Frontiers in Interpolation and Approximation
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Asymptotic Optimality in Time-Frequency Localization of Scaling Functions and Wavelets

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Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

For a triangular array of numbers $a_{n,j}$, $n = 2, 3, \dots$, $j = 0, 1, \dots, n$, the refinement equation with mask $P_n(z) = \sum_{j=0}^{n} a_{n,j} z^j$, has for each n, a unique solution ϕ_n, which when suitably normalized, converges to the Gaussian if $P_n(z)$ has all roots in a sector inside the left half-plane and satisfying $P_n(1) = 1$, $P_n(-1) = 0$. The Gaussian, being the extremal function that attains the optimal constant in the uncertainty product, passes this characteristic to ϕ_n, whose uncertainty product converges to the optimal constant. The object is to analyse this phenomena for ϕ_n as well as for the corresponding wavelets.

1 Introduction

For $n \geq 2$, let

$$P_n(z) = \sum_{j=0}^{n} a_{n,j} z^j, \tag{1.1}$$

be a polynomial with all its roots in the left half-plane $\{z : \Re\, z \leq 0\}$ and satisfying

$$P_n(1) = 1, \quad P_n(-1) = 0. \tag{1.2}$$

It is known that the *refinement equation*

$$\phi_n(x) = \sum_{j=0}^{n} 2a_{n,j}\phi_n(2x-j), \quad x \in \mathbb{R}, \tag{1.3}$$

has a unique solution satisfying $\int_{-\infty}^{\infty} \phi_n = 1$. Moreover it is shown in [5] that ϕ_n is continuous, non-negative and has support in $[0, n]$. In the special case $P_n(z) = 2^{-n}(z+1)^n$, ϕ_n is the uniform B-spline of degree $n-1$ with knots at $0, 1, \ldots, n$. We shall refer to ϕ_n as a *refinable function* with *symbol* P_n.

Now suppose that the roots of P_n are $-r_{n,j}$, $j = 1, \ldots, n$. It is shown in [2] that if we assume the stronger condition that for some β in $[0, \frac{\pi}{2})$,

$$|\arg r_{n,j}| \le \beta, \text{ for } j = 1, \ldots, n, \; n = 2, 3, \ldots,$$

and furthermore

$$\sigma_n^2 := \frac{1}{3}\sum_{j=1}^{n} \frac{r_{n,j}}{(1+r_{n,j})^2} \to \infty \quad \text{as } n \to \infty, \tag{1.4}$$

then a suitable shift and scaling of ϕ_n converges to the Gaussian function $G(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. To be precise, let

$$\mu_n := \sum_{j=1}^{n} \frac{1}{1+r_{n,j}} \tag{1.5}$$

and

$$\widetilde{\phi}_n(x) := \sigma_n \phi_n(\sigma_n x + \mu_n), \quad x \in \mathbb{R}. \tag{1.6}$$

Then $\widetilde{\phi}_n$ has mean zero and standard deviation 1 and $\lim_{n\to\infty} \widetilde{\phi}_n = G$ in $L^p(\mathbb{R})$, $1 \le p \le \infty$. Moreover we have convergence in the frequency domain, i.e. $\lim_{n\to\infty} \widehat{\widetilde{\phi}}_n(u) = \widehat{G}(u) = e^{-u^2/2}$, where the convergence is uniform on $\mathbb{R}$.

The above properties are useful for many applications, e.g. signal processing, because the Gaussian function gives optimal time-frequency localisation. Thus we may approximate the Gaussian function by a refinable function ϕ_n with compact support which has fast algorithms for practical implementation. A natural choice is the uniform B-spline of degree $n-1$, but it was shown in [2] that other choices of ϕ_n give, in a sense, faster rates of convergence to the Gaussian.

To make precise the optimal time-frequency localisation of the Gaussian, we recall *Heisenberg's Uncertainty Principle.* For an L^2 function ϕ for which $\int_{-\infty}^{\infty} x^j |\phi(x)|^2 dx$ exists, $j = 1, 2$, we write

$$\mu_\phi := \frac{\int_{-\infty}^{\infty} x|\phi(x)|^2 dx}{\|\phi\|_2^2} \tag{1.7}$$

and

$$\Delta_\phi := \frac{\left\{\int_{-\infty}^{\infty} (x-\mu_\phi)^2 |\phi(x)|^2 dx\right\}^{\frac{1}{2}}}{\|\phi\|_2}. \tag{1.8}$$

Similarly if $\int_{-\infty}^{\infty} u^j |\widehat{\phi}(u)|^2 du$ exists, $j = 1, 2$, we may define $\mu_{\widehat{\phi}}$ and $\Delta_{\widehat{\phi}}$. Thus Δ_ϕ is the standard deviation of the density function $|\phi|^2/\|\phi\|_2^2$ and gives a measure of the localisation of ϕ in the time domain. Similarly $\Delta_{\widehat{\phi}}$ is the standard deviation of $\left|\widehat{\phi}\right|^2 / \|\widehat{\phi}\|_2^2$ and measures the localisation of ϕ in the frequency domain. The *uncertainty product* $\Delta_\phi \Delta_{\widehat{\phi}}$ gives an overall measure of the time-frequency localisation of ϕ. Clearly for any constants c, $\mu \in \mathbb{R}$, and $\sigma > 0$, the function $c\phi(\sigma \cdot -\mu)$ has the same uncertainty product as ϕ. *Heisenberg's Uncertainty Principle* states that for any ϕ as above,

$$\Delta_\phi \Delta_{\widehat{\phi}} \geq \frac{1}{2}, \tag{1.9}$$

and equality holds if and only if $\phi = cG(\sigma \cdot -\mu)$ for some c, $\mu \in \mathbb{R}$, and $\sigma > 0$, (see [4] for a general discussion). So for the sequence (ϕ_n) of refinable functions, as above, to approach optimal time-frequency localisation, we would desire

$$\lim_{n \to \infty} \Delta_{\phi_n} \Delta_{\widehat{\phi}_n} = \frac{1}{2}.$$

This is proved in Section 3 under the further assumption $P_n'(-1) = 0$.

In applications such as signal processing it is also important to have good time-frequency localisation for the wavelet corresponding to the refinable function ϕ_n, as above, which is defined as follows. Let

$$\Phi_n(x) := \int_{-\infty}^{\infty} \phi_n(x+y)\phi_n(y)dy, \quad x \in \mathbb{R}, \tag{1.10}$$

$$Q_n(z) := \sum_{j=-n+1}^{n-1} \Phi_n(j)z^j, \quad z \in \mathbb{C} \setminus \{0\}. \tag{1.11}$$

Then the wavelet corresponding to ϕ_n is defined, up to multiplication by a constant, as $\psi_n(2\cdot -1)$, where ψ_n is defined by

$$\widehat{\psi}_n(u) = e^{inu} Q_n(e^{-i(u+\pi)}) P_n(e^{-i(u+\pi)}) \widehat{\phi}_n(u), \quad u \in \mathbb{R}. \tag{1.12}$$

We note that since $P_n(-1) = 0$, we have $\widehat{\psi}_n(0) = 0$. For a function ψ with $\widehat{\psi}(0) = 0$ (associated with a bandpass filter), the definition of the uncertainty product is modified to reflect the fact that $\widehat{\psi}$ treats positive and negative frequency bands separately. Let

$$\mu^+_{\widehat{\psi}} := \int_0^{\infty} u \left|\widehat{\psi}(u)\right|^2 du \Big/ \int_0^{\infty} \left|\widehat{\psi}(u)\right|^2 du, \tag{1.13}$$

$$\Delta^+_{\widehat{\psi}} := \left\{ \int_0^{\infty} (u - \mu^+_{\widehat{\psi}})^2 \left|\widehat{\psi}(u)\right|^2 du \Big/ \int_0^{\infty} \left|\widehat{\psi}(u)\right|^2 du \right\}^{\frac{1}{2}}, \tag{1.14}$$

where we assume these are well-defined. (Note that for a real-valued function ψ, $|\widehat{\psi}|$ is even and so $\int_0^{\infty} |\widehat{\psi}(u)|^2 du = \frac{1}{2}\|\widehat{\psi}\|_2^2$ and the definition of $\Delta^+_{\widehat{\psi}}$ is unaltered by replacing $\int_0^{\infty}$ by $\int_{-\infty}^{0}$ in (1.13) and (1.14).) Then a measure of the time-frequency localisation of ψ is given by $\Delta_\psi \Delta^+_{\widehat{\psi}}$.

It is shown in ([3], Theorem 1) that if $\psi \in L^2 \cap L^1$ is a real-valued symmetric or anti-symmetric function that satisfies $t\psi(t) \in L^2$, $\psi' \in L^2$ and $\widehat{\psi}(0) = 0$, then

$$\Delta_\psi \Delta^+_{\widehat{\psi}} > \frac{1}{2},$$

and the lower bound cannot be improved or attained.

A sequence of functions which are asymptotically optimal are the *modulated Gaussians*

$$G_n(x) := \sin(\lambda_n x) G(x), \quad n = 1, 2, \ldots,$$

where $\lambda_n \to \infty$ as $n \to \infty$. To see this we note that

$$2i\widehat{G}_n(u) = e^{-(u-\lambda_n)^2/2} - e^{-(u+\lambda_n)^2/2}.$$

Thus $\mu^+_{\widehat{G}_n} - \lambda_n \to 0$ as $n \to \infty$ and so

$$\begin{aligned}\lim_{n\to\infty}\left(\Delta^+_{\widehat{G}_n}\right)^2 &= \lim_{n\to\infty}\frac{\int_0^\infty (u-\lambda_n)^2 e^{-(u-\lambda_n)^2}du}{\int_0^\infty e^{-(u-\lambda_n)^2}du}\\ &= \frac{\int_{-\infty}^\infty u^2 e^{-u^2}du}{\int_{-\infty}^\infty e^{-u^2}du} = \frac{1}{2}.\end{aligned}$$

Also

$$\Delta^2_{G_n} = \frac{\int_0^\infty \left|\widehat{G}'_n(u)\right|^2 du}{\int_0^\infty \left|\widehat{G}_n(u)\right|^2 du}$$

and so

$$\lim_{n\to\infty}\Delta^2_{G_n} = \lim_{n\to\infty}\frac{\int_0^\infty (u-\lambda_n)^2 e^{-(u-\lambda_n)^2}du}{\int_0^\infty e^{-(u-\lambda_n)^2}du} = \frac{1}{2}.$$

Thus

$$\lim_{n\to\infty}\Delta_{G_n}\Delta^+_{\widehat{G}_n} = \frac{1}{2}.$$

In Section 4, we consider the wavelets ψ_n, as above. We assume further that P_n is reciprocal (which ensures ψ_n is symmetric or anti-symmetric), the roots of P_n are real (and hence negative), and $P'_n(-1) = 0$. It is then proved that for certain k_n, σ_n, α_n with $\sigma_n \to \infty$ as $n \to \infty$,

$$k_n\widehat{\psi}_n(u/\sigma_n) - e^{-\frac{1}{2}(u-\sigma_n\alpha_n)^2} \to 0$$

in $L^p(0,\infty)$, $1 \le p \le \infty$. Taking inverse Fourier transforms shows that the difference between $|k_n|\psi_n(\sigma_n x)$ and either $2\cos(\sigma_n\alpha_n x)G(x)$ or $2\sin(\sigma_n\alpha_n x)G(x)$ converges to 0 in $L^q(\mathbb{R})$, $2 \le q \le \infty$.

Finally we deduce that if we also have $P''_n(-1) = 0$, then

$$\lim_{n\to\infty}\Delta_{\psi_n}\Delta^+_{\widehat{\psi}_n} = \frac{1}{2},$$

and so the wavelets ψ_n have asymptotically optimal time-frequency location.

In [6] it was shown that the uniform B-spline converged to the Gaussian and the difference between the B-spline wavelets and modulated Gaussians converged to zero, both in time and frequency domains. In [3] this was extended to more general scaling functions ϕ_n and corresponding wavelets ψ_n and it was shown that the appropriate uncertainty products converged to the optimal value as $n \to \infty$ for both ϕ_n and ψ_n. It was assumed that the symbol P_n for ϕ_n was reciprocal and had real roots. Moreover it was assumed that $P_n(z)$ had a factor $(z+1)^{m_n}$ where $m_n \geq Cn$ for a constant C. These assumptions allowed the use of similar techniques to those used for the case of B-splines. In this paper we use the different techniques in [2] to allow us to relax all of these conditions for the case of ϕ_n, and for the case of ψ_n to relax the final condition to require $P_n(z)$ to have merely a factor of $(z+1)^3$. These together with the results in [2] will provide a more complete undertsanding of the asymptotic behaviour of scaling functions that approximate the Gaussian and the asymptotic properties of the corresponding wavelets.

2 A Preliminary Result

In this section we state a result that will be needed later. Taking Fourier transforms of (1.3) gives

$$\widehat{\phi}_n(u) = P_n(e^{-iu/2})\widehat{\phi}_n(u/2), \quad u \in \mathbb{R}, \tag{2.1}$$

and it follows that

$$\widehat{\phi}_n(u) = \prod_{k=1}^{\infty} P_n(e^{-iu/2^k}), \quad u \in \mathbb{R}, \tag{2.2}$$

where the infinite product converges uniformly. Recalling (1.2) and that the roots of P_n are $-r_{n,j}$, $j = 1, \ldots, n$, gives

$$\widehat{\phi}_n(u) = \prod_{j=1}^{n} \prod_{k=1}^{\infty} \frac{e^{-iu/2^k} + r_{n,j}}{1 + r_{n,j}}, \quad u \in \mathbb{R}. \tag{2.3}$$

In order to consider the convergence of $\widehat{\phi}_n(u)$ to $e^{-u^2/2}$, we need to consider the covergence of products as in (2.3). The following result actually covers more general products which will be encountered in Section 4. It extends a corresponding result in [2] on asymptotic normality and also gives convergence of derivatives. Although its proof is of the same form as that of Theorem 1.2 of [2], we sketch it here for completeness.

Theorem 2.1 *For $\gamma \in [0, \frac{\pi}{2})$, let D_γ comprise all $z \in \mathbb{C}$ satisfying the inequality*

$$\left| \Im \frac{z}{(1+z)^2} \right| \leq \tan\gamma \, \Re \frac{z}{(1+z)^2} \; .$$

Suppose that for $n = 1, 2, \ldots,$ f_n is defined by

$$f_n(u) = \prod_{j=1}^{r_n} \frac{e^{-iu} + \mu_{n,j}}{1 + \mu_{n,j}} \prod_{j=1}^{s_n} \prod_{k=1}^{\infty} \frac{e^{-iu/2^k} + \lambda_{n,j}}{1 + \lambda_{n,j}}, \quad u \in \mathbb{R}, \tag{2.4}$$

where $\mu_{n,j}$, $j = 1, \ldots, r_n$, and $\lambda_{n,j}$, $j = 1, \ldots, s_n$, $n = 1, 2, \ldots,$ lie in D_γ for some $\gamma \in [0, \frac{\pi}{2})$ and are bounded away from -1.

Suppose that for $\mu_n \in \mathbb{R}$ and $\sigma_n > 0$, with $\sigma_n \to \infty$ as $n \to \infty$,

$$\widetilde{f}_n(u) := e^{iu\mu_n/\sigma_n} f_n(u/\sigma_n) \tag{2.5}$$

satisfies $\widetilde{f}_n'(0) = 0$ and $\widetilde{f}_n''(0) = -1$. Then as $n \to \infty$,

$$\widetilde{f}_n(u) \to e^{-u^2/2} \qquad \widetilde{f}_n'(u) \to -ue^{-u^2/2},$$

locally uniformly on $\mathbb{R}$.

Proof. A simple calculation shows that

$$\mu_n = \sum_{j=1}^{r_n} \frac{1}{1+\mu_{n,j}} + \sum_{j=1}^{s_n} \frac{1}{1+\lambda_{n,j}}, \tag{2.6}$$

$$\sigma_n^2 = \sum_{j=1}^{r_n} \frac{\mu_{n,j}}{(1+\mu_{n,j})^2} + \frac{1}{3} \sum_{j=1}^{s_n} \frac{\lambda_{n,j}}{(1+\lambda_{n,j})^2}. \tag{2.7}$$

Thus the condition $\lim_{n\to\infty} \sigma_n = \infty$ implies $\lim_{n\to\infty}(r_n + s_n) = \infty$.

As in the proof of Theorem 1.2 of [2] we write

$$\log \widetilde{f}_n(u) = \frac{iu\mu_n}{\sigma_n} + \sum_{j=1}^{r_n} F(\mu_{n,j}, \frac{-iu}{\sigma_n}) + \sum_{j=1}^{s_n}\sum_{k=1}^{\infty} F(\lambda_{n,j}, \frac{-iu}{2^k\sigma_n}),$$

where

$$F(\mu, t) = \log\left(\frac{e^t+\mu}{1+\mu}\right),$$

and expanding $F(\mu, t)$ in a Taylor series about $t = 0$ gives

$$\log \widetilde{f}_n(u) = -\frac{u^2}{2} + \sum_{\nu=3}^{\infty} \sigma_n^{-\nu}(-iu)^\nu \left\{ \sum_{j=1}^{r_n} a_\nu(\mu_{n,j}) + \sum_{j=1}^{s_n} \frac{a_\nu(\lambda_{n,j})}{2^\nu - 1} \right\}, \tag{2.8}$$

where, under the conditions on $\mu_{n,j}$, $\lambda_{n,j}$, there is a constant $A > 0$ with

$$\begin{aligned} |a_\nu(\mu_{n,j})| &\le \frac{A^{\nu-2}}{\nu} \sec\gamma\, \Re\, \frac{\mu_{n,j}}{(1+\mu_{n,j})^2}, \\ |a_\nu(\lambda_{n,j})| &\le \frac{A^{\nu-2}}{\nu} \sec\gamma\, \Re\, \frac{\lambda_{n,j}}{(1+\lambda_{n,j})^2}. \end{aligned}$$

Thus, from (2.8), (2.6) and (2.7),

$$\begin{aligned} \left|\log \widetilde{f}_n(u) + \frac{u^2}{2}\right| &\le \sec\gamma \sum_{\nu=3}^{\infty} \frac{\sigma_n^{-\nu}|u|^\nu}{\nu} A^{\nu-2} \\ &\quad \times \left\{ \sum_{j=1}^{r_n} \Re\, \frac{\mu_{n,j}}{(1+\mu_{n,j})^2} + \frac{1}{2^\nu - 1} \sum_{j=1}^{s_n} \Re\, \frac{\lambda_{n,j}}{(1+\lambda_{n,j})^2} \right\} \\ &\le \sec\gamma \sum_{\nu=3}^{\infty} \frac{|u|^\nu}{\nu} \left(\frac{A}{\sigma_n}\right)^{\nu-2} \\ &\le \sec\gamma\, \frac{A|u|^3}{\sigma_n} \left(1 - \frac{A|u|}{\sigma_n}\right)^{-1} \end{aligned}$$

whenever $A|u| < \sigma_n$. Since $\sigma_n \to \infty$ as $n \to \infty$, $\lim_{n\to\infty} \widetilde{f}_n(u) = e^{-u^2/2}$ locally uniformly on $\mathbb{R}$.

Differentiating (2.8) gives

$$\frac{\widetilde{f}_n'(u)}{\widetilde{f}_n(u)} = -u - i\sum_{\nu=3}^{\infty} \sigma_n^{-\nu}(-iu)^{\nu-1}\left\{\sum_{j=1}^{r_n} a_\nu(\mu_{n,j}) + \sum_{j=1}^{s_n} \frac{a_\nu(\lambda_{n,j})}{2^\nu - 1}\right\}$$

and, as before,

$$\begin{aligned}\left|\frac{\widetilde{f}_n'(u)}{\widetilde{f}_n(u)} + u\right| &\le \sec\gamma \sum_{\nu=3}^{\infty} \sigma_n^{-\nu}|u|^{\nu-1}A^{\nu-2} \\ &\quad \times \left\{\sum_{j=1}^{r_n} \Re \frac{\mu_{n,j}}{(1+\mu_{n,j})^2} + \frac{1}{2^\nu - 1}\sum_{j=1}^{s_n} \Re \frac{\lambda_{n,j}}{(1+\lambda_{n,j})^2}\right\} \\ &\le \sec\gamma \sum_{\nu=3}^{\infty} |u|^{\nu-1}\left(\frac{A}{\sigma_n}\right)^{\nu-2} \\ &= \sec\gamma\, \frac{A|u|^2}{\sigma_n}\left(1 - \frac{A|u|}{\sigma_n}\right)^{-1}\end{aligned}$$

whenever $A|u| < \sigma_n$. Thus $\lim_{n\to\infty} \widetilde{f}_n'(u) = -ue^{-u^2/2}$ locally uniformly in $\mathbb{R}$. □

As noted in [2], the set D_γ contains the sector $|\arg z| \le \gamma$ and for $z = \pm re^{i\theta}$, $r > 0$, $\gamma \le \theta \le \pi$, z lies in D_γ if and only if

$$\frac{\sin(\frac{\theta-\gamma}{2})}{\sin(\frac{\theta+\gamma}{2})} \le r \le \frac{\sin(\frac{\theta+\gamma}{2})}{\sin(\frac{\theta-\gamma}{2})}.$$

In particular, D_γ contains the unit circle $r = 1$.

3 Asymptotic Optimality of Refinable Functions

We suppose that (ϕ_n) is a sequence of refinable functions as in Section 1.

Theorem 3.1 *If* $P_n'(-1) = 0$, *then*

$$\lim_{n\to\infty} \Delta_{\phi_n}\Delta_{\widehat{\phi}_n} = \frac{1}{2}. \tag{3.1}$$

PROOF. We recall that $\lim_{n\to\infty}\widetilde{\phi}_n = G$ in $L^p(\mathbb{R})$, $1 \le p \le \infty$. From (1.6) we see that (3.1) is equivalent to

$$\lim_{n\to\infty} \Delta_{\widetilde{\phi}_n} \Delta_{\widehat{\widetilde{\phi}}_n} = \frac{1}{2}. \tag{3.2}$$

We also recall that $\widetilde{\phi}_n$ is defined so that

$$\int_{-\infty}^{\infty} x^2 \widetilde{\phi}_n(x)dx = \int_{-\infty}^{\infty} x^2 G(x)dx = 1. \tag{3.3}$$

Firstly we shall show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} x^2 \widetilde{\phi}_n(x)^2 dx = \int_{-\infty}^{\infty} x^2 G(x)^2 dx. \tag{3.4}$$

Take $\epsilon > 0$ and choose $A > 1$ so that

$$\int_{|x|>A} x^2 G(x)dx < \epsilon. \tag{3.5}$$

Choose N so that for all $n > N$ and $|x| \le A$,

$$|x^2 \widetilde{\phi}_n(x)^k - x^2 G(x)^k| < \frac{\epsilon}{2A}, \quad k = 1, 2. \tag{3.6}$$

Take any $n > N$. Then

$$\left| \int_{-A}^{A} x^2 \widetilde{\phi}_n(x)dx - \int_{-A}^{A} x^2 G(x)dx \right| < \epsilon$$

and so by (3.3),

$$\left| \int_{|x|>A} x^2 \widetilde{\phi}_n(x)dx - \int_{|x|>A} x^2 G(x)dx \right| < \epsilon.$$

So by (3.5),

$$\int_{|x|>A} x^2 \tilde{\phi}_n(x)dx < 2\varepsilon.$$

Thus for large enough n,

$$\int_{|x|>A} x^2 \widetilde{\phi}_n(x)^2 dx < \frac{1}{\sqrt{2\pi}} \int_{|x|>A} x^2 \widetilde{\phi}_n(x)dx < \varepsilon,$$

and so

$$\left| \int_{|x|>A} x^2 \tilde{\phi}_n(x)^2 dx - \int_{|x|>A} x^2 G(x)^2 dx \right| < 2\epsilon.$$

By (3.6), for all large enough n,

$$\left| \int_{-A}^{A} x^2 \widetilde{\phi}_n(x)^2 dx - \int_{-A}^{A} x^2 G(x)^2 dx \right| < \epsilon$$

and (3.4) follows.

Also for large enough n,

$$\left| \int_{-A}^{A} x \widetilde{\phi}_n(x)^2 dx - \int_{-A}^{A} x G(x)^2 dx \right| < \epsilon,$$

$$\int_{|x|>A} |x| \widetilde{\phi}_n(x)^2 dx < \int_{|x|>A} x^2 \widetilde{\phi}_n(x)^2 dx < \epsilon,$$

and as before,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} x \widetilde{\phi}_n(x)^2 dx = \int_{-\infty}^{\infty} x G(x)^2 dx = 0. \tag{3.7}$$

Since

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \widetilde{\phi}_n(x)^2 dx = \int_{-\infty}^{\infty} G(x)^2 dx,$$

(3.4) and (3.7) give

$$\lim_{n\to\infty} \Delta_{\tilde{\phi}_n} = \Delta_G = \frac{1}{\sqrt{2}}.$$

To complete the proof we need to show that $\lim_{n\to\infty} \Delta_{\widehat{\tilde{\phi}}_n} = \Delta_{\widehat{G}}$ and since $|\widehat{\tilde{\phi}}_n(u)|^2$ is even and $\lim_{n\to\infty} \|\widehat{\tilde{\phi}}_n\|_2 = \|\widehat{G}\|_2$, it remains to show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} u^2 |\widehat{\tilde{\phi}}_n(u)|^2 du = \int_{-\infty}^{\infty} u^2 e^{-u^2} du. \tag{3.8}$$

Let $\widehat{\Phi}_n(u) = |\widehat{\tilde{\phi}}_n(u)|^2$ and $f(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}$, so that $\widehat{f}(u) = e^{-u^2}$. Then (3.8) is equivalent to

$$\lim_{n\to\infty} \Phi_n''(0) = f''(0). \tag{3.9}$$

Recall that ϕ_n is the refinable function with symbol $P_n(z)$, which we may write as $(z+1)^2 p_n(z)$. Then Φ_n is, up to scaling, the refinable function with symbol $P_n(z)P_n(z^{-1}) = (z+1)^2(z^{-1}+1)^2 p_n(z)p_n(z^{-1})$. It follows that

$$\Phi_n''(x) = \eta_n(x+1) - 2\eta_n(x) + \eta_n(x-1),$$

where η_n is the refinable function with symbol $(z+1)(z^{-1}+1)p_n(z)p_n(z^{-1})$, up to scaling. Since this polynomial has all roots in the left half-plane, it follows from [5], and [1], that η_n satisfies the following property: for any sequence (λ_j) in $\mathbb{R}$, the number of inflections of the function $\sum_{j=-\infty}^{\infty} \lambda_j \eta_n(\cdot - j)$ is bounded by the number of inflections in the polygonal arc with vertices (j, λ_j), $j \in \mathbb{Z}$. Thus the function Φ_n'' has at most four inflections.

Now $\lim_{n\to\infty} \widehat{\widehat{\phi}}_n(u) = e^{-u^2/2}$ locally uniformly on $\mathbb{R}$ and so $\lim_{n\to\infty} (\Phi_n'')\widehat{\ }(u) = (f'')\widehat{\ }(u)$ locally uniformly on $\mathbb{R}$. Now f'' has exactly four inflections and it follows as in the proof of Lemma 5.1 of [2] that Φ_n'' converges to f'' uniformly on a neighbounded of 0. So (3.9) holds. □

4 Asymptotic Optimality of Wavelets

Throughout this section we shall assume that the roots of P_n in (1.1) are real and negative, and that P_n is reciprocal, i.e. the roots are invariant under the transformation $z \to z^{-1}$. It follows that ϕ_n is symmetric, i.e. $\phi_n(x) = \phi_n(n-x)$, $x \in \mathbb{R}$.

We define Φ_n and Q_n as in (1.10) and (1.11). It is easily seen that Φ_n is a refinable function with mask $P_n(z)P_n(z^{-1})$. It follows that Φ_n is continuous with support $[-n, n]$ and satisfies $\Phi_n(-x) = \Phi_n(x)$, $x \in \mathbb{R}$. The Laurent polynomial Q_n is called the Euler-Frobenius polynomial corresponding to ϕ_n. From the work of [5] we know that Q_n has real negative roots. As in (1.12) we define the wavelet, up to multiplication by a constant as $\psi_n(2\ \cdot\ -1)$, where ψ_n is defined by

$$\widehat{\psi}_n(u) = e^{inu} Q_n(e^{-i(u+\pi)}) P_n(e^{-i(u+\pi)}) \widehat{\phi}_n(u), \quad u \in \mathbb{R}. \tag{4.1}$$

Suppose that the roots of P_n are $-\lambda_{n,1}, \ldots, -\lambda_{n,n}$. By (1.2) we may suppose $\lambda_{n,1} = 1$. We write

$$\tilde{\lambda}_{n,j} := \frac{1}{2}(\lambda_{n,j} + \lambda_{n,j}^{-1}), \quad j = 1, \ldots, n,$$

and for simplicity drop the first subscript n when it is unambiguous to do so. Note that $\tilde{\lambda}_j \geq 1$, $j = 1, \ldots, n$, and

$$(e^{-iu} + \lambda_j)(e^{-iu} + \lambda_j^{-1}) = 2e^{-iu}(\tilde{\lambda}_j + \cos u) \ .$$

Also

$$\begin{aligned} e^{-iu} + 1 &= 2e^{-iu/2} \cos \frac{u}{2} \\ &= \sqrt{2} e^{-iu/2} (1 + \cos u)^{1/2}, \quad -\pi \leq u \leq \pi, \end{aligned}$$

and therefore

$$P_n(e^{-iu}) = e^{-inu/2} \prod_{j=1}^{n} \left(\frac{\tilde{\lambda}_j + \cos u}{\tilde{\lambda}_j + 1} \right)^{1/2}, \quad -\pi \leq u \leq \pi.$$

Similarly, if the roots of Q_n are $-\mu_1, \ldots, -\mu_{n-1}, -\mu_1^{-1}, \ldots, -\mu_{n-1}^{-1}$, and setting $\tilde{\mu}_j := \frac{1}{2}(\mu_j + \mu_j^{-1})$, $j = 1, \ldots, n-1$, then for a constant $A_n > 0$,

$$Q_n(e^{-iu}) = A_n \prod_{j=1}^{n-1} (\tilde{\mu}_j + \cos u).$$

So from (4.1) and (2.2),

$$\widehat{\psi}_n(u) = B_n e^{-in\pi/2} f_n(u), \quad u \in \mathbb{R}, \tag{4.2}$$

where $B_n > 0$, $f_n(u)$ is real,

$$\begin{aligned} |f_n(u)| &= \prod_{j=1}^{n-1} (\tilde{\mu}_j - \cos u) \prod_{j=1}^{n} (\tilde{\lambda}_j - \cos u)^{1/2} \\ &\quad \times \prod_{k=1}^{\infty} \prod_{j=1}^{n} \left(\frac{\tilde{\lambda}_j + \cos(2^{-k} u)}{\tilde{\lambda}_j + 1} \right)^{1/2}, \quad u \in \mathbb{R}, \end{aligned} \tag{4.3}$$

$f_n(u) > 0$, $0 < u < 2\pi$, $f_n(0) = f_n(2\pi) = 0$, $f_n(-u) = (-1)^n f_n(u)$.

We shall make the following mild assumption on the growth rate of $\tilde{\lambda}_{n,j}$ with n. There are constants $K > 0$ and $0 < \sigma \leq 1$ such that

$$\left| \{ j : \tilde{\lambda}_{n,j} \leq K \} \right| \geq \sigma n, \quad n = 2, 3, \ldots. \tag{4.4}$$

This can be reformulated as follows.

Lemma 4.1 *Condition (4.4) holds if and only if there is a constant* $C > 0$ *with*

$$\sum_{j=1}^{n} \frac{1}{(\tilde{\lambda}_{n,j}+1)^2} \geq Cn, \quad n = 2, 3, \ldots. \tag{4.5}$$

Proof. If (4.4) holds, then

$$\sum_{j=1}^{n} \frac{1}{(\tilde{\lambda}_{n,j}+1)^2} \geq \frac{\sigma n}{(K+1)^2}, \quad n = 2, 3, \ldots,$$

which gives (4.5).

Conversely, suppose that (4.4) does not hold. Take any $\epsilon > 0$. Then there exists n such that

$$|\{j : \tilde{\lambda}_{n,j} \leq \epsilon^{-1/2}\}| < \epsilon n.$$

If $\tilde{\lambda}_{n,j} > \epsilon^{-1/2}$, then $(\tilde{\lambda}_{n,j}+1)^{-2} < \epsilon$ and so

$$\sum_{j=1}^{n} \frac{1}{(\tilde{\lambda}_{n,j}+1)^2} < 2n\epsilon.$$

Thus there is no constant C for which (4.5) holds. □

We can now study the shape of the graph of f_n in $[0, 2\pi]$, which will lead to the asymptotic behaviour of $\widehat{\psi}_n$.

Lemma 4.2 *There is a number* $\alpha > 0$ *such that* f_n *in (4.2) has a unique local maximum in* $[0, 2\pi]$ *at* α_n *with*

$$\frac{2\pi}{3} < \alpha_n < \pi - \alpha, \quad n = 2, 3, \ldots.$$

Proof. For $n = 2, 3, \ldots$, putting $F_n(u) := \log f_n(u)$, $0 < u < 2\pi$, gives by (4.3),

$$F_n'(u) = \sum_{j=1}^{n-1} \frac{\sin u}{\tilde{\mu}_j - \cos u} + \frac{1}{2}\sum_{j=1}^{n} \frac{\sin u}{\tilde{\lambda}_j - \cos u} - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{2^{-k}\sin(2^{-k}u)}{\tilde{\lambda}_j + \cos(2^{-k}u)}. \tag{4.6}$$

For $\pi \le u < 2\pi$, all these terms are negative and so

$$F_n'(u) < 0, \quad \pi \le u < 2\pi.$$

For $0 < u \le \frac{2\pi}{3}$,

$$\begin{aligned} F_n'(u) &> \frac{1}{2}\sum_{j=1}^{n} \frac{\sin u}{\tilde{\lambda}_j - \cos u} - \frac{1}{2}\sum_{j=1}^{n} \frac{\sin(u/2)}{\tilde{\lambda}_j + \cos(u/2)} \\ &= \frac{1}{2}\sum_{j=1}^{n} \frac{\tilde{\lambda}_j(\sin u - \sin(u/2)) + \sin(3u/2)}{(\tilde{\lambda}_j - \cos u)(\tilde{\lambda}_j + \cos(u/2))} \ge 0, \end{aligned}$$

since $\sin u \ge \sin(u/2)$.

Now by condition (4.4), for $0 < u \le \pi$,

$$F_n'(u) \le \frac{3n \sin u}{2(1 - \cos u)} - \frac{1}{4}\sigma n \frac{\sin(u/2)}{K + \cos(u/2)} = nG(u),$$

say. Since $G(\pi) < 0$, we can choose $\alpha > 0$ with $G(u) < 0$ for $\pi - \alpha \le u \le \pi$. Thus for $n = 2, 3, \ldots$,

$$F_n'(u) < 0, \quad \pi - \alpha \le u \le \pi.$$

Finally we note that for $\frac{\pi}{2} \le u \le \pi$,

$$\begin{aligned} F_n''(u) &= \sum_{j=1}^{n-1} \frac{\tilde{\mu}_j \cos u - 1}{(\tilde{\mu}_j - \cos u)^2} + \frac{1}{2}\sum_{j=1}^{n} \frac{\tilde{\lambda}_j \cos u - 1}{(\tilde{\lambda}_j - \cos u)^2} \\ &\quad - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{\tilde{\lambda}_j \cos(2^{-k}u) + 1}{2^{2k}(\tilde{\lambda}_j + \cos(2^{-k}u))^2} < 0, \end{aligned} \tag{4.7}$$

since all terms are negative.

Thus we have shown that for $n = 2, 3, \ldots$, $F_n'(u) > 0$, $0 < u \le \frac{2\pi}{3}$, $F_n'(u) < 0$, $\pi - \alpha \le u < 2\pi$, and $F_n''(u) < 0$, $\frac{\pi}{2} \le u \le \pi$. Hence F_n has a unique local maximum in $(0, 2\pi)$ at α_n with $\frac{2\pi}{3} < \alpha_n < \pi - \alpha$. The result follows. □

It will be convenient to renormalise the wavelet ψ_n by defining

$$\Psi_n(x) = \frac{(-1)^{[n/2]}}{B_n f_n(\alpha_n)} \psi_n(x), \quad x \in \mathbb{R},$$

where B_n is as in (4.2), from which it follows that for even n,

$$\widehat{\Psi}_n(u) = \frac{f_n(u)}{f_n(\alpha_n)}, \tag{4.8}$$

while for odd n,

$$\widehat{\Psi}_n(u) = \frac{f_n(u)}{i f_n(\alpha_n)}.$$

We next define $\sigma_n > 0$ by

$$\sigma_n^2 = -F_n''(\alpha_n). \tag{4.9}$$

By (4.7) we see that

$$\sigma_n^2 > \sum_{j=1}^{n-1} \frac{1}{(\tilde{\mu}_j + 1)^2} + \frac{2}{3} \sum_{j=1}^{n} \frac{1}{(\tilde{\lambda}_j + 1)^2}.$$

So by condition (4.4) and Lemma 4.1,

$$\sigma_n \geq a\sqrt{n}, \quad n = 2, 3, \ldots, \tag{4.10}$$

for some $a > 0$. Also from (4.7),

$$\sigma_n < \sqrt{2n}, \quad n = 2, 3, \ldots. \tag{4.11}$$

Theorem 4.3 *If j_n denotes the number 1 or i as n is even or odd, then*

$$\begin{aligned} \lim_{n\to\infty} j_n \widehat{\Psi}_n\left(u/\sigma_n + \alpha_n\right) &= e^{-u^2/2}, \\ \lim_{n\to\infty} \frac{j_n}{\sigma_n} \widehat{\Psi}_n'\left(u/\sigma_n + \alpha_n\right) &= -ue^{-u^2/2}, \end{aligned}$$

where the convergence is locally uniform on $\mathbb{R}$.

PROOF. We consider even n, the case for odd n differing only by a factor of i. Putting $\widetilde{f}_n(u) := \widehat{\Psi}_n(u/\sigma_n + \alpha_n)$, we see from (4.8), Lemma 4.2 and (4.9) that

$$\widetilde{f}_n(0) = 1, \ \widetilde{f}_n'(0) = 0, \ \widetilde{f}_n''(0) = -1.$$

We also have from (4.1),

$$\begin{aligned}\widetilde{f}_n(u) &= e^{i(2n-1)u/\sigma_n} \prod_{j=1}^{n-1} \frac{(e^{-iu/\sigma_n} - \mu_j e^{i\alpha_n})(e^{-iu/\sigma_n} - \mu_j^{-1} e^{i\alpha_n})}{(1-\mu_j e^{i\alpha_n})(1-\mu_j^{-1} e^{i\alpha_n})} \\ &\quad \times \prod_{j=1}^{n} \frac{e^{-iu/\sigma_n} - \lambda_j e^{i\alpha_n}}{1-\lambda_j e^{i\alpha_n}} \prod_{j=1}^{n}\prod_{k=1}^{\infty} \frac{e^{iu/2^k\sigma_n} + \lambda_j e^{i\alpha_n/2^k}}{1+\lambda_j e^{i\alpha_n/2^k}},\end{aligned}$$

which is of form (2.5), (2.4) in Theorem 2.1. In order to apply Theorem 2.1 we need that for $n = 2, 3, \ldots$, $-\mu_j e^{i\alpha_n}, -\mu_j^{-1} e^{i\alpha_n}$, $j = 1, \ldots, n-1$, $-\lambda_j e^{i\alpha_n}$, $j = 1, \ldots, n$, $\lambda_j e^{i\alpha_n/2^k}$, $j = 1, \ldots, n$, $k = 1, 2, \ldots$, all lie in D_γ for some γ in $[0, \frac{\pi}{2})$ and are bounded away from -1. From Lemma 4.2, $\pi - \alpha_n < \frac{\pi}{3}$ and $\frac{\alpha_n}{2^k} < \frac{\pi}{2} - \frac{\alpha}{2}$, $k = 1, 2, \ldots$, and hence all the above numbers lie in the sector $|\arg z| \leq \gamma$ for $\gamma = \max\{\frac{\pi}{3}, \frac{\pi}{2} - \frac{\alpha}{2}\}$. Since D_γ contains this sector, the condition is satisfied.

We see from (4.10) that $\sigma_n \to \infty$ as $n \to \infty$. Then Theorem 2.1 gives the desired result. □

In Theorem 4.3 we consider convergence of $\widehat{\Psi}_n'$ because this will be needed later for convergence of the uncertainty products. In order to extend Theorem 4.3 to convergence in $L^P(\mathbb{R})$ we shall need to apply the Dominated Convergence Theorem. For this are shall need the further condition that P_n has a double root at -1, i.e. $P_n'(-1) = 0$.

Lemma 4.4 *Let*

$$\mu_\lambda(u) = \prod_{k=1}^{\infty} \frac{\lambda + \cos(2^{-k}u)}{\lambda + 1}.$$

Then for any $K > 1$, there is a constant $\rho < 1$ such that $\mu_\lambda(u) \leq \rho\mu_\lambda(\pi)$, $|\mu_\lambda'(u)| \leq \rho\mu_\lambda(\pi)$, for $u \geq \frac{3\pi}{2}$ and $1 \leq \lambda \leq K$.

Proof. Fix $\lambda \geq 1$ and let $R(u) = \frac{\lambda + \cos u}{\lambda + 1}$. For $\pi \leq u \leq 2\pi$, $R(u/2^k) \leq R(\pi/2^k)$, $k = 1, 2, \ldots$, and so

$$\frac{\mu_\lambda(u)}{\mu_\lambda(\pi)} = \prod_{k=1}^{\infty} \frac{R(2^{-k}u)}{R(2^{-k}\pi)} \leq \frac{R(u/2)}{R(\pi/2)} = \frac{\lambda + \cos(u/2)}{\lambda}.$$

Thus

$$\mu_\lambda(u) \le \mu_\lambda(\pi), \quad \pi \le u \le 2\pi, \tag{4.12}$$

and

$$\mu_\lambda(u) \le \left(1 - \frac{1}{\lambda\sqrt{2}}\right)\mu_\lambda(\pi), \quad \frac{3\pi}{2} \le u \le 2\pi.$$

For $2\pi \le u \le 3\pi$,

$$\mu_\lambda(u) = R(u/2)\mu_\lambda(u/2) \le \frac{\lambda}{\lambda+1}\mu_\lambda(\pi),$$

by (4.12). So for any $K > 1$ there is a constant $\rho < 1$ such that for $1 \le \lambda \le K$, $\frac{3\pi}{2} \le u \le 3\pi$,

$$\mu_\lambda(u) \le \rho\mu_\lambda(\pi). \tag{4.13}$$

Now for all $u \ge 3\pi$,

$$\mu_\lambda(u) = R(u/2)\mu_\lambda(u/2) \le \mu_\lambda(u/2)$$

and successive application of this gives (4.13) for all $u \ge \frac{3\pi}{2}$.

Now $\mu'_\lambda(u) = -\sum_{k=1}^{\infty} 2^{-k}\nu_k(u)$, where

$$\nu_k(u) := \prod_{\substack{j=1 \\ j\ne k}}^{\infty} \frac{\lambda + \cos(2^{-j}u)}{\lambda+1} \cdot \frac{\sin(2^{-k}u)}{\lambda+1}\,.$$

For $k \ge 1$, $\pi \le u \le 2\pi$,

$$\frac{|\nu_k(u)|}{\mu_\lambda(\pi)} = \prod_{\substack{j=1 \\ j\ne k}}^{\infty} \frac{R(2^{-j}u)}{R(2^{-j}\pi)} \cdot \frac{\sin(2^{-k}u)}{\lambda + \cos(2^{-k}u)} \;\le\; \frac{1}{\lambda} \;\le\; 1.$$

For $\frac{3\pi}{2} \le u \le 2\pi$,

$$\frac{|\nu_1(u)|}{\mu_\lambda(\pi)} \le \frac{\sin(3\pi/4)}{\lambda} \le \frac{1}{\sqrt{2}}\,.$$

So

$$|\mu'_\lambda(u)| \le \mu_\lambda(\pi), \quad \pi \le u \le \frac{3\pi}{2},$$

$$|\mu'_\lambda(u)| \le \rho\mu_\lambda(\pi), \quad \frac{3\pi}{2} \le u \le 2\pi,$$

for a constant $\rho < 1$. Now since $\mu_\lambda(u) = R(u/2)\mu_\lambda(u/2)$,

$$\begin{aligned} |\mu'_\lambda(u)| &= \left| \frac{1}{2}R'(u/2)\mu_\lambda(u/2) + \frac{1}{2}R(u/2)\mu'_\lambda(u/2) \right| \\ &\le \frac{1}{4}\left|\mu_\lambda(u/2)\right| + \frac{1}{2}\left|\mu'_\lambda(u/2)\right|. \end{aligned}$$

So recalling (4.12),

$$|\mu'_\lambda(u)| \le \frac{3}{4}\mu_\lambda(\pi), \quad u \ge 2\pi.$$

□

Lemma 4.5 *If $P'_n(-1) = 0$, then for constants $A > 0$, $0 < \rho < 1$, $0 < \sigma \le 1$, $u \ge \frac{3\pi}{2}$, $n = 2, 3, \ldots$,*

$$\begin{aligned} |\widehat{\phi}_n(u)| &\le A\rho^{\sigma n}u^{-2}|\widehat{\phi}_n(\pi)|, \\ |\widehat{\phi}'_n(u)| &\le An\rho^{\sigma n}u^{-2}|\widehat{\phi}_n(\pi)|. \end{aligned}$$

Proof. We may write

$$P_n(e^{-iu}) = \prod_{j=1}^{(n-m)/2} \left(\frac{\tilde{\lambda}_j + \cos u}{\tilde{\lambda}_j + 1} \right) \cos^m\left(\frac{u}{2}\right),$$

for $m = 2$ or 3. We note that

$$\prod_{k=1}^{\infty} \cos\left(\frac{u}{2^{k+1}}\right) = \prod_{k=1}^{\infty} \frac{\sin(2^{-k}u)}{2\sin(2^{-k-1}u)} = \lim_{n\to\infty} \frac{\sin(u/2)}{2^n \sin(2^{-n-1}u)} = \frac{\sin(u/2)}{u/2}.$$

Recalling (2.2), Lemma 4.4 and condition (4.4) gives the result. □

Lemma 4.6 *If $P'_n(-1) = 0$, then there is a cosntant $C > 0$ such that for $u \ge \sigma_n(3\pi/2 - \alpha_n)$, $n = 2, 3, \ldots$, $\left|\widehat{\Psi}_n\left(u/\sigma_n + \alpha_n\right)\right|$ and $\left|\widehat{\Psi}'_n\left(u/\sigma_n + \alpha_n\right)\right|$ are bounded by $C(1+u)^{-2}$.*

Proof. For all u,

$$|Q_n(e^{-i(u+\pi)})P_n(e^{-i(u+\pi)})| \leq Q_n(1)P_n(1),$$

and so by Lemma 4.5, on recalling (4.1), for $u \geq \frac{3\pi}{2}$,

$$|\widehat{\psi}_n(u)| \leq A\rho^{\sigma n}u^{-2}|\widehat{\psi}_n(\pi)|.$$

Since $|\widehat{\Psi}_n(\pi)| \leq |\widehat{\Psi}_n(\alpha_n)| = 1$, for $u \geq \sigma_n(\frac{3\pi}{2} - \alpha_n)$,

$$\begin{aligned} \left|\widehat{\Psi}_n\left(u/\sigma_n + \alpha_n\right)\right| &\leq A\rho^{\sigma n}\left(u/\sigma_n + \alpha_n\right)^{-2} \\ &\leq C(1+u)^{-2}, \end{aligned}$$

for some $C > 0$, by (4.11).

Similarly we see that for a constant $B > 0$,

$$\left|\widehat{\Psi}'_n\left(u/\sigma_n + \alpha_n\right)\right| \leq Bn\rho^{\sigma n}\left(u/\sigma_n + \alpha_n\right)^{-2}$$

and the result follows. □

Remark

The above method of proof shows that if P_n has a root at $z = -1$ of multiplicity m, then Lemma 4.6 holds with $(1+u)^{-2}$ replaced by $(1+u)^{-m}$.

Lemma 4.7 *There are constants $a,\ b > 0$ such that for $-\sigma_n\alpha_n \leq u \leq \sigma_n(\frac{3\pi}{2} - \alpha_n)$, $n = 2, 3, \ldots$, $\left|\widehat{\Psi}_n\left(u/\sigma_n + \alpha_n\right)\right|$ and $\left|\frac{1}{\sigma_n}\Psi'_n\left(u/\sigma_n + \alpha_n\right)\right| \leq ae^{-bu^2}$.*

Proof. As in the proof of Lemma 4.2 we put $F_n(u) = \log f_n(u)$, $0 < u < 2\pi$. Recall from (4.7) that for $\frac{\pi}{2} \leq u \leq \pi$,

$$\begin{aligned} |F''_n(u)| &= \sum_{j=1}^{n-1} \frac{1 - \tilde{\mu}_j \cos u}{(\tilde{\mu}_j - \cos u)^2} + \frac{1}{2}\sum_{j=1}^{n} \frac{1 - \tilde{\lambda}_j \cos u}{(\tilde{\lambda}_j - \cos u)^2} \\ &\quad + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{1 + \tilde{\lambda}_j \cos(2^{-k}u)}{2^{2k}(\tilde{\lambda}_j + \cos(2^{-k}u))^2}. \qquad (4.14) \end{aligned}$$

We also recall condition (4.4) and note that for $\mu > K$, $\frac{\pi}{2} \le u < \pi$,

$$\frac{1-\mu\cos u}{(\mu-\cos u)^2} < \frac{1-K\cos u}{(K-\cos u)^2} \le \frac{1}{\sigma n}\sum_{\substack{j=1\\ \tilde\lambda_j\le K}}^{n} \frac{1-\tilde\lambda_j\cos u}{(\tilde\lambda_j-\cos u)^2}.$$

It $G_n(u)$ denotes the right-hand side of (4.14) with the summations taken only for $\tilde\mu_j \le K$, $\tilde\lambda_j \le K$, then for $\frac{\pi}{2} \le u \le \pi$,

$$|F_n''(u)| \le \left(1+\frac{2}{\sigma}\right) G_n(u).$$

Since $|G_n(u)| \le |F_n''(u)|$, $\frac{\pi}{2} \le u \le \pi$, there is a constant $C > 0$ with

$$\frac{|F_n''(u)|}{|F_n''(v)|} \le \left(1+\frac{2}{\sigma}\right)\frac{G_n(u)}{G_n(v)} \le C, \quad \frac{\pi}{2} \le u,\ v \le \pi,$$

since

$$(K+1)^{-2} \le \frac{1+\lambda x}{(\lambda+x)^2} \le K+1, \quad 0 \le x \le 1,\ 1 \le \lambda \le K.$$

As $F_n'(\alpha_n) = 0$,

$$F_n(u) \le F_n(\alpha_n) + \frac{1}{2}C^{-1}(u-\alpha_n)^2 F_n''(\alpha_n), \quad \frac{\pi}{2} \le u \le \pi.$$

Recalling that $\sigma_n^2 = -F_n''(\alpha_n)$ we have

$$f_n(u) \le f_n(\alpha_n)\exp\left(-\frac{1}{2}C^{-1}(u-\alpha_n)^2\sigma_n^2\right), \quad \frac{\pi}{2} \le u \le \pi,$$

and hence

$$\left|\widehat\Psi_n\left(u/\sigma_n+\alpha_n\right)\right| \le e^{-\frac{1}{2}C^{-1}u^2}, \quad \sigma_n\left(\pi/2-\alpha_n\right) \le u \le \sigma_n(\pi-\alpha_n). \tag{4.15}$$

Also for $\frac{\pi}{2} \le u \le \pi$,

$$|F_n'(u)| \le C|u-\alpha_n||F_n''(\alpha_n)|$$

and so

$$|f_n'(u)| = |f_n(u)||F_n'(u)| \le C\sigma_n^2|u-\alpha_n||f_n(u)|,$$

which gives

$$\left|\widehat{\Psi}'_n\,(u/\sigma_n+\alpha_n)\right| \le C|u|\left|\widehat{\Psi}_n\,(u/\sigma_n+\alpha_n)\right|, \quad \sigma_n\left(\frac{\pi}{2}-\alpha_n\right) \le u \le \sigma_n(\pi-\alpha_n). \tag{4.16}$$

By Lemma 4.2 we may choose α, $0<\alpha<\frac{\pi}{6}$, with $\pi-\alpha_n>\alpha$, $n=2,3,\ldots$. Let $E(u)=\exp(-\alpha^2u^2/8C\pi^2)$. For $\pi\le u\le\frac{3\pi}{2}$, $f_n(u)\le f_n(\pi)$ and so for $\sigma_n(\pi-\alpha_n)\le u\le\sigma_n(\frac{3\pi}{2}-\alpha_n)$, (4.15) gives

$$\begin{aligned}\widehat{\Psi}_n\,(u/\sigma_n+\alpha_n) &\le \exp\left(-\frac{1}{2}C^{-1}(\pi-\alpha_n)^2\sigma_n^2\right)\\ &= E(2\pi\sigma_n\alpha^{-1}(\pi-\alpha_n))\\ &\le E(2\pi\sigma_n)\le E(u),\end{aligned} \tag{4.17}$$

since $\pi-\alpha_n\ge\alpha$ and $u\le\sigma_n(\frac{3\pi}{2}-\alpha_n)<2\pi\sigma_n$.

Now from (4.6) and (4.10), there is a constant $A>0$ such that for $\pi\le u\le\frac{3\pi}{2}$,

$$|F'_n(u)|\le a^2A_n\le A\sigma_n^2.$$

Thus

$$|f'_n(u)|\le A\sigma_n^2f_n(u), \quad \pi\le u\le\frac{3\pi}{2}$$

and so for $\sigma_n(\pi-\alpha_n)\le u\le\sigma_n(\frac{3\pi}{2}-\alpha_n)$,

$$\begin{aligned}\left|\widehat{\Psi}'_n\,(u/\sigma_n+\alpha_n)\right| &\le A\sigma_n^2\left|\widehat{\Psi}_n\,(u/\sigma_n+\alpha_n)\right|\\ &\le A\alpha^{-2}u^2\left|\widehat{\Psi}_n\,(u/\sigma_n+\alpha_n)\right|,\end{aligned} \tag{4.18}$$

since $u\ge\sigma_n(\pi-\alpha_n)>\sigma_n\alpha$.

For $0\le u\le\frac{\pi}{2}$, $f_n(u)\le f_n(\frac{\pi}{2})$ and so for $-\sigma_n\alpha_n\le u\le\sigma_n(\frac{\pi}{2}-\alpha_n)$, (4.15) gives

$$\begin{aligned}\left|\widehat{\Psi}_n\,(u/\sigma_n+\alpha_n)\right| &\le \exp\left(-\frac{1}{2}C^{-1}\left(\frac{\pi}{2}-\alpha_n\right)^2\sigma_n^2\right)\\ &= E\left(2\pi\sigma_n\alpha^{-1}\left(\frac{\pi}{2}-\alpha_n\right)\right)\\ &\le E(2\pi\sigma_n)\le E(u),\end{aligned} \tag{4.19}$$

since $\alpha_n-\frac{\pi}{2}\ge\frac{\pi}{6}\ge\alpha$ and $|u|\le\sigma_n\alpha_n<2\pi\sigma_n$.

Now we see from (4.6) that for $0 \leq u \leq \frac{\pi}{2}$,

$$|f_n'(u)| \leq \left\{ \sum_{j=1}^{n-1} \frac{\sin u}{\tilde{\mu}_j - \cos u} + \frac{1}{2} \sum_{j=1}^{n} \frac{\sin u}{\tilde{\lambda}_j - \cos u} \right\} f_n(u).$$

Noting that

$$\frac{\sin u}{\lambda} \prod_{k=1}^{\infty} \frac{\lambda + \cos(2^{-k}u)}{\lambda + 1} \left(\prod_{k=1}^{\infty} \frac{\lambda + \cos(2^{-k-1}\pi)}{\lambda + 1} \right)^{-1}$$

is bounded for $\lambda \geq 1$, $0 \leq u \leq \frac{\pi}{2}$, and recalling from Lemma 4.2 that f_n is increasing in $[0, \frac{\pi}{2}]$, we see that for a constant K,

$$|f_n'(u)| \leq K n f_n(\pi/2), \quad 0 \leq u \leq \pi/2.$$

So by (4.10) there is a constant B such that for $-\sigma_n \alpha_n \leq u \leq \sigma_n(\frac{\pi}{2} - \alpha_n)$,

$$\left| \widehat{\Psi}_n' \left(u/\sigma_n + \alpha_n \right) \right| \leq B \sigma_n^2 \widehat{\Psi}_n \left(\pi/2\sigma_n + \alpha_n \right) \leq B \frac{36}{\pi^2} u^2 E(u), \quad (4.20)$$

as in (4.19) and since $|u| \geq \sigma_n(\alpha_n - \frac{\pi}{2}) \geq \sigma_n \pi/6$.

The result now follows from (4.15)–(4.20). □

Lemmas 4.6 and 4.7 allow us to apply the Dominated Convergence Theorem to prove the following

Theorem 4.8 *Suppose $P_n'(-1) = 0$. If j_n denotes the number 1 or i as n is even or odd, then as $n \to \infty$,*

$$j_n \widehat{\Psi}_n \left(u/\sigma_n \right) - e^{-\frac{1}{2}(u - \sigma_n \alpha_n)^2} \to 0,$$

$$\frac{j_n}{\sigma_n} \widehat{\Psi}_n'(u/\sigma_n) + (u - \sigma_n \alpha_n) e^{-\frac{1}{2}(u - \sigma_n \alpha_n)^2} \to 0,$$

in $L^p(0, \infty)$, $1 \leq p \leq \infty$.

Proof. We need consider only even n. Let

$$h_n(u) = \begin{cases} \widehat{\Psi}_n \left(u/\sigma_n + \alpha_n \right), & u \geq -\sigma_n \alpha_n, \\ 0, & \text{otherwise.} \end{cases}$$

From Theorem 4.3, $\lim_{n\to\infty} h_n(u) = e^{-u^2/2}$ and $\lim_{n\to\infty} h'_n(u) = -ue^{-u^2/2}$ locally uniformly. By Lemmas 4.5 and 4.7, h_n and h'_n are dominated by

$$h(u) = \max\{C(1+|u|)^{-2}, ae^{-bu^2}\}, \quad u \in \mathbb{R}.$$

So by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} h_n(u) = e^{-u^2/2}, \quad \lim_{n\to\infty} h'_n(u) = -ue^{-u^2/2}, \tag{4.21}$$

in $L^p(\mathbb{R})$, $1 \le p < \infty$. Also since h, $e^{-u^2/2}$ and $ue^{-u^2/2}$ all tend to 0 as $u \to \pm\infty$, it follows that the convergence in (4.21) is also uniform. The result follows. □

Since $\widehat{\Psi}_n(-u) = (-1)^n \widehat{\Psi}_n(u)$, $u \in \mathbb{R}$, we have a corresponding result on $(-\infty, 0)$. Adding these together and taking inverse Fourier transforms gives the following

Theorem 4.9 *Suppose $P'_n(-1) = 0$. As $n \to \infty$, for even n,*

$$\sigma_n \Psi_n(\sigma_n x) - 2\cos(\sigma_n \alpha_n x) G(x) \to 0,$$

and for odd n,

$$\sigma_n \Psi_n(\sigma_n x) - 2\sin(\sigma_n \alpha_n x) G(x) \to 0,$$

where the convergence is in $L^q(\mathbb{R})$, $2 \le q \le \infty$.

Finally we show that the wavelets Ψ_n have asymptotically optimal time-frequency localisation, as deseribed in Section 1 (see (1.13), (1.14)).

Theorem 4.10 *If $P'_n(-1) = 0$, then*

$$\lim_{n\to\infty} \Delta_{\Psi_n} \Delta^{+}_{\widehat{\Psi}_n} = \frac{1}{2}.$$

PROOF. Since $\Psi_n(-x) = (-1)^n \Psi_n(x)$, $x \in \mathbb{R}$, we see that μ_Ψ, as in (1.7), is 0 and so from (1.8),

$$\begin{aligned}\Delta^2_{\Psi_n} &= \frac{\int_{-\infty}^{\infty} x^2 |\Psi_n(x)|^2 dx}{\int_{-\infty}^{\infty} |\Psi_n(x)|^2 dx} \\ &= \frac{\int_{-\infty}^{\infty} |\widehat{\Psi}_n'(u)|^2 du}{\int_{-\infty}^{\infty} |\widehat{\Psi}_n(u)|^2 du} \\ &= \frac{\int_0^{\infty} |\widehat{\Psi}_n'(u)|^2 du}{\int_0^{\infty} |\widehat{\Psi}_n(u)|^2 du} .\end{aligned}$$

By Theorem 4.8, as $n \to \infty$,

$$\int_0^\infty \left|\widehat{\Psi}_n\left(u/\sigma_n\right)\right|^2 du - \int_0^\infty e^{-(u-\sigma_n\alpha_n)^2} du \to 0$$

and so

$$\sigma_n \int_0^\infty |\widehat{\Psi}_n(u)|^2 du \to \int_{-\infty}^\infty e^{-u^2} du, \tag{4.22}$$

since $\sigma_n \to \infty$. Also by Theorem 4.8, as $n \to \infty$,

$$\frac{1}{\sigma_n^2}\int_0^\infty \left|\widehat{\Psi}_n'\left(u/\sigma_n\right)\right|^2 du - \int_0^\infty (u-\sigma_n\alpha_n)^2 e^{-(u-\sigma_n\alpha_n)^2} du \to 0$$

and so

$$\frac{1}{\sigma_n}\int_0^\infty |\hat{\Psi}_n'(u)|^2 du \to \int_{-\infty}^\infty u^2 e^{-u^2} du.$$

Thus

$$\lim_{n\to\infty} \sigma_n^{-2}\Delta^2_{\Psi_n} = \frac{\int_{-\infty}^\infty u^2 e^{-u^2} du}{\int_{-\infty}^\infty e^{-u^2} du} = \frac{1}{2}. \tag{4.23}$$

By Theorem 4.3 and Lemmas 4.6 and 4.7, for $j = 1, 2$,

$$(u-\sigma_n\alpha_n)^j \left|\widehat{\Psi}_n\left(u/\sigma_n\right)\right|^2 - (u-\sigma_n\alpha_n)^j e^{-(u-\sigma_n\alpha_n)^2} \to 0 \text{ as } n \to \infty$$

in $L^1(0,\infty)$. Thus as $n \to \infty$,

$$\int_0^\infty (u-\sigma_n\alpha_n)\left|\widehat{\Psi}_n\left(u/\sigma_n\right)\right|^2 du \to 0, \tag{4.24}$$

$$\int_0^\infty (u-\sigma_n\alpha_n)^2 \left|\widehat{\Psi}_n\,(u/\sigma_n)\right|^2 du \to \int_{-\infty}^\infty u^2 e^{-u^2} du. \qquad (4.25)$$

Now

$$\begin{aligned}\int_0^\infty u\left|\widehat{\Psi}_n(u)\right|^2 du &= \frac{1}{\sigma_n^2}\int_0^\infty u\left|\widehat{\Psi}_n\,(u/\sigma_n)\right|^2 du \\ &= \frac{1}{\sigma_n^2}\int_0^\infty (u-\sigma_n\alpha_n)\left|\widehat{\Psi}_n\,(u/\sigma_n)\right|^2 du + \frac{\alpha_n}{\sigma_n}\int_0^\infty \left|\widehat{\Psi}_n\,(u/\sigma_n)\right|^2 du.\end{aligned}$$

Thus from (1.13),

$$\mu^+_{\widehat{\Psi}_n} = \frac{\int_0^\infty (u-\sigma_n\alpha_n)|\widehat{\Psi}_n(u/\sigma_n)|^2 du}{\sigma_n\int_0^\infty |\widehat{\Psi}_n(u/\sigma_n)|^2 du} + \alpha_n$$

and hence from (4.22) and (4.24),

$$\lim_{n\to\infty} \sigma_n(\mu^+_{\widehat{\Psi}_n} - \alpha_n) = 0. \qquad (4.26)$$

By (1.14),

$$\begin{aligned}\sigma_n^2\left(\Delta^+_{\widehat{\Psi}_n}\right)^2 &= \frac{\sigma_n^2\int_0^\infty (u-\mu^+_{\widehat{\Psi}_n})^2|\widehat{\Psi}_n(u)|^2 du}{\int_0^\infty |\widehat{\Psi}_n(u)|^2 du} \\ &= \frac{\int_0^\infty (u-\sigma_n\mu^+_{\widehat{\Psi}_n})^2|\widehat{\Psi}_n(u/\sigma_n)|^2 du}{\int_0^\infty |\widehat{\Psi}_n(u/\sigma_n)|^2 du} \\ &\to \frac{\int_{-\infty}^\infty u^2 e^{-u^2} du}{\int_{-\infty}^\infty e^{-u^2} du} = \frac{1}{2}, \qquad (4.27)\end{aligned}$$

as $n \to \infty$, by (4.26), (4.25) and (4.22). The result then follows from (4.23) and (4.27). □

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