$$

and L_r is the rth univariate Legendre polynomial (5). Note that expression (21) has the same basic structure as (17).

We shall first verify that the polynomials $P_{n,r}(u,v,w)$, $r=0,\ldots,n$, are orthogonal to all polynomials of degree $<n$ over the domain triangle T.

Lemma 5. *For $r=0,\ldots,n$ and $i=0,\ldots,n-r-1$, $q_{n,r}(w)$ is orthogonal to $(1-w)^{2r+i+1}$ on $w\in[0,1]$, and hence*

$$\int_0^1 q_{n,r}(w)\,p(w)(1-w)^{2r+1}\,\mathrm{d}w=0 \tag{22}$$

for every polynomial $p(w)$ of degree $\leqslant n-r-1$.

Proof. Using the result $\int_0^1(1-t)^r t^s\,\mathrm{d}t=r!s!/(r+s+1)!$, we have

$$\begin{aligned}
&\int_0^1 q_{n,r}(w)(1-w)^{2r+i+1}\,\mathrm{d}w\\
&\quad=\sum_{j=0}^{n-r}(-1)^j\binom{n+r+1}{j}\binom{n-r}{j}\int_0^1 w^j(1-w)^{n+r+i-j+1}\,\mathrm{d}w\\
&\quad=\frac{1}{n+r+i+2}\sum_{j=0}^{n-r}(-1)^j\frac{\binom{n+r+1}{j}\binom{n-r}{j}}{\binom{n+r+i+1}{j}}=\frac{1}{n+r+i+2}\,\frac{\binom{i}{n-r}}{\binom{n+r+i+1}{n-r}}.
\end{aligned}$$

In the last step we use identity (7.1) from (Gould, 1972). Since the binomial coefficient $\binom{i}{n-r}$ vanishes for $0 \leqslant i \leqslant n-r-1$ and $1, 1-w, \dots, (1-w)^{n-r-1}$ form a basis for polynomials of degree $\leqslant n-r-1$, we obtain the relation (22). □

Theorem 1. *For $n \geqslant 1$ and $0 \leqslant r \leqslant n$, we have $P_{n,r}(u,v,w) \in \mathcal{L}_n$.*

Proof. Consider the polynomials defined by

$$Q_{s,m}(u,v,w) = L_s\left(\frac{u}{1-w}\right)(1-w)^m w^{n-m-1}$$

for $s = 0, \dots, m$ and $m = 0, \dots, n-1$, whose span includes

$$p\left(\frac{u}{1-w}\right)(1-w)^m w^{n-m-1} \quad \text{for } m = 0, \dots, n-1,$$

for any polynomial p of degree $\leqslant m$, and hence also

$$b_j^m\left(\frac{u}{1-w}\right)(1-w)^m w^{n-m-1} = b_j^m(u,v) w^{n-m-1}$$

for $j = 0, \dots, m$ and $m = 0, \dots, n-1$. Since the latter polynomials span all of Π_{n-1}, we need to show that

$$\iint_T P_{n,r}(u,v,w) Q_{s,m}(u,v,w)\,\mathrm{d}A = 0$$

for $s = 0, \dots, m$ and $m = 0, \dots, n-1$. Now for suitable functions $f(u,w)$,

$$\iint_T f\left(\frac{u}{1-w}, w\right)\mathrm{d}A = \Delta \int_0^1 \int_0^{1-w} f\left(\frac{u}{1-w}, w\right)\mathrm{d}u\,\mathrm{d}w = \Delta \int_0^1 \int_0^1 f(t,w)(1-w)\,\mathrm{d}t\,\mathrm{d}w,$$

on making the substitution $t = u/(1-w)$. Thus,

$$\begin{aligned}
&\iint_T P_{n,r}(u,v,w) Q_{s,m}(u,v,w)\,\mathrm{d}A \\
&\quad = \Delta \int_0^1 \int_0^1 L_r(t)(1-w)^r q_{n,r}(w) L_s(t)(1-w)^{m+1} w^{n-m-1}\,\mathrm{d}t\,\mathrm{d}w \\
&\quad = \Delta \int_0^1 L_r(t) L_s(t)\,\mathrm{d}t \int_0^1 q_{n,r}(w) w^{n-m-1}(1-w)^{r+m+1}\,\mathrm{d}w.
\end{aligned}$$

When $m < r$, we have $s < r$, and the first integral vanishes. Hence, it suffices to show that

$$\int_0^1 q_{n,r}(w) w^{n-m-1}(1-w)^{r+m+1}\,\mathrm{d}w = 0$$

for $r \leqslant m \leqslant n-1$ or, equivalently, that

$$\int_0^1 q_{n,r}(w)\,p(w)(1-w)^{2r+1}\,\mathrm{d}w = 0$$

for each polynomial $p(w)$ of degree $\leqslant n-r-1$, which holds by Lemma 5. □

Finally, we show that the polynomials $P_{n,r}(u,v,w)$ of any eiven degree n are all mutually orthogonal to each other.

Theorem 2. *For $r \neq s$, we have $P_{n,r}(u,v,w) \perp P_{n,s}(u,v,w)$.*

Proof. For $r \neq s$, invoking (2) gives

$$\begin{aligned}
&\iint_T P_{n,r}(u,v,w)\,P_{n,s}(u,v,w)\,\mathrm{d}A \\
&\quad = \Delta \int_0^1 \int_0^{1-w} L_r\left(\frac{u}{1-w}\right) L_s\left(\frac{u}{1-w}\right)(1-w)^{r+s} q_{n,r}(w) q_{n,s}(w)\,\mathrm{d}u\,\mathrm{d}w \\
&\quad = \Delta \int_0^1 L_r(t) L_s(t)\,\mathrm{d}t \int_0^1 q_{n,r}(w) q_{n,s}(w)(1-w)^{r+s+1}\,\mathrm{d}w = 0. \qquad \square
\end{aligned}$$

Hence, the bivariate polynomials

$$P_{n,r}(u,v,w) \quad \text{for } r = 0,\ldots,n \text{ and } n = 0,1,2,\ldots \tag{23}$$

defined by (19) or (21) constitute an orthogonal system, ordered by degree, over the domain triangle T. Since this system is, in fact, a particular case (expressed in Bernstein form) of the known system (17), Theorems 1 and 2 are consequences of the fact that the latter forms an orthogonal basis.

3.7. Bernstein form of the orthogonal polynomials

For practical computations, the Bernstein–Bézier form

$$P_{n,r}(u,v,w) = \sum_{|\alpha|=n} a_\alpha^{n,r} b_\alpha^n(u,v,w)$$

of the orthogonal polynomials (23) is very useful. We shall now derive closed-form expressions for the Bernstein coefficients $a_\alpha^{n,r}$ and formulate a recursion relation that allows us to efficiently compute them.

Clearly, since (19) is of maximum degree $n-r$ in w, we must have

$$a_{ijk}^{n,r} = 0 \quad \text{for } k > n-r. \tag{24}$$

Otherwise, for $0 \leqslant k \leqslant n-r$, the remaining coefficients are determined from (19) by the requirement

$$\sum_{i+j=n-k} a_{ijk}^{n,r} b_{ijk}^n(u,v,w) = \binom{n+r+1}{k} b_k^{n-r}(w, u+v) \sum_{i=0}^{r} (-1)^{k+i} \binom{r}{i} b_i^r(u,v).$$

Comparing powers of w on both sides, we have

$$\begin{aligned}\sum_{i=0}^{n-k} a_{ijk}^{n,r}\binom{n}{k} b_i^{n-k}(u,v) &= (-1)^{k+r}\binom{n+r+1}{k}\binom{n-r}{k}(u+v)^{n-r-k}\sum_{i=0}^{r}(-1)^{r+i}\binom{r}{i} b_i^r(u,v)\\ &= (-1)^{k+r}\binom{n+r+1}{k}\binom{n-r}{k}\sum_{i=0}^{n-k}\gamma_i^{r,n-k} b_i^{n-k}(u,v),\end{aligned}$$

where we make use of (20) and (6) in the last step. Hence, we obtain

$$a_{ijk}^{n,r} = (-1)^{k+r}\frac{\binom{n+r+1}{k}\binom{n-r}{k}}{\binom{n}{k}}\gamma_i^{r,n-k} \quad \text{for } 0 \leqslant k \leqslant n-r, \tag{25}$$

where $\gamma_i^{r,n-k}$ are the coefficients of the degree-r Legendre polynomial (5) in the Bernstein basis of degree $n-k$, as defined by expression (7).

We now derive a recurrence relation for the coefficients $a_{ijk}^{n,r}$ of $P_{n,r}(u,v,w)$. Applying Lemma 3 with $q = P_{n,r}(u,v,w)$ and $p = b_{ijk}^{n-1}(u,v,w)$, i.e.,

$$\begin{aligned}p(u,v,w) &= \frac{(n-1)!}{i!j!k!}u^i v^j w^k\\ &= \frac{i+1}{n}b_{i+1,j,k}^n(u,v,w) + \frac{j+1}{n}b_{i,j+1,k}^n(u,v,w) + \frac{k+1}{n}b_{i,j,k+1}^n(u,v,w),\end{aligned}$$

where $i+j+k = n-1$, yields

$$(i+1)a_{i+1,j,k}^{n,r} + (j+1)a_{i,j+1,k}^{n,r} + (k+1)a_{i,j,k+1}^{n,r} = 0, \tag{26}$$

and since we know from (25) that

$$a_{i,n-i,0}^{n,r} = (-1)^r\gamma_i^{r,n} \quad \text{for } i = 0,\ldots,n, \tag{27}$$

we can use (26) to generate $a_{ijk}^{n,r}$ recursively on k. Note also that the familiar (univariate) degree elevation algorithm (Farouki and Rajan, 1988),

$$\gamma_i^{r,n} = \frac{i}{n}\gamma_{i-1}^{r,n-1} + \left(1-\frac{i}{n}\right)\gamma_i^{r,n-1} \quad \text{for } i = 0,\ldots,n,$$

can be used to derive $\gamma_i^{r,n}$ recursively from

$$\gamma_i^{r,r} = (-1)^{r+i}\binom{r}{i} \quad \text{for } i = 0,\ldots,r.$$

Fig. 1 shows the Bernstein coefficients computed from (24) and (25), or (26) and (27), for the cases $n \leqslant 3$. Note that these coefficients are symmetric (even r) and antisymmetric (odd r) with respect to the i and j indices:

$$a_{ijk}^{n,r} = (-1)^r a_{jik}^{n,r} \quad \text{for } i+j+k = n. \tag{28}$$

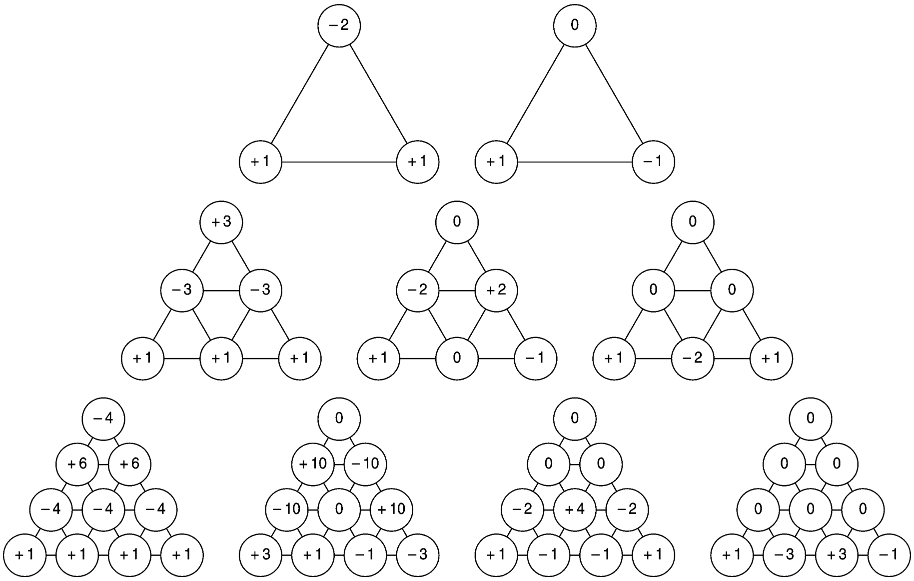

Fig. 1. Bernstein coefficients of the linear $(P_{1,0}, P_{1,1})$, quadratic $(P_{2,0}, P_{2,1}, P_{2,2})$, and cubic $(P_{3,0}, P_{3,1}, P_{3,2}, P_{3,3})$ orthogonal basis functions (in the case of $P_{3,1}$ the coefficients have been scaled by a factor of 3 to give integer values). The constant basis function (not shown here) is simply $P_{0,0} = 1$. Note the symmetry/antisymmetry properties of the coefficients, as expressed by (28).

4. Application to surface smoothing

To illustrate the utility of the orthogonal bivariate basis, we briefly sketch its application to a surface "smoothing" problem. Consider a triangular surface patch of total degree N, expressed in terms of both the standard Bernstein basis and the orthogonal basis over the domain triangle T,

$$\mathbf{r}(u, v, w) = \sum_{|\alpha|=N} \mathbf{p}_\alpha b_\alpha^N(u, v, w) = \sum_{n=0}^{N} \sum_{r=0}^{n} \mathbf{q}_{n,r} P_{n,r}(u, v, w). \tag{29}$$

Once the quantities $a_\alpha^{n,r}$ in Section 3.7 are computed, the linear relationship between the $\frac{1}{2}(N+1)(N+2)$ control points $\mathbf{p}_\alpha$ and vector coefficients $\mathbf{q}_{n,r}$ is known.

We assume that M linear interpolation constraints have been prescribed for the surface: if the boundary curves are specified, for instance, this amounts to $3N$ (vector) constraints. These linear constraints are written in the form

$$F_s(\mathbf{q}_{0,0} \ldots \mathbf{q}_{N,N}) = \sum_{n=0}^{N} \sum_{r=0}^{n} \mathbf{c}_{n,r}^s \cdot \mathbf{q}_{n,r} = 0, \quad s = 1, \ldots, M \tag{30}$$

for suitable coefficients $\mathbf{c}_{n,r}^s$. The freedoms remaining after satisfaction of the interpolation constraints will be used to minimize a "surface smoothness" integral, typically (Kolb et al., 1995) of the form

$$J(\mathbf{q}_{0,0} \ldots \mathbf{q}_{N,N}) = \iint_T \Psi\big(\mathbf{r}(u, v, w)\big)\, \mathrm{d}A, \tag{31}$$

the functional Ψ being quadratic in $\mathbf{r}(u, v, w)$ and its derivatives. Introducing Lagrange multipliers $\lambda_1, \ldots, \lambda_M$ the necessary conditions for minimization of (31) are obtained by equating to zero the derivatives of

$$J(\mathbf{q}_{0,0} \ldots \mathbf{q}_{N,N}) + \sum_{s=1}^{M} \lambda_s F_s(\mathbf{q}_{0,0} \ldots \mathbf{q}_{N,N})$$

with respect to $\mathbf{q}_{0,0} \ldots \mathbf{q}_{N,N}$. Substituting the second expression from (29) in (31) we find that, by orthogonality of the basis functions, a single non-zero term remains when we take the derivative with respect to $\mathbf{q}_{n,r}$—namely, that which contains the square of $P_{n,r}(u, v, w)$. Denoting the resulting coefficient of $\mathbf{q}_{n,r}$ by $C_{n,r}$ we thus obtain a sparse linear system of (vector) equations

$$C_{n,r}\mathbf{q}_{n,r} + \sum_{s=1}^{M} \lambda_s \mathbf{c}_{n,r}^s = \mathbf{0} \quad \text{for } r = 0, \ldots, n,\ n = 0, \ldots, N. \tag{32}$$

Together with Eqs. (30), this system is easily solved for $\lambda_1, \ldots, \lambda_M$ and $\mathbf{q}_{0,0} \ldots \mathbf{q}_{N,N}$ to obtain the "smoothest" surface—according to the chosen functional (31)—that satisfies the interpolation conditions. The smoothness can be further improved by increasing the surface degree N in (29), resulting in a simple augmentation of the linear system defined by (30) and (32).

5. The multivariate case

We now extend the orthogonal basis construction for bivariate polynomials over triangular domains to the case of multivariate polynomials in d variables over a d-dimensional simplex, for $d > 2$. Again, we emphasize that the bases presented here are known (Dunkl and Xu, 2001; Koornwinder and Schwartz, 1997) and our contribution is thus to give explicit constructions of their Bernstein forms. For this purpose, we first recall some notations for multivariate polynomials in Bernstein form.

In barycentric coordinates, the d-dimensional unit simplex is given by

$$T_d = \left\{ u = (u_0, \ldots, u_d)\colon\ u_j \geqslant 0,\ \sum_{j=0}^{d} u_j = 1 \right\}$$

and the *Bernstein basis* for polynomials of degree n over T_d is defined by

$$b_\alpha^n(u) = \binom{n}{\alpha} u^\alpha = \frac{n!}{\alpha_0! \cdots \alpha_d!} u_0^{\alpha_0} \cdots u_d^{\alpha_d}, \tag{33}$$

where

$$\alpha = (\alpha_0, \ldots, \alpha_d) \in \mathbb{N}_0^{d+1} \quad \text{and} \quad |\alpha| = \alpha_0 + \cdots + \alpha_d = n.$$

Any polynomial P of degree n at most can be written in the Bernstein–Bézier representation

$$P(u) = \sum_{|\alpha|=n} p_\alpha b_\alpha^n(u). \tag{34}$$

The vector $p = (p_\alpha\colon\ |\alpha| = n)$ contains the *Bernstein coefficients* of P. We denote the set of all Bernstein coefficients of a given degree n by

$$\mathbb{P}_n := \left\{ (p_\alpha\colon\ |\alpha| = n)\colon\ p_\alpha \in \mathbb{R},\ |\alpha| = n \right\}.$$

As in Section 3, we denote by Π_n the space of all polynomials of degree $\leqslant n$ over T_d, and by $\mathcal{L}_n$ the space of polynomials of degree n that are orthogonal to every polynomial of degree $< n$ over T_d. Eq. (34) defines an isomorphism between polynomials $P \in \Pi_n$ and coefficients $p \in \mathbb{P}_n$. When we use upper- and lower-case letters henceforth, we assume they are connected by (34).

We first seek a generalization of Lemma 3. For that purpose, we recall the *Bernstein–Durrmeyer operator* $V_n : C(T_d) \to \Pi_n$ defined by

$$V_n f = \sum_{|\alpha|=n} \frac{\langle f, b_\alpha^n\rangle}{\langle 1, b_\alpha^n\rangle} b_\alpha^n = \frac{(n+d)!}{n!} \sum_{|\alpha|=n} \langle f, b_\alpha^n\rangle b_\alpha^n.$$

The following result has been proved in (Derriennic, 1985, Theorem II.1):

Theorem 3. *The space $\mathcal{L}_m$ is an eigenspace of V_n with the eigenvalue*

$$\lambda_{m,n} := \begin{cases} \dfrac{(n+d)!n!}{(n+m+d)!(n-m)!} & m \leqslant n, \\ 0 & m > n. \end{cases}$$

We also recall the *degree elevation* operator $R : \mathbb{P}_n \to \mathbb{P}_{n+1}$, which satisfies

$$P(u) = \sum_{|\alpha|=n+1} (Rp)_\alpha b_\alpha^{n+1}(u),$$

where

$$Rp = \left(\sum_{j=0}^{d} \frac{\alpha_j}{n+1} p_{\alpha-\varepsilon_j} \colon\ |\alpha| = n+1 \right) \tag{35}$$

and ε_j for $j = 0, \ldots, n$ denote the canonical unit multi-indices—i.e., $\varepsilon_0 = (1, 0, 0, \ldots, 0)$, $\varepsilon_1 = (0, 1, 0, \ldots, 0)$, $\ldots$, etc.

Consider now the inner product $\langle \cdot, \cdot \rangle_n : \mathbb{P}_n \times \mathbb{P}_n \to \mathbb{R}$ defined by

$$\langle p, q\rangle_n = \sum_{|\alpha|=n} p_\alpha q_\alpha. \tag{36}$$

Lemma 6. *Let $P \in \mathcal{L}_m$ and $Q \in \Pi_n$ have Bernstein coefficients $p \in \mathbb{P}_m$ and $q \in \mathbb{P}_n$, respectively, where $m \leqslant n$. Then*

$$\langle P, Q\rangle = \frac{n!\lambda_{m,n}}{(n+d)!} \big\langle R^{n-m} p, q \big\rangle_n. \tag{37}$$

Proof. By the definition of the degree elevation operator and Theorem 3, we obtain

$$\sum_{|\alpha|=n} \big(R^{n-m} p\big)_\alpha b_\alpha^n = P = \frac{1}{\lambda_{m,n}} V_n P = \frac{(n+d)!}{n!\lambda_{m,n}} \sum_{|\alpha|=n} \langle P, b_\alpha^n\rangle b_\alpha^n,$$

and by comparing coefficients we find that

$$R^{n-m} p = \left(\frac{(n+d)!}{n!\lambda_{m,n}} \langle P, b_\alpha^n\rangle \colon\ |\alpha| = n \right).$$

Hence, we have

$$\langle P, Q\rangle = \sum_{|\alpha|=n} q_\alpha \langle P, b_\alpha^n\rangle = \sum_{|\alpha|=n} q_\alpha \frac{n!\lambda_{m,n}}{(n+d)!}\big(R^{n-m}p\big)_\alpha = \frac{n!\lambda_{m,n}}{(n+d)!}\big\langle R^{n-m}p, q\big\rangle_n. \qquad \square$$

Now if we take Bernstein coefficients $p \in \mathbb{P}_{n-1}$ and $q \in \mathbb{P}_n$, we can apply degree elevation to compute

$$\langle Rp, q\rangle_n = \big\langle p, R^t q\big\rangle_{n-1}$$

where R^t, the formal adjoint of R, is defined by

$$\sum_{|\alpha|=n-1} p_\alpha\big(R^t q\big)_\alpha = \sum_{|\alpha|=n} q_\alpha (Rp)_\alpha = \sum_{|\alpha|=n} q_\alpha \sum_{j=0}^{d} \frac{\alpha_j}{n} p_{\alpha-\varepsilon_j} = \sum_{|\alpha|=n-1} \sum_{j=0}^{d} q_{\alpha+\varepsilon_j} \frac{\alpha_j+1}{n} p_\alpha,$$

that is,

$$R^t q = \left(\sum_{j=0}^{d} \frac{\alpha_j+1}{n} q_{\alpha+\varepsilon_j}\colon\ |\alpha| = n-1\right). \tag{38}$$

Note that the linear operator R^t is a *degree reduction* operator. Moreover, R^t can be used to characterize the space $\mathcal{L}_n$ as follows:

Lemma 7. *For $P \in \Pi_n$ with Bernstein coefficients $p \in \mathbb{P}_n$, we have*

$$P \in \mathcal{L}_n \quad \Leftrightarrow \quad R^t p = 0.$$

Proof. To prove necessity, assume that $P \in \mathcal{L}_n$ has Bernstein coefficients $p \in \mathbb{P}_n$ and note that, for any $Q \in \Pi_{n-1}$ with Bernstein coefficients $q \in \mathbb{P}_{n-1}$, Lemma 6 yields

$$0 = \langle P, Q\rangle = \frac{n!\lambda_{n,n}}{(n+d)!}\langle p, Rq\rangle_n = \frac{n!\lambda_{n,n}}{(n+d)!}\big\langle R^t p, q\big\rangle_{n-1}.$$

Specifically, choosing the polynomials $Q = \beta_\alpha^n$ for each $\alpha \in \mathbb{N}_0^{d+1}$ with $|\alpha| = n-1$, we obtain $R^t p = 0$.

Conversely, assume that $R^t p = 0$. From (38) we observe this implies, for $k = 0, \ldots, n-1$ and $\hat\alpha \in \mathbb{N}_0^d$ with $|\hat\alpha| = n-k-1$, that

$$p_{(\hat\alpha,k+1)} = -\sum_{j=0}^{d-1} \frac{\hat\alpha_j+1}{k+1}\, p_{(\hat\alpha+\varepsilon_j,k)} = -\frac{n-k}{k+1}\big(R^t \hat p_k\big)_{\hat\alpha}, \tag{39}$$

where $\hat p_k = (p_{(\hat\alpha,k)}\colon\ |\hat\alpha| = n-k)$. That is,

$$\hat p_{k+1} = -\frac{n-k}{k+1}\, R^t \hat p_k. \tag{40}$$

In other words, the conditions $R^t p = 0$ and the values $p_{(\hat\alpha,0)}$ for $\hat\alpha \in \mathbb{N}_0^d$ with $|\hat\alpha| = n$, determine p completely. In particular,

$$\dim\big\{p \in \mathbb{P}_n\colon\ R^t p = 0\big\} \leqslant \binom{n+d-1}{d-1} = \dim \mathcal{L}_n.$$

However, by what we proved before, we also know that

$$\mathcal{L}_n \sim \left\{ p \in \mathbb{P}_n\colon \sum_{|\alpha|=n} p_\alpha b_\alpha^n \in \mathcal{L}_n \right\} \subseteq \left\{ p \in \mathbb{P}_n\colon\ R^t p = 0 \right\},$$

and since the dimensions of the above two finite-dimensional vector spaces coincide, they must be identical. □

Moreover, Eq. (39) has another immediate consequence:

Corollary 1. *For any given* $(d-1)$*-variate polynomial* $\widehat{P} \in \Pi_n$*, there exists a unique* d*-variate polynomial* $P \in \mathcal{L}_n$ *such that*

$$\widehat{P}(\hat{u}) \equiv P(\hat{u}, 0) \quad \textit{for } \hat{u} \in T_{d-1}.$$

Proof. We observe that $\widehat{P} = P(\cdot, 0)$ is equivalent to $p_{(\hat{\alpha},0)} = \hat{p}_{\hat{\alpha}}$ for each $\hat{\alpha} \in \mathbb{N}_0^d$ with $|\hat{\alpha}| = n$, and by (39), which is in turn equivalent to $P \in \mathcal{L}_n$, this defines p uniquely. □

This corollary shows that our construction agrees with the “standard” theory (Askey, 1975) of orthogonal polynomials. Univariate orthogonal polynomials are often not normalized to have *integral* 1, but rather in such a way that the value at one end of the integration interval is 1 (which is the only orthogonal polynomial in zero variables). This normalization ensures that the orthogonal polynomials possess rational coefficients. Our construction extends this mode of normalization by demanding mutual orthogonality for all polynomials, and that the restriction of any orthogonal polynomial in d variables to one face of the simplex T_d is again an orthogonal polynomial in $d-1$ variables.

The next result—which shows that, for orthogonal polynomials, R^t is “almost” an inverse of R—will be crucial for what follows.

Lemma 8. *Let* $P \in \mathcal{L}_n$ *have Bernstein coefficients* $p \in \mathbb{P}_n$*, and let* $j, k \in \mathbb{N}$ *satisfy* $j \leqslant k$*. Then*

$$\left(R^t\right)^j R^k p = \mu_{n,j,k}\, R^{k-j} p, \tag{41}$$

where

$$\mu_{n,j,k} = \frac{(n+d+k)!}{(n+d+k-j)!}\,\frac{(n+k-j)!}{(n+k)!}\,\frac{\lambda_{n+k-j,n}}{\lambda_{n+k,n}}.$$

Proof. Choose Bernstein coefficients $p(\gamma)$ for $\gamma \in \mathbb{N}_0^d$ with $|\gamma| \leqslant n$, so that the associated polynomials

$$P_\gamma = \sum_{|\alpha|=n} p_\alpha(\gamma) b_\alpha^n$$

form an orthonormal basis for Π_n. Then by Lemma 6 we have, for $|\gamma| = n$ and $|\eta| \leqslant n$,

$$\begin{aligned}
\left\langle R^{n+k-j-|\eta|} p(\eta), \left(R^t\right)^j R^k p(\gamma) \right\rangle_{n+k-j} &= \left\langle R^{n+k-|\eta|} p(\eta), R^k p(\gamma) \right\rangle_{n+k} \\
&= \frac{(n+d+k)!}{(n+k)!\lambda_{n+k,|\eta|}} \langle P_\eta, P_\gamma \rangle \\
&= \frac{(n+d+k)!}{(n+k)!\lambda_{n+k,n}} \delta_{\eta,\gamma}.
\end{aligned}$$

On the other hand, another application of Lemma 6 also yields

$$\left\langle R^{n+k-j-|\eta|}p(\eta), R^{k-j}p(\gamma)\right\rangle_{n+k-j} = \frac{(n+k-j+d)!}{(n+k-j)!\lambda_{n+k-j,n}}\delta_{\eta,\gamma},$$

and since the vectors $R^{n-|\eta|}p$, $|\eta| \leqslant n$, are a basis for $\mathbb{P}_n$, we conclude that

$$\left(R^t\right)^j R^k p(\gamma) = \frac{(n+d+k)!}{(n+d+k-j)!}\,\frac{(n+k-j)!}{(n+k)!}\,\frac{\lambda_{n+k-j,n}}{\lambda_{n+k,n}}R^{k-j}p(\gamma),$$

which proves the lemma. □

We now describe a method to construct d-variate orthogonal polynomials $P_\alpha^d \in \Pi_{|\alpha|}$ for $\alpha \in \mathbb{N}_0^d$, which satisfy

$$\int_{T^d} P_\alpha^d(u)P_\beta^d(u)\,\mathrm{d}u = K_\alpha \delta_{\alpha,\beta} \quad \text{for } \alpha, \beta \in \mathbb{N}_0^d,$$

where $K_\alpha > 0$ for $\alpha \in \mathbb{N}_0^d$. This method will be recursive with respect to d, where the cases $d = 1$ and 2 have already been dealt with. Let $p(d,\alpha) \in \mathbb{P}_{|\alpha|}$ denote the Bernstein coefficients of these orthogonal polynomials, i.e.,

$$P_\alpha^d = \sum_{|\beta|=|\alpha|} p_\beta(d,\alpha) b_\beta^{|\alpha|}, \quad \alpha \in \mathbb{N}_0^d.$$

With each $\alpha \in \mathbb{N}_0^d$, we associate an $\hat{\alpha} = (\alpha_1, \ldots, \alpha_{d-1}) \in \mathbb{N}_0^{d-1}$ and, following Corollary 1, we "initialize" the polynomial P_α^d so that

$$P_\alpha^d\big|_{u_d=0} = P_{\hat{\alpha}}^{d-1}.$$

Clearly, the construction will work as described in (40), that is, we set

$$\hat{p}_0(d,\alpha) := R^{|\alpha|-|\hat{\alpha}|}p(d-1,\hat{\alpha}), \tag{42}$$

$$\hat{p}_{k+1}(d,\alpha) := -\frac{n-k}{k+1}R^t\hat{p}_k(d,\alpha), \quad k = 0, \ldots, |\alpha|-1. \tag{43}$$

Hence, for $k = 0, \ldots, n$, we have

$$\hat{p}_k(d,\alpha) = (-1)^k\left(\prod_{j=1}^k \frac{n-j+1}{j}\right)\left(R^t\right)^k R^{\alpha_d}p(d-1,\hat{\alpha}). \tag{44}$$

This construction has an appealing geometric interpretation: the initializing equation (42) ensures that P_α^d coincides with $P_{\hat{\alpha}}^{d-1}$ on the face $u_{d+1} = 0$ of the simplex T_{d+1}, while the second equation (43) guarantees that $R^t p(d,\alpha) = 0$, which in turn means that $P_\alpha^d \in \mathcal{L}_{|\alpha|}$. Moreover, Eq. (44) and Lemma 8 imply, for $k = 0, \ldots, \alpha_d$, that

$$\hat{p}_k(d,\alpha) = (-1)^k\left(\prod_{j=1}^k \frac{n-j+1}{j}\right)\mu_{|\hat{\alpha}|,k,\alpha_d}R^{\alpha_d-k}p(d-1,\hat{\alpha}). \tag{45}$$

In particular,

$$\hat{p}_{\alpha_d}(d,\alpha) = (-1)^{\alpha_d}\left(\prod_{j=1}^{k}\frac{n-j+1}{j}\right)\mu_{|\hat{\alpha}|,\alpha_d,\alpha_d}\, p(d-1,\hat{\alpha}),$$

and since $P_{\hat{\alpha}}^{d-1} \in \mathcal{L}_{|\hat{\alpha}|}$, and hence $R^t p(d-1,\hat{\alpha}) = 0$, it follows that

$$\hat{p}_k(d,\alpha) = 0 \quad \text{for } k = \alpha_d + 1, \ldots, n. \tag{46}$$

We are now ready to show that this construction gives an orthogonal basis.

Theorem 4. *The polynomials P_α^d, $|\alpha| = n$, with Bernstein coefficients $p(d,\alpha)$ defined by* (42) *and* (43), *form an orthogonal basis for $\mathcal{L}_n$, $n \in \mathbb{N}$.*

Proof. The proof proceeds by induction on d, the case $d = 1$ being trivial. For $d > 1$ we first remark that, by construction, we have $P_\alpha^d \in \mathcal{L}_{|\alpha|}$, so all we have to show is that $\langle P_\alpha^d, P_\beta^d \rangle = 0$ whenever $\alpha \neq \beta$ and $|\alpha| = |\beta|$.

Suppose now $\alpha, \beta \in \mathbb{N}_0^d$ are such that $|\alpha| = |\beta|$ and $\alpha \neq \beta$. Note this also implies that $\hat{\alpha} \neq \hat{\beta}$, since if $\hat{\alpha} = \hat{\beta}$ and $|\alpha| = |\beta|$, then

$$\alpha_d = |\alpha| - |\hat{\alpha}| = |\beta| - |\hat{\beta}| = \beta_d$$

and therefore $\alpha = \beta$. Furthermore, we can assume without loss of generality that $\alpha_d \leqslant \beta_d$. Then, by (46), we have

$$\big\langle \hat{p}_k(d,\alpha), \hat{p}_k(d,\beta)\big\rangle_{n-k} = 0 \quad \text{for } k = \alpha_d + 1, \ldots, n. \tag{47}$$

On the other hand, when $k \leqslant \alpha_d$ we first note that this implies $n - k \geqslant |\hat{\alpha}|$, and hence $\lambda_{n-k,|\hat{\alpha}|} > 0$. Taking this into account, we obtain from (45) and Lemma 6 that

$$\begin{aligned}
&\big\langle \hat{p}_k(d,\alpha), \hat{p}_k(d,\beta)\big\rangle_{n-k} \\
&\quad = \left(\prod_{j=1}^{k}\frac{n-j+1}{j}\right)^2 \mu_{|\hat{\alpha}|,k,\alpha_d}\,\mu_{|\hat{\beta}|,k,\beta_d}\big\langle R^{\alpha_d-k} p(d-1,\hat{\alpha}), R^{\beta_d-k} p(d-1,\hat{\beta})\big\rangle_{n-k} \\
&\quad = \left(\prod_{j=1}^{k}\frac{n-j+1}{j}\right)^2 \mu_{|\hat{\alpha}|,k,\alpha_d}\,\mu_{|\hat{\beta}|,k,\beta_d}\frac{(n-k+d-1)!}{(n-k)!\lambda_{n-k,|\hat{\alpha}|}}\big\langle P_{\hat{\alpha}}^{d-1}, P_{\hat{\beta}}^{d-1}\big\rangle = 0.
\end{aligned} \tag{48}$$

Thus, (47) and (48) yield

$$\big\langle \hat{p}_k(d,\alpha), \hat{p}_k(d,\beta)\big\rangle_{n-k} = 0 \quad \text{for } k = 0, \ldots, n$$

and, consequently,

$$\big\langle p(d,\alpha), p(d,\beta)\big\rangle_n = \sum_{k=0}^{n}\big\langle \hat{p}_k(d,\alpha), \hat{p}_k(d,\beta)\big\rangle_{n-k} = 0. \tag{49}$$

But (49), the fact that $P_\alpha^d, P_\beta^d \in \mathcal{L}_n$, and a final application of Lemma 6 yield

$$\big\langle P_\alpha^d, P_\beta^d\big\rangle = \frac{n!\lambda_{n,n}}{(n+d)!}\big\langle p(d,\alpha), p(d,\beta)\big\rangle_n = 0,$$

which proves orthogonality. □

6. Other extensions and generalizations

Our construction of orthogonal bases for triangular and higher-dimensional simplex domains are just preliminary results, and many other aspects of this problem deserve further investigation. We have emphasized the importance of the degree ordering (16) of the basis functions, for example, in applications where least-squares approximants of increasing degree are to be computed. As noted in Section 1, this ordering precludes orthogonal bases that have symmetry (or antisymmetry) with respect to the domain triangle or simplex.

For some applications, an orthogonal basis that comprises homogeneous polynomials of fixed total degree[2] n may be preferred, and in this context we expect that a greater degree of symmetry and structure in the Bernstein representations of the orthogonal basis functions will become apparent. This consideration is not merely of aesthetic interest: it may also influence stability considerations—see, for example, (Farouki, 2000b) in the univariate case.

Finally, we mention the problem of constructing orthogonal bases under prescribed boundary constraints. Least-squares polynomial approximations of given functions over bounded domains do not, in general, match values of the given function on the domain boundary. Hence, piecewise least-squares approximants computed using standard orthogonal bases do not exhibit even C^0 continuity (Farouki, 2000a; Lachance, 1988)—this is usually unacceptable. In the univariate case, the classical orthogonal (e.g., Chebyshev or Legendre) polynomials can be modified to yield best approximants that are constrained to match function values and derivatives to a prescribed order at the ends of the approximation interval. Such an adaptation should also be possible for our bivariate (and multivariate) systems of orthogonal polynomials over simplexes. However, a detailed investigation of this problem exceeds our current goals.

7. Closure

We have described a scheme to construct a degree-ordered orthogonal basis for polynomials in the barycentric coordinates over any triangular domain T. Although the basis coincides with known orthogonal systems over triangles, its construction in the Bernstein form offers numerically-stable algorithms that allow the coefficients to be computed by either closed-form expressions or simple recursion relations. The coefficients exhibit an intuitive structure, closely related to the behavior of the univariate Legendre polynomials. The scheme also admits a natural generalization to the construction of orthogonal bases over simplexes in an arbitrary number of dimensions. We expect that these orthogonal bases will find diverse applications in problems concerned with the approximation or interpolation of bivariate and multivariate data.

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[2] We refer here to the *true* degree of the basis functions, rather than the trivial approach of simply degree elevating the basis functions defined in Section 3 to a fixed maximum value n.

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