Asymptotic Optimality in Time-Frequency Localization of Scaling Functions and Wavelets

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Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

For a triangular array of numbers $a_{n,j}$, $n = 2, 3, \ldots$, $j = 0, 1, \ldots, n$, the refinement equation with mask $P_n(z) = \sum_{j=0}^{n} a_{n,j} z^j$, has for each $n$, a unique solution $\phi_n$, which when suitably normalized, converges to the Gaussian if $P_n(z)$ has all roots in a sector inside the left half-plane and satisfying $P_n(1) = 1, P_n(-1) = 0$. The Gaussian, being the extremal function that attains the optimal constant in the uncertainty product, passes this characteristic to $\phi_n$, whose uncertainty product converges to the optimal constant. The object is to analyze this phenomena for $\phi_n$ as well as for the corresponding wavelets.

1 Introduction

For $n \geq 2$, let

$$P_n(z) = \sum_{j=0}^{n} a_{n,j} z^j,$$

be a polynomial with all its roots in the left half-plane $\{ z : \Re z \leq 0 \}$ and satisfying

$$P_n(1) = 1, \quad P_n(-1) = 0.$$
It is known that the refinement equation

\[ \phi_n(x) = \sum_{j=0}^{n} 2a_{n,j} \phi_n(2x-j), \quad x \in \mathbb{R}, \]  

(1.3)

has a unique solution satisfying \( \int_{-\infty}^{\infty} \phi_n = 1 \). Moreover it is shown in [5] that \( \phi_n \) is continuous, non-negative and has support in \([0, n]\).

In the special case \( P_n(z) = 2^{-n}(z + 1)^n \), \( \phi_n \) is the uniform B-spline of degree \( n - 1 \) with knots at \( 0, 1, \ldots, n \). We shall refer to \( \phi_n \) as a refirable function with symbol \( P_n \).

Now suppose that the roots of \( P_n \) are \( -r_{n,j} \), \( j = 1, \ldots, n \). It is shown in [2] that if we assume the stronger condition that for some \( \beta \) in \([0, \frac{\pi}{2}]\),

\[ |\arg r_{n,j}| \leq \beta, \quad \text{for } j = 1, \ldots, n, \quad n = 2, 3, \ldots, \]

and furthermore

\[ \sigma_n^2 := \frac{1}{3} \sum_{j=1}^{n} \frac{r_{n,j}}{(1 + r_{n,j})^2} \rightarrow \infty \quad \text{as } n \rightarrow \infty, \]  

(1.4)

then a suitable shift and scaling of \( \phi_n \) converges to the Gaussian function \( G(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \). To be precise, let

\[ \mu_n := \sum_{j=1}^{n} \frac{1}{1 + r_{n,j}} \]  

(1.5)

and

\[ \bar{\phi}_n(x) := \sigma_n \phi_n(\sigma_n x + \mu_n), \quad x \in \mathbb{R}. \]  

(1.6)

Then \( \bar{\phi}_n \) has mean zero and standard deviation 1 and \( \lim_{n \to \infty} \bar{\phi}_n = G \) in \( L^p(\mathbb{R}) \), \( 1 \leq p \leq \infty \). Moreover we have convergence in the frequency domain, i.e. \( \lim_{n \to \infty} \hat{\phi}_n(u) = \hat{G}(u) = e^{-u^2/2} \), where the convergence is uniform on \( \mathbb{R} \).

The above properties are useful for many applications, e.g. signal processing, because the Gaussian function gives optimal time-frequency localisation. Thus we may approximate the Gaussian function by a refifiable function \( \phi_n \) with compact support which has fast
algorithms for practical implementation. A natural choice is the uniform B-spline of degree \( n - 1 \), but it was shown in [2] that other choices of \( \phi_n \) give, in a sense, faster rates of convergence to the Gaussian.

To make precise the optimal time-frequency localisation of the Gaussian, we recall **Heisenberg’s Uncertainty Principle.** For an \( L^2 \) function \( \phi \) for which \( \int_{-\infty}^{\infty} x^j |\phi(x)|^2 dx \) exists, \( j = 1, 2 \), we write

\[
\mu_\phi := \frac{\int_{-\infty}^{\infty} x |\phi(x)|^2 dx}{\|\phi\|_2^2}
\]

and

\[
\Delta_\phi := \left\{ \frac{\int_{-\infty}^{\infty} (x - \mu_\phi)^2 |\phi(x)|^2 dx}{\|\phi\|_2^2} \right\}^{\frac{1}{2}}
\]

Similarly if \( \int_{-\infty}^{\infty} u^j |\hat{\phi}(u)|^2 du \) exists, \( j = 1, 2 \), we may define \( \mu_{\hat{\phi}} \) and \( \Delta_{\hat{\phi}} \). Thus \( \Delta_\phi \) is the standard deviation of the density function \( |\phi|^2/\|\phi\|_2^2 \) and gives a measure of the localisation of \( \phi \) in the time domain. Similarly \( \Delta_{\hat{\phi}} \) is the standard deviation of \( |\hat{\phi}|^2/\|\hat{\phi}\|_2^2 \) and measures the localisation of \( \phi \) in the frequency domain. The **uncertainty product** \( \Delta_\phi \Delta_{\hat{\phi}} \) gives an overall measure of the time-frequency localisation of \( \phi \). Clearly for any constants \( c, \mu \in \mathbb{R} \), and \( \sigma > 0 \), the function \( c\phi(\sigma \cdot -\mu) \) has the same uncertainty product as \( \phi \). **Heisenberg’s Uncertainty Principle** states that for any \( \phi \) as above,

\[
\Delta_\phi \Delta_{\hat{\phi}} \geq \frac{1}{2},
\]

and equality holds if and only if \( \phi = cG(\sigma \cdot -\mu) \) for some \( c, \mu \in \mathbb{R} \), and \( \sigma > 0 \), (see [4] for a general discussion). So for the sequence \( (\phi_n) \) of refinable functions, as above, to approach optimal time-frequency localisation, we would desire

\[
\lim_{n \to \infty} \Delta_{\phi_n} \Delta_{\hat{\phi_n}} = \frac{1}{2}.
\]

This is proved in Section 3 under the further assumption \( P'_n(-1) = 0 \).

In applications such as signal processing it is also important to have good time-frequency localisation for the wavelet corresponding
to the refinable function $\phi_n$, as above, which is defined as follows. Let
\[
\Phi_n(x) := \int_{-\infty}^{\infty} \phi_n(x+y)\phi_n(y)dy, \quad x \in \mathbb{R}, \quad (1.10)
\]
\[
Q_n(z) := \sum_{j=-n+1}^{n-1} \Phi_n(j)z^j, \quad z \in \mathbb{C} \setminus \{0\}. \quad (1.11)
\]
Then the wavelet corresponding to $\phi_n$ is defined, up to multiplication by a constant, as $\psi_n(2 \cdot -1)$, where $\psi_n$ is defined by
\[
\hat{\psi}_n(u) = e^{inu}Q_n(e^{-i(u+\pi)})P_n(e^{-i(u+\pi)})\hat{\phi}_n(u), \quad u \in \mathbb{R}. \quad (1.12)
\]
We note that since $P_n(-1) = 0$, we have $\hat{\psi}_n(0) = 0$. For a function $\psi$ with $\hat{\psi}(0) = 0$ (associated with a bandpass filter), the definition of the uncertainty product is modified to reflect the fact that $\hat{\psi}$ treats positive and negative frequency bands separately. Let
\[
\mu_{\psi}^\pm := \int_{0}^{\infty} u \left| \hat{\psi}(u) \right|^2 du / \int_{0}^{\infty} \left| \hat{\psi}(u) \right|^2 du,
\]
\[
\Delta_{\psi}^\pm := \left\{ \int_{0}^{\infty} (u - \mu_{\psi}^\pm)^2 \left| \hat{\psi}(u) \right|^2 du / \int_{0}^{\infty} \left| \hat{\psi}(u) \right|^2 du \right\}^{\frac{1}{2}}, \quad (1.14)
\]
where we assume these are well-defined. (Note that for a real-valued function $\psi$, $|\hat{\psi}|$ is even and so $\int_{0}^{\infty} \left| \hat{\psi}(u) \right|^2 du = \frac{1}{2}||\hat{\psi}||_2^2$ and the definition of $\Delta_{\psi}^\pm$ is unaltered by replacing $\int_{0}^{\infty}$ by $\int_{-\infty}^{0}$ in (1.13) and (1.14).) Then a measure of the time-frequency localization of $\psi$ is given by $\Delta_{\psi} \Delta_{\psi}^\pm$.

It is shown in ([3], Theorem 1) that if $\psi \in L^2 \cap L^1$ is a real-valued symmetric or anti-symmetric function that satisfies $t\psi(t) \in L^2$, $\psi' \in L^2$ and $\hat{\psi}(0) = 0$, then
\[
\Delta_{\psi} \Delta_{\psi}^\pm > \frac{1}{2},
\]
and the lower bound cannot be improved or attained.

A sequence of functions which are asymptotically optimal are the modulated Gaussians
\[
G_n(x) := \sin(\lambda_n x)G(x), \quad n = 1, 2, \ldots,
\]
where \( \lambda_n \to \infty \) as \( n \to \infty \). To see this we note that
\[
2i \hat{G}_n(u) = e^{-(u-\lambda_n)^2/2} - e^{-(u+\lambda_n)^2/2}.
\]
Thus \( \mu^+_n \approx \lambda_n \to 0 \) as \( n \to \infty \) and so
\[
\lim_{n \to \infty} \left( \Delta^+_n \right)^2 = \lim_{n \to \infty} \frac{\int_0^\infty (u-\lambda_n)^2 e^{-(u-\lambda_n)^2} du}{\int_0^\infty e^{-(u-\lambda_n)^2} du} = \frac{1}{2}.
\]
Also
\[
\Delta^2_n = \frac{\int_0^\infty |\hat{G}_n(u)|^2 du}{\int_0^\infty |\hat{G}_n(u)|^2 du}
\]
and so
\[
\lim_{n \to \infty} \Delta^2_n = \lim_{n \to \infty} \frac{\int_0^\infty (u-\lambda_n)^2 e^{-(u+\lambda_n)^2} du}{\int_0^\infty e^{-(u-\lambda_n)^2} du} = \frac{1}{2}.
\]
Thus
\[
\lim_{n \to \infty} \Delta_n \Delta^+_n = \frac{1}{2}.
\]

In Section 4, we consider the wavelets \( \psi_n \), as above. We assume further that \( P_n \) is reciprocal (which ensures \( \psi_n \) is symmetric or anti-symmetric), the roots of \( P_n \) are real (and hence negative), and \( P_n(-1) = 0 \). It is then proved that for certain \( k_n, \sigma_n, \alpha_n \) with \( \sigma_n \to \infty \) as \( n \to \infty \),
\[
k_n \hat{\psi}_n(u/\sigma_n) - e^{-\frac{1}{2}(u-\sigma_n \alpha_n)^2} \to 0
\]
in \( L^p(0, \infty), 1 \leq p \leq \infty \). Taking inverse Fourier transforms shows that the difference between \( |k_n| \hat{\psi}_n(\sigma_n x) \) and either \( 2 \cos(\sigma_n \alpha_n x) G(x) \) or \( 2 \sin(\sigma_n \alpha_n x) G(x) \) converges to 0 in \( L^q(\mathbb{R}), 2 \leq q \leq \infty \).

Finally we deduce that if we also have \( P_n^u(-1) = 0 \), then
\[
\lim_{n \to \infty} \Delta \psi_n \Delta^+_\psi_n = \frac{1}{2}.
\]
and so the wavelets $\psi_n$ have asymptotically optimal time-frequency location.

In [6] it was shown that the uniform $B$-spline converged to the Gaussian and the difference between the $B$-spline wavelets and modulated Gaussians converged to zero, both in time and frequency domains. In [3] this was extended to more general scaling functions $\phi_n$ and corresponding wavelets $\psi_n$ and it was shown that the appropriate uncertainty products converged to the optimal value as $n \to \infty$ for both $\phi_n$ and $\psi_n$. It was assumed that the symbol $P_n$ for $\phi_n$ was reciprocal and had real roots. Moreover it was assumed that $P_n(z)$ had a factor $(z + 1)^m$ where $m \geq Cn$ for a constant $C$. These assumptions allowed the use of similar techniques to those used for the case of $B$-splines. In this paper we use the different techniques in [2] to allow us to relax all of these conditions for the case of $\phi_n$, and for the case of $\psi_n$ to relax the final condition to require $P_n(z)$ to have merely a factor of $(z + 1)^3$. These together with the results in [2] will provide a more complete understanding of the asymptotic behaviour of scaling functions that approximate the Gaussian and the asymptotic properties of the corresponding wavelets.

## 2 A Preliminary Result

In this section we state a result that will be needed later. Taking Fourier transforms of (1.3) gives

$$\hat{\phi}_n(u) = P_n(e^{-iu/2})\hat{\phi}_n(u/2), \quad u \in \mathbb{R}, \quad (2.1)$$

and it follows that

$$\hat{\psi}_n(u) = \prod_{k=1}^{\infty} P_n(e^{-iu/2^k}), \quad u \in \mathbb{R}, \quad (2.2)$$

where the infinite product converges uniformly. Recalling (1.2) and that the roots of $P_n$ are $-r_{n,j}$, $j = 1, \ldots, n$, gives

$$\hat{\psi}_n(u) = \prod_{j=1}^{n} \prod_{k=1}^{\infty} \frac{e^{-iu/2^k} + r_{n,j}}{1 + r_{n,j}}, \quad u \in \mathbb{R}. \quad (2.3)$$
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In order to consider the convergence of $\widehat{e}_n (u)$ to $e^{-u^2/2}$, we need to consider the convergence of products as in (2.3). The following result actually covers more general products which will be encountered in Section 4. It extends a corresponding result in [2] on asymptotic normality and also gives convergence of derivatives. Although its proof is of the same form as that of Theorem 1.2 of [2], we sketch it here for completeness.

**Theorem 2.1** For $\gamma \in [0, \frac{\pi}{2})$, let $D_\gamma$ comprise all $z \in \mathbb{C}$ satisfying the inequality

$$\left| \Im \left( \frac{z}{(1 + z)^2} \right) \right| \leq \tan \gamma \Re \left( \frac{z}{(1 + z)^2} \right).$$

Suppose that for $n = 1, 2, \ldots$, $f_n$ is defined by

$$f_n(u) = \prod_{j=1}^{r_n} \frac{e^{-iu + \mu_{n,j}}}{1 + \mu_{n,j}} \prod_{j=1}^{s_n} \frac{e^{-iu/2^j + \lambda_{n,j}}}{1 + \lambda_{n,j}}, \quad u \in \mathbb{R},$$

where $\mu_{n,j}, j = 1, \ldots, r_n,$ and $\lambda_{n,j}, j = 1, \ldots, s_n$, $n = 1, 2, \ldots,$ lie in $D_\gamma$ for some $\gamma \in [0, \frac{\pi}{2})$ and are bounded away from $-1$.

Suppose that for $\mu_n \in \mathbb{R}$ and $\sigma_n > 0$, with $\sigma_n \to \infty$ as $n \to \infty$,

$$\tilde{f}_n(u) := e^{i\mu_n/\sigma_n} f_n(u/\sigma_n)$$

satisfies $\tilde{f}_n(0) = 0$ and $\tilde{f}_n'(0) = -1$. Then as $n \to \infty$,

$$\bar{f}_n(u) \to e^{-u^2/2} \quad \tilde{f}_n(u) \to -ue^{-u^2/2},$$

locally uniformly on $\mathbb{R}$.

**Proof.** A simple calculation shows that

$$\mu_n = \sum_{j=1}^{r_n} \frac{1}{1 + \mu_{n,j}} + \sum_{j=1}^{s_n} \frac{1}{1 + \lambda_{n,j}},$$

$$\sigma_n^2 = \sum_{j=1}^{r_n} \frac{\mu_{n,j}}{(1 + \mu_{n,j})^2} + \frac{1}{3} \sum_{j=1}^{s_n} \frac{\lambda_{n,j}}{(1 + \lambda_{n,j})^2}.$$  

Thus the condition $\lim_{n \to \infty} \sigma_n = \infty$ implies $\lim_{n \to \infty} (r_n + s_n) = \infty$. 


As in the proof of Theorem 1.2 of [2] we write
\[
\log f_n(u) = \frac{iu \mu_n}{\sigma_n} + \sum_{j=1}^{r_n} F(\mu_{n,j}, \frac{-iu}{\sigma_n}) + \sum_{j=1}^{s_n} \sum_{k=1}^{\infty} F(\lambda_{n,j}, \frac{-iu}{2^k \sigma_n}),
\]
where
\[
F(\mu, t) = \log \left( \frac{e^t + \mu}{1 + \mu} \right),
\]
and expanding \(F(\mu, t)\) in a Taylor series about \(t = 0\) gives
\[
\log f_n(u) = -\frac{u^2}{2} + \sum_{\nu = 3}^{\infty} \sigma_n^{-\nu} (-iu)^\nu \left\{ \sum_{j=1}^{r_n} a_\nu(\mu_{n,j}) + \sum_{j=1}^{s_n} \frac{a_\nu(\lambda_{n,j})}{2^\nu - 1} \right\},
\]
where, under the conditions on \(\mu_{n,j}, \lambda_{n,j}\), there is a constant \(A > 0\) with
\[
|a_\nu(\mu_{n,j})| \leq \frac{A^{\nu-2}}{\nu} \sec \gamma \Re \left( \frac{\mu_{n,j}}{(1 + \mu_{n,j})^2} \right),
\]
\[
|a_\nu(\lambda_{n,j})| \leq \frac{A^{\nu-2}}{\nu} \sec \gamma \Re \left( \frac{\lambda_{n,j}}{(1 + \lambda_{n,j})^2} \right).
\]
Thus, from (2.8), (2.6) and (2.7),
\[
\left| \log f_n(u) + \frac{u^2}{2} \right| \leq \sec \gamma \sum_{\nu = 3}^{\infty} \sigma_n^{-\nu} \frac{|u|^\nu}{\nu} A^{\nu-2} \times \left\{ \sum_{j=1}^{r_n} \Re \left( \frac{\mu_{n,j}}{(1 + \mu_{n,j})^2} \right) + \frac{1}{2^\nu - 1} \sum_{j=1}^{s_n} \Re \left( \frac{\lambda_{n,j}}{(1 + \lambda_{n,j})^2} \right) \right\}
\]
\[
\leq \sec \gamma \sum_{\nu = 3}^{\infty} \frac{|u|^\nu}{\nu} \left( \frac{A}{\sigma_n} \right)^{\nu-2}
\]
\[
\leq \sec \gamma \frac{|A|^2}{\sigma_n} \left( 1 - \frac{|A| |u|}{\sigma_n} \right)^{-1}
\]
whenever \(|A| |u| < \sigma_n\). Since \(\sigma_n \to \infty\) as \(n \to \infty\), \(\lim_{n \to \infty} f_n(u) = e^{-u^2/2}\) locally uniformly on \(\mathbb{R}\).
Differentiating (2.8) gives
\[
\frac{f'_n(u)}{f_n(u)} = -u - i \sum_{\nu = 3}^{\infty} \sigma_n^{-\nu} (-iu)^{\nu - 1} \left\{ \sum_{j=1}^{r_n} a_{\nu}(\mu_{n,j}) + \sum_{j=1}^{s_n} a_{\nu}(\lambda_{n,j}) \right\}
\]
and, as before,
\[
\left| \frac{f'_n(u)}{f_n(u)} + u \right| \leq \sec \gamma \sum_{\nu = 3}^{\infty} \sigma_n^{-\nu} |u|^{\nu - 1} A^{\nu - 2} \times \left\{ \sum_{j=1}^{r_n} \Re \frac{\mu_{n,j}}{(1 + \mu_{n,j})^2} + \frac{1}{2\nu - 1} \sum_{j=1}^{s_n} \Re \frac{\lambda_{n,j}}{(1 + \lambda_{n,j})^2} \right\}
\]
\[
\leq \sec \gamma \sum_{\nu = 3}^{\infty} |u|^{\nu - 1} \left( \frac{A}{\sigma_n} \right)^{\nu - 2} = \sec \gamma \frac{A|u|^2}{\sigma_n} \left( 1 - \frac{A|u|}{\sigma_n} \right)^{-1}
\]
whenever \( A|u| < \sigma_n \). Thus \( \lim_{n \to \infty} f'_n(u) = -ue^{-u^2/2} \) locally uniformly in \( \mathbb{R} \).

As noted in [2], the set \( D_\gamma \) contains the sector \( |\arg z| \leq \gamma \) and for \( z = \pm re^{i\theta} \), \( r > 0 \), \( \gamma \leq \theta \leq \pi \), \( z \) lies in \( D_\gamma \) if and only if
\[
\frac{\sin \left( \frac{\theta - \gamma}{2} \right)}{\sin \left( \frac{\theta + \gamma}{2} \right)} \leq r \leq \frac{\sin \left( \frac{\theta + \gamma}{2} \right)}{\sin \left( \frac{\theta - \gamma}{2} \right)}.
\]
In particular, \( D_\gamma \) contains the unit circle \( r = 1 \).

## 3 Asymptotic Optimality of Refinable Functions

We suppose that \( (\phi_n) \) is a sequence of refinable functions as in Section 1.

**Theorem 3.1** If \( P'_n(-1) = 0 \), then
\[
\lim_{n \to \infty} \Delta_{\phi_n} \Delta_{\phi_n} = \frac{1}{2}, \quad (3.1)
\]
Proof. We recall that \( \lim_{n \to \infty} \bar{\phi}_n = G \) in \( L^p(\mathbb{R}) \), \( 1 \leq p \leq \infty \). From (1.6) we see that (3.1) is equivalent to

\[
\lim_{n \to \infty} \Delta_{\bar{\phi}_n} = \frac{1}{2}.
\] (3.2)

We also recall that \( \bar{\phi}_n \) is defined so that

\[
\int_{-\infty}^{\infty} x^2 \bar{\phi}_n(x) \, dx = \int_{-\infty}^{\infty} x^2 G(x) \, dx = 1.
\] (3.3)

Firstly we shall show that

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} x^2 \bar{\phi}_n(x)^2 \, dx = \int_{-\infty}^{\infty} x^2 G(x)^2 \, dx.
\] (3.4)

Take \( \epsilon > 0 \) and choose \( A > 1 \) so that

\[
\int_{|x| > A} x^2 G(x) \, dx < \epsilon.
\] (3.5)

Choose \( N \) so that for all \( n > N \) and \( |x| \leq A \),

\[
|x^2 \bar{\phi}_n(x)^k - x^2 G(x)^k| < \frac{\epsilon}{2A}, \quad k = 1, 2.
\] (3.6)

Take any \( n > N \). Then

\[
\left| \int_{-A}^{A} x^2 \bar{\phi}_n(x) \, dx - \int_{-A}^{A} x^2 G(x) \, dx \right| < \epsilon
\]

and so by (3.3),

\[
\left| \int_{|x| > A} x^2 \bar{\phi}_n(x) \, dx - \int_{|x| > A} x^2 G(x) \, dx \right| < \epsilon.
\]

So by (3.5),

\[
\int_{|x| > A} x^2 \bar{\phi}_n(x) \, dx < 2\epsilon.
\]

Thus for large enough \( n \),

\[
\int_{|x| > A} x^2 \bar{\phi}_n(x)^2 \, dx < \frac{1}{\sqrt{2\pi}} \int_{|x| > A} x^2 \bar{\phi}_n(x) \, dx < \epsilon,
\]
and so
\[ \left| \int_{|x| > A} x^2 \hat{\phi}_n(x)^2 \, dx - \int_{|x| > A} x^2 G(x)^2 \, dx \right| < 2\epsilon. \]

By (3.6), for all large enough \( n \),
\[ \left| \int_{-A}^{A} x^2 \hat{\phi}_n(x)^2 \, dx - \int_{-A}^{A} x^2 G(x)^2 \, dx \right| < \epsilon \]
and (3.4) follows.

Also for large enough \( n \),
\[ \left| \int_{-A}^{A} x \hat{\phi}_n(x)^2 \, dx - \int_{-A}^{A} x G(x)^2 \, dx \right| < \epsilon, \]
\[ \int_{|x| > A} |x| \hat{\phi}_n(x)^2 \, dx < \int_{|x| > A} x^2 \hat{\phi}_n(x)^2 \, dx < \epsilon, \]
and as before,
\[ \lim_{n \to \infty} \int_{-\infty}^{\infty} x \hat{\phi}_n(x^2) \, dx = \int_{-\infty}^{\infty} x G(x)^2 \, dx = 0. \quad (3.7) \]

Since
\[ \lim_{n \to \infty} \int_{-\infty}^{\infty} \hat{\phi}_n(x)^2 \, dx = \int_{-\infty}^{\infty} G(x)^2 \, dx, \]
(3.4) and (3.7) give
\[ \lim_{n \to \infty} \Delta \hat{\phi}_n = \Delta_G = \frac{1}{\sqrt{2}}. \]

To complete the proof we need to show that \( \lim_{n \to \infty} \Delta \hat{\phi}_n = \Delta_G \) and since \( \hat{\phi}_n(u)^2 \) is even and \( \lim_{n \to \infty} || \hat{\phi}_n ||_2 = || \hat{G} ||_2 \), it remains to show that
\[ \lim_{n \to \infty} \int_{-\infty}^{\infty} u^2 |\hat{\phi}_n(u)|^2 \, du = \int_{-\infty}^{\infty} u^2 e^{-u^2} \, du. \quad (3.8) \]

Let \( \hat{\Phi}_n(u) = |\hat{\phi}_n(u)|^2 \) and \( f(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4} \), so that \( \hat{f}(u) = e^{-u^2} \).

Then (3.8) is equivalent to
\[ \lim_{n \to \infty} \Phi_n''(0) = f''(0). \quad (3.9) \]
Recall that $\phi_n$ is the refinable function with symbol $P_n(z)$, which we may write as $(z+1)^2p_n(z)$. Then $\Phi_n$ is, up to scaling, the refinable function with symbol $P_n(z)P_n(z^{-1}) = (z+1)^2(z^{-1}+1)^2p_n(z)p_n(z^{-1})$. It follows that

$$
\Phi''(x) = \eta_n(x+1) - 2\eta_n(x) + \eta_n(x-1),
$$

where $\eta_n$ is the refinable function with symbol $(z+1)(z^{-1}+1)p_n(z)p_n(z^{-1})$, up to scaling. Since this polynomial has all roots in the left half-plane, it follows from [5], and [1], that $\eta_n$ satisfies the following property: for any sequence $(\lambda_j)$ in $\mathbb{R}$, the number of inflections of the function $\sum_{j=-\infty}^{\infty} \lambda_j \eta_n(\cdot - j)$ is bounded by the number of inflections in the polygonal arc with vertices $(j, \lambda_j)$, $j \in \mathbb{Z}$. Thus the function $\Phi''$ has at most four inflections.

Now $\lim_{n \to \infty} \phi_n(u) = e^{-u^2/2}$ locally uniformly on $\mathbb{R}$ and so $\lim_{n \to \infty} (\Phi''(u))^\wedge = (f''(u))^\wedge$ locally uniformly on $\mathbb{R}$. Now $f''$ has exactly four inflections and it follows as in the proof of Lemma 5.1 of [2] that $\Phi''$ converges to $f''$ uniformly on a neighboured of 0. So (3.9) holds. $\square$

## 4 Asymptotic Optimality of Wavelets

Throughout this section we shall assume that the roots of $P_n$ in (1.1) are real and negative, and that $P_n$ is reciprocal, i.e. the roots are invariant under the transformation $z \to z^{-1}$. It follows that $\phi_n$ is symmetric, i.e. $\phi_n(x) = \phi_n(n-x)$, $x \in \mathbb{R}$.

We define $\Phi_n$ and $Q_n$ as in (1.10) and (1.11). It is easily seen that $\Phi_n$ is a refinable function with mask $P_n(z)P_n(z^{-1})$. It follows that $\Phi_n$ is continuous with support $[-n, n]$ and satisfies $\Phi_n(-x) = \Phi_n(x)$, $x \in \mathbb{R}$. The Laurent polynomial $Q_n$ is called the Euler-Frobenius polynomial corresponding to $\phi_n$. From the work of [5] we know that $Q_n$ has real negative roots. As in (1.12) we define the wavelet, up to multiplication by a constant as $\psi_n(2 \cdot -1)$, where $\psi_n$ is defined by

$$
\hat{\psi}_n(u) = e^{inu}Q_n(e^{-i(u+\pi)})P_n(e^{-i(u+\pi)})\hat{\phi}_n(u), \quad u \in \mathbb{R}.
$$

(4.1)

Suppose that the roots of $P_n$ are $-\lambda_{n,1}, \ldots, -\lambda_{n,n}$. By (1.2) we may suppose $\lambda_{n,1} = 1$. We write

$$
\lambda_{n,j} := \frac{1}{2}(\lambda_{n,j} + \lambda_{n,j}^{-1}), \quad j = 1, \ldots, n,
$$
and for simplicity drop the first subscript $n$ when it is unambiguous to do so. Note that $\lambda_j \geq 1$, $j = 1, \ldots, n$, and 

$$(e^{-iu} + \lambda_j)(e^{-iu} + \lambda_j^{-1}) = 2e^{-iu}(\lambda_j + \cos u).$$

Also 

$$e^{-iu} + 1 = 2e^{-iu/2} \cos \frac{u}{2} = \sqrt{2}e^{-iu/2}(1 + \cos u)^{1/2}, \quad -\pi \leq u \leq \pi,$$

and therefore 

$$P_n(e^{-iu}) = e^{-iu/2} \prod_{j=1}^{n} \left( \frac{\lambda_j + \cos u}{\lambda_j + 1} \right)^{1/2}, \quad -\pi \leq u \leq \pi.$$ 

Similarly, if the roots of $Q_n$ are $-\mu_1, \ldots, -\mu_{n-1}, -\mu_1^{-1}, \ldots, -\mu_{n-1}^{-1}$, and setting $\bar{\mu}_j := \frac{1}{2}(\mu_j + \mu_j^{-1})$, $j = 1, \ldots, n - 1$, then for a constant $A_n > 0$,

$$Q_n(e^{-iu}) = A_n \prod_{j=1}^{n-1} (\bar{\mu}_j + \cos u).$$

So from (4.1) and (2.2),

$$\hat{\psi}_n(u) = B_n e^{-iu\pi/2} f_n(u), \quad u \in \mathbb{R}, \quad (4.2)$$

where $B_n > 0$, $f_n(u)$ is real,

$$|f_n(u)| = \prod_{j=1}^{n-1} (\bar{\mu}_j - \cos u) \prod_{j=1}^{n} (\lambda_j - \cos u)^{1/2} 

\times \prod_{k=1}^{\infty} \prod_{j=1}^{n} \left( \frac{\lambda_j + \cos (2^{-k} u)}{\lambda_j + 1} \right)^{1/2}, \quad u \in \mathbb{R}, (4.3)$$

$f_n(u) > 0$, $0 < u < 2\pi$, $f_n(0) = f_n(2\pi) = 0$, $f_n(-u) = (-1)^n f_n(u)$.

We shall make the following mild assumption on the growth rate of $\lambda_{n,j}$ with $n$. There are constants $K > 0$ and $0 < \sigma \leq 1$ such that

$$\left| \{j : \lambda_{n,j} \leq K \} \right| \geq \sigma n, \quad n = 2, 3, \ldots, \quad (4.4)$$

This can be reformulated as follows.
Lemma 4.1  Condition (4.4) holds if and only if there is a constant $C > 0$ with
\[
\sum_{j=1}^{n} \frac{1}{(\lambda_{n,j} + 1)^2} \geq Cn, \quad n = 2, 3, \ldots \tag{4.5}
\]

PROOF. If (4.4) holds, then
\[
\sum_{j=1}^{n} \frac{1}{(\lambda_{n,j} + 1)^2} \geq \frac{\sigma n}{(K + 1)^2}, \quad n = 2, 3, \ldots,
\]
which gives (4.5).

Conversely, suppose that (4.4) does not hold. Take any $\epsilon > 0$. Then there exists $n$ such that
\[
|\{j : \hat{\lambda}_{n,j} \leq \epsilon^{-1/2}\}| < \epsilon n.
\]
If $\hat{\lambda}_{n,j} > \epsilon^{-1/2}$, then $(\hat{\lambda}_{n,j} + 1)^{-2} < \epsilon$ and so
\[
\sum_{j=1}^{n} \frac{1}{(\hat{\lambda}_{n,j} + 1)^2} < 2\epsilon n.
\]
Thus there is no constant $C$ for which (4.5) holds. \qed

We can now study the shape of the graph of $f_n$ in $[0, 2\pi]$, which will lead to the asymptotic behaviour of $\psi_n$.

Lemma 4.2  There is a number $\alpha > 0$ such that $f_n$ in (4.2) has a unique local maximum in $[0, 2\pi]$ at $\alpha_n$ with
\[
\frac{2\pi}{3} < \alpha_n < \pi - \alpha, \quad n = 2, 3, \ldots.
\]

PROOF. For $n = 2, 3, \ldots$, putting $F_n(u) := \log f_n(u)$, $0 < u < 2\pi$, gives by (4.3),
\[
F_n'(u) = \sum_{j=1}^{n-1} \frac{\sin u}{\tilde{\mu}_j - \cos u} + \frac{1}{2} \sum_{j=1}^{n} \frac{\sin u}{\lambda_j - \cos u} - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{\infty} \frac{2^{-k} \sin(2^{-k} u)}{\lambda_j + \cos(2^{-k} u)}, \tag{4.6}
\]
For \( \pi \leq u < 2\pi \), all these terms are negative and so

\[
F'_n(u) < 0, \quad \pi \leq u < 2\pi.
\]

For \( 0 < u \leq \frac{2\pi}{3} \),

\[
F'_n(u) > \frac{1}{2} \sum_{j=1}^{n} \sin u \left( \frac{\sin(u/2)}{\lambda_j - \cos u} \right) - \frac{1}{2} \sum_{j=1}^{n} \sin(u/2)
\]

\[
= \frac{1}{2} \sum_{j=1}^{n} \frac{\lambda_j \sin(u - \sin(u/2)) + \sin(3u/2)}{\lambda_j - \cos u(\lambda_j + \cos(u/2))} \geq 0,
\]

since \( \sin u \geq \sin(u/2) \).

Now by condition (4.4), for \( 0 < u \leq \pi \),

\[
F'_n(u) \leq \frac{3n \sin u}{2(1 - \cos u)} - \frac{1}{4} \sigma^n \frac{\sin(u/2)}{K + \cos(u/2)} = nG(u),
\]

say. Since \( G(\pi) < 0 \), we can choose \( \alpha > 0 \) with \( G(\alpha) \leq 0 \) for \( \pi - \alpha \leq u \leq \pi \). Thus for \( n = 2, 3, \ldots \),

\[
F'_n(u) < 0, \quad \pi - \alpha \leq u \leq \pi.
\]

Finally we note that for \( \frac{\pi}{2} \leq u \leq \pi \),

\[
F''_n(u) = \sum_{j=1}^{n-1} \frac{\lambda_j \cos u - 1}{(\lambda_j - \cos u)^2} + \frac{1}{2} \sum_{j=1}^{n} \frac{\hat{\lambda}_j \cos u - 1}{(\lambda_j - \cos u)^2} - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{\infty} \frac{2^{2k} \lambda_j \cos(2^{-k}u)}{2^{2k}(\lambda_j + \cos(2^{-k}u))^2} < 0,
\]

since all terms are negative.

Thus we have shown that for \( n = 2, 3, \ldots \), \( F'_n(u) > 0, 0 < u \leq \frac{2\pi}{3} \), \( F'_n(u) < 0, \pi - \alpha \leq u < 2\pi \), and \( F''_n(u) < 0, \frac{\pi}{2} \leq u \leq \pi \). Hence \( F_n \) has a unique local maximum in \((0, 2\pi)\) at \( \alpha_n \) with \( \frac{4\pi}{3} < \alpha_n < \pi - \alpha \). The result follows. \( \square \)

It will be convenient to renormalise the wavelet \( \psi_n \) by defining

\[
\Psi_n(x) = \frac{(-1)^{[b/2]}}{B_n f_n(\alpha_n)} \psi_n(x), \quad x \in \mathbb{R},
\]
where $B_n$ is as in (4.2), from which it follows that for even $n,$

$$\hat{\Psi}_n(u) = \frac{f_n(u)}{f_n(\alpha_n)},$$

(8)

while for odd $n,$

$$\hat{\Psi}_n(u) = \frac{f_n(u)}{i f_n(\alpha_n)}.$$

We next define $\sigma_n > 0$ by

$$\sigma_n^2 = -\Gamma''(\alpha_n).$$

(9)

By (4.7) we see that

$$\sigma_n^2 > \sum_{j=1}^{n-1} \frac{1}{(\hat{\mu}_j + 1)^2} + \frac{2}{3} \sum_{j=1}^n \frac{1}{(\lambda_j + 1)^2}.$$

So by condition (4.4) and Lemma 4.1,

$$\sigma_n \geq a \sqrt{n}, \quad n = 2, 3, \ldots,$$

(10)

for some $a > 0.$ Also from (4.7),

$$\sigma_n < \sqrt{2n}, \quad n = 2, 3, \ldots$$

(11)

**Theorem 4.3** If $j_n$ denotes the number 1 or i as $n$ is even or odd, then

$$\lim_{n \to \infty} j_n \hat{\Psi}_n(u/\sigma_n + \alpha_n) = e^{-u^2/2},$$

$$\lim_{n \to \infty} j_n \frac{\hat{\Psi}_n(u/\sigma_n + \alpha_n)}{\sigma_n} = -ue^{-u^2/2},$$

where the convergence is locally uniform on $\mathbb{R}$.

**Proof.** We consider even $n,$ the case for odd $n$ differing only by a factor of i. Putting $\tilde{f}_n(u) := \hat{\Psi}_n(u/\sigma_n + \alpha_n),$ we see from (4.8), Lemma 4.2 and (4.9) that

$$\tilde{f}_n(0) = 1, \quad \tilde{f}'_n(0) = 0, \quad \tilde{f}''_n(0) = -1.$$
We also have from (4.1),
\[
\tilde{f}_n(u) = e^{i(2n-1)u/\sigma_n} \prod_{j=1}^{n-1} \frac{(e^{-iu/\sigma_n} - \mu_j e^{i\alpha_n})(e^{-iu/\sigma_n} - \mu_j^{-1} e^{i\alpha_n})}{(1 - \mu_j e^{i\alpha_n})(1 - \mu_j^{-1} e^{i\alpha_n})} \times \prod_{j=1}^{n} \frac{e^{-iu/\sigma_n} - \lambda_j e^{i\alpha_n}}{1 - \lambda_j e^{i\alpha_n}} \prod_{j=1}^{n} \prod_{k=1}^{\infty} \frac{e^{iu/2^k \sigma_n} + \lambda_j e^{i\alpha_n/2^k}}{1 + \lambda_j e^{i\alpha_n/2^k}},
\]
which is of form (2.5), (2.4) in Theorem 2.1. In order to apply Theorem 2.1 we need that for \( n = 2, 3, \ldots, -\mu_j e^{i\alpha_n}, -\mu_j^{-1} e^{i\alpha_n}, j = 1, \ldots, n - 1, -\lambda_j e^{i\alpha_n}, j = 1, \ldots, n, \lambda_j e^{i\alpha_n/2^k}, j = 1, \ldots, n, k = 1, 2, \ldots, \) all lie in \( D_\gamma \) for some \( \gamma \in [0, \frac{\pi}{2}] \) and are bounded away from \(-1\). From Lemma 4.2, \( \pi - \alpha_n < \frac{\pi}{2} \) and \( \frac{\pi}{2} < \frac{\pi}{2} - \frac{\gamma}{4}, k = 1, 2, \ldots, \) and hence all the above numbers lie in the sector \( |\arg z| \leq \gamma \) for \( \gamma = \max\{\frac{\pi}{4}, \frac{\pi}{2} - \frac{\gamma}{4}\} \). Since \( D_\gamma \) contains this sector, the condition is satisfied.

We see from (4.10) that \( \sigma_n \to \infty \) as \( n \to \infty \). Then Theorem 2.1 gives the desired result. \( \square \)

In Theorem 4.3 we consider convergence of \( \tilde{\Psi}_n^\prime \) because this will be needed later for convergence of the uncertainty products. In order to extend Theorem 4.3 to convergence in \( L^p(\mathbb{R}) \) we shall need to apply the Dominated Convergence Theorem. For this are shall need the further condition that \( P_n^\prime \) has a double root at \(-1\), i.e. \( P_n^\prime(-1) = 0 \).

**Lemma 4.4** Let
\[
\mu_\lambda(u) = \prod_{k=1}^{\infty} \frac{\lambda + \cos(2^{-k} u)}{\lambda + 1}.
\]
Then for any \( K > 1 \), there is a constant \( \rho < 1 \) such that \( \mu_\lambda(u) \leq \rho \mu_\lambda(\pi), \quad |\mu_\lambda(u)| \leq \rho \mu_\lambda(\pi), \) for \( u \geq \frac{\pi}{2} \) and \( 1 \leq \lambda \leq K \).

**Proof.** Fix \( \lambda \geq 1 \) and let \( R(u) = \frac{\lambda + \cos u}{\lambda + 1} \). For \( \pi \leq u \leq 2\pi, \) \( R(u/2^k) \leq R(\pi/2^k), k = 1, 2, \ldots, \) and so
\[
\frac{\mu_\lambda(u)}{\mu_\lambda(\pi)} = \prod_{k=1}^{\infty} \frac{R(2^{-k} u)}{R(2^{-k} \pi)} \leq \frac{R(u/2)}{R(\pi/2)} = \frac{\lambda + \cos(u/2)}{\lambda}.
\]
Thus
\[ \mu_\lambda(u) \leq \mu_\lambda(\pi), \quad \pi \leq u \leq 2\pi, \quad (4.12) \]
and
\[ \mu_\lambda(u) \leq \left( 1 - \frac{1}{\lambda^{\sqrt{2}}} \right) \mu_\lambda(\pi), \quad \frac{3\pi}{2} \leq u \leq 2\pi. \]

For \(2\pi \leq u \leq 3\pi\),
\[ \mu_\lambda(u) = R(u/2)\mu_\lambda(u/2) \leq \frac{\lambda}{\lambda + 1} \mu_\lambda(\pi), \]
by (4.12). So for any \(K > 1\) there is a constant \(\rho < 1\) such that for
\[ 1 \leq \lambda \leq K, \quad \frac{\pi}{3} \leq u \leq 3\pi, \]
\[ \mu_\lambda(u) \leq \rho \mu_\lambda(\pi). \quad (4.13) \]

Now for all \(u \geq 3\pi\),
\[ \mu_\lambda(u) = R(u/2)\mu_\lambda(u/2) \leq \mu_\lambda(u/2) \]
and successive application of this gives (4.13) for all \(u \geq \frac{3\pi}{2}\).

Now \(\mu'_\lambda(u) = -\sum_{k=1}^{\infty} 2^{-k} \nu_k(u)\), where
\[ \nu_k(u) = \prod_{j \neq k}^{\infty} \frac{\lambda + \cos(2^{-j}u)}{\lambda + 1} \cdot \frac{\sin(2^{-k}u)}{\lambda + 1}. \]

For \(k \geq 1, \pi \leq u \leq 2\pi\),
\[ \left| \frac{\nu_k(u)}{\mu_\lambda(\pi)} \right| = \prod_{j \neq k}^{\infty} \frac{R(2^{-j}u)}{R(2^{-j}\pi)} \cdot \frac{\sin(2^{-k}u)}{\lambda + \cos(2^{-k}u)} \leq \frac{1}{\lambda} \leq 1. \]

For \(\frac{3\pi}{2} \leq u \leq 2\pi\),
\[ \left| \frac{\nu_1(u)}{\mu_\lambda(\pi)} \right| \leq \frac{\sin(3\pi/4)}{\lambda} \leq \frac{1}{\sqrt{2}}. \]

So
\[ |\mu'_\lambda(u)| \leq \mu_\lambda(\pi), \quad \pi \leq u \leq \frac{3\pi}{2}, \]
\[ |\mu'_\lambda(u)| \leq \rho \mu_\lambda(\pi), \quad \frac{3\pi}{2} \leq u \leq 2\pi, \]
for a constant \( \rho < 1 \). Now since \( \mu_\lambda(u) = R(u/2)\mu_\lambda(u/2) \),
\[
|\mu'_\lambda(u)| = \left| \frac{1}{2} R'(u/2)\mu_\lambda(u/2) + \frac{1}{2} R(u/2)\mu'_\lambda(u/2) \right|
\leq \frac{1}{4} |\mu_\lambda(u/2)| + \frac{1}{2} |\mu'_\lambda(u/2)| .
\]
So recalling (4.12),
\[
|\mu'_\lambda(u)| \leq \frac{3}{4} \mu_\lambda(\pi), \quad u \geq 2\pi.
\]
\hspace{1cm} \square

**Lemma 4.5** If \( P'_n(-1) = 0 \), then for constants \( A > 0, 0 < \rho < 1, 0 < \sigma \leq 1, u \geq \frac{3\pi}{2}, n = 2, 3, \ldots \),
\[
|\hat{\phi}'_n(u)| \leq A \rho^\sigma u^{-2} |\hat{\phi}_n(\pi)|,
|\hat{\phi}''_n(u)| \leq A n \rho^\sigma u^{-2} |\hat{\phi}_n(\pi)| .
\]

**Proof.** We may write
\[
P_n(e^{-iu}) = \prod_{j=1}^{(n-m)/2} \left( \frac{\hat{\lambda}_j + \cos u}{\hat{\lambda}_j + 1} \right) \cos^n \left( \frac{u}{2} \right),
\]
for \( m = 2 \) or \( 3 \). We note that
\[
\prod_{k=1}^{\infty} \cos \left( \frac{u}{2k+1} \right) = \prod_{k=1}^{\infty} \frac{\sin(2^{-k}u)}{2 \sin(2^{-k-1}u)} = \lim_{n \to \infty} \frac{\sin(u/2)}{2^n \sin(2^{-n-1}u)} = \frac{\sin(u/2)}{u/2} .
\]
Recalling (2.2), Lemma 4.4 and condition (4.4) gives the result. \hspace{1cm} \square

**Lemma 4.6** If \( P'_n(-1) = 0 \), then there is a constant \( C > 0 \) such that for \( u \geq \sigma_n(3\pi/2 - \alpha_n), n = 2, 3, \ldots \), \( |\hat{\phi}_n(u/\sigma_n + \alpha_n)| \) and \( |\hat{\Phi}'_n(u/\sigma_n + \alpha_n)| \) are bounded by \( C(1+u)^{-2} \).
PROOF. For all $u$, 

$$|Q_n(e^{-i(u+\pi)})P_n(e^{-i(u+\pi)})| \leq Q_n(1)P_n(1),$$

and so by Lemma 4.5, on recalling (4.1), for $u \geq \frac{3\pi}{2}$,

$$|\tilde{\Psi}_n(u)| \leq A\rho^n u^{-2}|\tilde{\Psi}_n(\pi)|.$$

Since $|\tilde{\Psi}_n(\pi)| \leq |\tilde{\Psi}_n(\alpha_n)| = 1$, for $u \geq \sigma_n\left(\frac{3\pi}{2} - \alpha_n\right)$,

$$|\tilde{\Psi}_n(u/\sigma_n + \alpha_n)| \leq A\rho^n (u/\sigma_n + \alpha_n)^{-2} \leq C(1 + u)^{-2},$$

for some $C > 0$, by (4.11).

Similarly we see that for a constant $B > 0$,

$$|\tilde{\Psi}_n'(u/\sigma_n + \alpha_n)| \leq Bn\rho^n (u/\sigma_n + \alpha_n)^{-2}$$

and the result follows. \hfill \Box

Remark

The above method of proof shows that if $P_n$ has a root at $z = -1$ of multiplicity $m$, then Lemma 4.6 holds with $(1 + u)^{-2}$ replaced by $(1 + u)^{-m}$.

Lemma 4.7 There are constants $a$, $b > 0$ such that for $-\sigma_n \alpha_n \leq u \leq \sigma_n\left(\frac{3\pi}{2} - \alpha_n\right)$, $n = 2, 3, \ldots$, $|\tilde{\Psi}_n(u/\sigma_n + \alpha_n)|$ and $|\frac{1}{\sigma_n}\psi_n(u/\sigma_n + \alpha_n)| \leq ae^{-bu^2}$.

PROOF. As in the proof of Lemma 4.2 we put $F_n(u) = \log f_n(u)$, $0 < u < 2\pi$. Recall from (4.7) that for $\frac{\pi}{2} \leq u \leq \pi$,

$$|F_n''(u)| = \sum_{j=1}^{n-1} \frac{1 - \mu_j \cos u}{(\mu_j - \cos u)^2} + \frac{1}{2} \sum_{j=1}^{n} \frac{1 - \lambda_j \cos u}{(\lambda_j - \cos u)^2} + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{\infty} \frac{1 + \lambda_j \cos(2^{-k}u)}{2^{2k}(\lambda_j + \cos(2^{-k}u))^2}. \quad (4.14)$$
We also recall condition (4.4) and note that for \( \mu > K \), \( \frac{\pi}{2} \leq u < \pi \),

\[
\frac{1 - \mu \cos u}{(\mu - \cos u)^2} < \frac{1 - K \cos u}{(K - \cos u)^2} \leq \frac{1}{\sigma n} \sum_{\lambda_j \leq K} 1 - \lambda_j \cos u \quad (\lambda_j - \cos u)^2.
\]

It \( G_n(u) \) denotes the right-hand side of (4.14) with the summations taken only for \( \tilde{\lambda}_j \leq K \), \( \lambda_j \leq K \), then for \( \frac{\pi}{2} \leq u \leq \pi \),

\[
|F^{nn}_n(u)| \leq \left( 1 + \frac{2}{\sigma} \right) G_n(u).
\]

Since \( |G_n(u)| \leq |F^{nn}_n(u)| \), \( \frac{\pi}{2} \leq u \leq \pi \), there is a constant \( C > 0 \) with

\[
\frac{|F^{nn}_n(u)|}{|F^{nn}_n(v)|} \leq \left( 1 + \frac{2}{\sigma} \right) \frac{G_n(u)}{G_n(v)} \leq C, \quad \frac{\pi}{2} \leq u, \ v \leq \pi,
\]

since

\[
(K + 1)^{-2} \leq \frac{1 + \lambda x}{(\lambda + x)^2} \leq K + 1, \quad 0 \leq x \leq 1, \quad 1 \leq \lambda \leq K.
\]

As \( F^{n}_n(\alpha_n) = 0 \),

\[
F_n(u) \leq F_n(\alpha_n) + \frac{1}{2} C^{-1} (u - \alpha_n)^2 F^{nn}_n(\alpha_n), \quad \frac{\pi}{2} \leq u \leq \pi.
\]

Recalling that \( \sigma^2_n = -F^{nn}_n(\alpha_n) \) we have

\[
f_n(u) \leq f_n(\alpha_n) \exp \left( -\frac{1}{2} C^{-1} (u - \alpha_n)^2 \sigma^2_n \right), \quad \frac{\pi}{2} \leq u \leq \pi,
\]

and hence

\[
\left| \tilde{\Psi}_n \left( u/\sigma_n + \alpha_n \right) \right| \leq e^{-\frac{1}{2} C^{-1} u^2}, \quad \sigma_n (\pi/2 - \alpha_n) \leq u \leq \sigma_n (\pi - \alpha_n).
\]

Also for \( \frac{\pi}{2} \leq u \leq \pi \),

\[
|F^{n}_n(u)| \leq C |u - \alpha_n||F^{nn}_n(\alpha_n)|
\]

and so

\[
|f'_n(u)| = |f_n(u)||F'_n(\alpha_n)| \leq C \sigma^2_n |u - \alpha_n||f_n(u)|,
\]
which gives
\[
\left| \hat{\Psi}''(u/\sigma_n + \alpha_n) \right| \leq C|u| \left| \hat{\Psi}_n(u/\sigma_n + \alpha_n) \right|, \quad \sigma_n \left( \frac{\pi}{2} - \alpha_n \right) \leq u \leq \sigma_n(\pi - \alpha_n).
\]
(4.16)

By Lemma 4.2 we may choose \( \alpha, 0 < \alpha < \frac{\pi}{6} \), with \( \pi - \alpha_n > \alpha \), \( n = 2, 3, \ldots \) Let \( E(u) = \exp(-\alpha^2 u^2/8Cn^2) \). For \( \pi \leq u \leq \frac{2\pi}{n} \), \( f_n(u) \leq f_n(\pi) \) and so for \( \sigma_n(\pi - \alpha_n) \leq u \leq \sigma_n\left( \frac{3\pi}{2} - \alpha_n \right) \), (4.15) gives
\[
\hat{\Psi}_n(u/\sigma_n + \alpha_n) \leq \exp\left( \frac{1}{2} C^{-1}(\pi - \alpha_n)^2 \sigma_n^2 \right)
\]
\[
= E(2\pi\sigma_n\alpha^{-1}(\pi - \alpha_n))
\]
\[
\leq E(2\pi\sigma_n) \leq E(u),
\]
(4.17)

since \( \pi - \alpha_n \geq \alpha \) and \( u \leq \sigma_n\left( \frac{3\pi}{2} - \alpha_n \right) < 2\pi\sigma_n \).

Now from (4.6) and (4.10), there is a constant \( A > 0 \) such that
\[
|F_n'(u)| \leq a^2 A_n \leq A\sigma_n^2.
\]

Thus
\[
|f_n'(u)| \leq A\sigma_n^2 f_n(u), \quad \pi \leq u \leq \frac{3\pi}{2}
\]

and so for \( \sigma_n(\pi - \alpha_n) \leq u \leq \sigma_n\left( \frac{3\pi}{2} - \alpha_n \right) \),
\[
\left| \hat{\Psi}''(u/\sigma_n + \alpha_n) \right| \leq A\sigma_n^2 \left| \hat{\Psi}_n(u/\sigma_n + \alpha_n) \right|
\]
\[
\leq A\alpha^{-2} u^2 \left| \hat{\Psi}_n(u/\sigma_n + \alpha_n) \right|,
\]
(4.18)

since \( u \geq \sigma_n(\pi - \alpha_n) > \sigma_n \alpha \).

For \( 0 \leq u \leq \frac{\pi}{2} \), \( f_n(u) \leq f_n\left( \frac{\pi}{2} \right) \) and so for \( -\sigma_n \alpha_n \leq u \leq \sigma_n\left( \frac{\pi}{2} - \alpha_n \right) \), (4.15) gives
\[
\left| \hat{\Psi}_n(u/\sigma_n + \alpha_n) \right| \leq \exp\left( \frac{1}{2} C^{-1} \left( \frac{\pi}{2} - \alpha_n \right)^2 \sigma_n^2 \right)
\]
\[
= E\left( 2\pi\sigma_n\alpha^{-1}\left( \frac{\pi}{2} - \alpha_n \right) \right)
\]
\[
\leq E(2\pi\sigma_n) \leq E(u),
\]
(4.19)

since \( \alpha_n - \frac{\pi}{2} \geq \frac{\pi}{6} \geq \alpha \) and \( |u| \leq \sigma_n \alpha_n < 2\pi\sigma_n \).
Asymptotic Optimality of Scaling Functions and Wavelets

Now we see from (4.6) that for \(0 \leq u \leq \frac{\pi}{2}\),

\[
|f'_n(u)| \leq \left\{ \sum_{j=1}^{n-1} \frac{\sin u}{\mu_j - \cos u} + \frac{1}{2} \sum_{j=1}^{n} \frac{\sin u}{\lambda_j - \cos u} \right\} f_n(u).
\]

Noting that

\[
\frac{\sin u}{\lambda} \prod_{k=1}^{\infty} \frac{\lambda + \cos(2^{-k}u)}{\lambda + 1} \left( \prod_{k=1}^{\infty} \frac{\lambda + \cos(2^{-k-1}\pi)}{\lambda + 1} \right)^{-1}
\]

is bounded for \(\lambda \geq 1, 0 \leq u \leq \frac{\pi}{2}\), and recalling from Lemma 4.2 that \(f_n\) is increasing in \([0, \frac{\pi}{2}]\), we see that for a constant \(K\),

\[
|f'_n(u)| \leq Kn f_n(\pi/2), \quad 0 \leq u \leq \pi/2.
\]

So by (4.10) there is a constant \(B\) such that for \(-\sigma_n \alpha_n \leq u \leq \sigma_n(\frac{\pi}{2} - \alpha_n)\),

\[
\left| \hat{\Psi}_n'(u/\sigma_n + \alpha_n) \right| \leq B\sigma_{2n}^2 \hat{\Psi}_n(\pi/2\sigma_n + \alpha_n) \leq B \frac{36}{\pi^2} u^2 E(u), \quad (4.20)
\]

as in (4.19) and since \(|u| \geq \sigma_n(\alpha_n - \hat{\alpha}) \geq \sigma_n \pi/6\).

The result now follows from (4.15)–(4.20).

Lemma 4.6 and 4.7 allow us to apply the Dominated Convergence Theorem to prove the following

**Theorem 4.8** Suppose \(L'_n(-1) = 0\). If \(j_n\) denotes the number 1 or \(i\) as \(n\) is even or odd, then as \(n \to \infty\),

\[
j_n \hat{\Psi}_n(u/\sigma_n) - e^{-\frac{1}{2} (u-\sigma_n \alpha_n)^2} \to 0,
\]

\[
j_n \frac{\hat{\Psi}_n'(u/\sigma_n)}{\sigma_n} + (u - \sigma_n \alpha_n) e^{-\frac{1}{2} (u-\sigma_n \alpha_n)^2} \to 0,
\]

in \(L^p(0, \infty), 1 \leq p \leq \infty\).

**Proof.** We need consider only even \(n\). Let

\[
h_n(u) = \begin{cases} \hat{\Psi}_n(u/\sigma_n + \alpha_n), & u \geq -\sigma_n \alpha_n, \\ 0, & \text{otherwise}. \end{cases}
\]
From Theorem 4.3, \( \lim_{n \to \infty} h_n(u) = e^{-u^2/2} \) and \( \lim_{n \to \infty} h'_n(u) = -ue^{-u^2/2} \) locally uniformly. By Lemmas 4.5 and 4.7, \( h_n \) and \( h'_n \) are dominated by

\[
h(u) = \max\{C(1 + |u|)^{-2}, ae^{-bu^2}\}, \quad u \in \mathbb{R}.
\]

So by the Dominated Convergence Theorem,

\[
\lim_{n \to \infty} h_n(u) = e^{-u^2/2}, \quad \lim_{n \to \infty} h'_n(u) = -ue^{-u^2/2},
\]

in \( L^p(\mathbb{R}), 1 \leq p < \infty \). Also since \( h_n, e^{-u^2/2} \) and \( ue^{-u^2/2} \) all tend to 0 as \( u \to \pm \infty \), it follows that the convergence in (4.21) is also uniform. The result follows. \( \Box \)

Since \( \hat{\Psi}_n(-u) = (-1)^n \hat{\Psi}_n(u), u \in \mathbb{R} \), we have a corresponding result on \( (-\infty, 0) \). Adding these together and taking inverse Fourier transforms gives the following

**Theorem 4.9** Suppose \( P'_n(-1) = 0 \). As \( n \to \infty \), for even \( n \),

\[
\sigma_n \Psi_n(\sigma_n x) - 2 \cos(\sigma_n \alpha_n x) G(x) \to 0,
\]

and for odd \( n \),

\[
\sigma_n \Psi_n(\sigma_n x) - 2 \sin(\sigma_n \alpha_n x) G(x) \to 0,
\]

where the convergence is in \( L^q(\mathbb{R}), 2 \leq q \leq \infty \).

Finally we show that the wavelets \( \Psi_n \) have asymptotically optimal time-frequency localisation, as described in Section 1 (see (1.13), (1.14)).

**Theorem 4.10** If \( P'_n(-1) = 0 \), then

\[
\lim_{n \to \infty} \Delta_{\Psi_n} \Delta_{\Psi_n}^* = \frac{1}{2}.
\]
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Proof. Since \( \Psi_n(-x) = (-1)^n \Psi_n(x) \), \( x \in \mathbb{R} \), we see that \( \mu_\Psi \), as in (1.7), is \( 0 \) and so from (1.8),

\[
\Delta^2_{\Psi_n} = \frac{\int_{-\infty}^{\infty} x^2 |\Psi_n(x)|^2 \, dx}{\int_{-\infty}^{\infty} |\Psi_n(x)|^2 \, dx} = \frac{\int_{-\infty}^{\infty} |\hat{\Psi}_n(u)|^2 \, du}{\int_{-\infty}^{\infty} |\Psi_n(u)|^2 \, du} = \frac{\int_{0}^{\infty} |\hat{\Psi}_n(u)|^2 \, du}{\int_{0}^{\infty} |\Psi_n(u)|^2 \, du}.
\]

By Theorem 4.8, as \( n \to \infty \),

\[
\int_{0}^{\infty} |\hat{\Psi}_n(u/\sigma_n)|^2 \, du - \int_{0}^{\infty} e^{-(u - \sigma_n \alpha_n)^2} \, du \to 0
\]

and so

\[
\sigma_n \int_{0}^{\infty} |\hat{\Psi}_n(u)|^2 \, du \to \int_{-\infty}^{\infty} e^{-u^2} \, du,
\]

since \( \sigma_n \to \infty \). Also by Theorem 4.8, as \( n \to \infty \),

\[
\frac{1}{\sigma_n^2} \int_{0}^{\infty} |\hat{\Psi}_n'(u/\sigma_n)|^2 \, du - \int_{0}^{\infty} (u - \sigma_n \alpha_n)^2 e^{-(u - \sigma_n \alpha_n)^2} \, du \to 0
\]

and so

\[
\frac{1}{\sigma_n} \int_{0}^{\infty} |\hat{\Psi}_n'(u)|^2 \, du \to \int_{-\infty}^{\infty} u^2 e^{-u^2} \, du.
\]

Thus

\[
\lim_{n \to \infty} \sigma_n^{-2} \Delta^2_{\Psi_n} = \frac{\int_{-\infty}^{\infty} u^2 e^{-u^2} \, du}{\int_{-\infty}^{\infty} e^{-u^2} \, du} = \frac{1}{2}.
\]

By Theorem 4.3 and Lemmas 4.6 and 4.7, for \( j = 1, 2 \),

\[
(u - \sigma_n \alpha_n)^j |\hat{\Psi}_n(u/\sigma_n)|^2 - (u - \sigma_n \alpha_n)^j e^{-(u - \sigma_n \alpha_n)^2} \to 0 \text{ as } n \to \infty
\]

in \( L^1(0, \infty) \). Thus as \( n \to \infty \),

\[
\int_{0}^{\infty} (u - \sigma_n \alpha_n) |\hat{\Psi}_n(u/\sigma_n)|^2 \, du \to 0,
\]

(4.24)
\[
\int_{0}^{\infty} (u - \sigma_n \alpha_n)^2 \left| \hat{\Psi}_n (u/\sigma_n) \right|^2 du \rightarrow \int_{-\infty}^{\infty} u^2 e^{-u^2} du. \quad (4.25)
\]

Now
\[
\int_{0}^{\infty} u \left| \hat{\Psi}_n (u) \right|^2 du = \frac{1}{\sigma_n^2} \int_{0}^{\infty} u \left| \hat{\Psi}_n (u/\sigma_n) \right|^2 du
\]
\[
= \frac{1}{\sigma_n^2} \int_{0}^{\infty} (u - \sigma_n \alpha_n) \left| \hat{\Psi}_n (u/\sigma_n) \right|^2 du + \frac{\alpha_n}{\sigma_n} \int_{0}^{\infty} \left| \hat{\Psi}_n (u/\sigma_n) \right|^2 du.
\]
Thus from (1.13),
\[
\mu_{\hat{\Phi}_n} = \frac{\int_{0}^{\infty} (u - \sigma_n \alpha_n) \left| \hat{\Psi}_n (u/\sigma_n) \right|^2 du}{\sigma_n \int_{0}^{\infty} \left| \hat{\Psi}_n (u/\sigma_n) \right|^2 du} + \alpha_n
\]
and hence from (4.22) and (4.24),
\[
\lim_{n \rightarrow \infty} \sigma_n (\mu_{\hat{\Phi}_n} - \alpha_n) = 0. \quad (4.26)
\]

By (1.14),
\[
\sigma_n^2 \left( \Delta_{\hat{\Phi}_n} \right)^2 = \frac{\sigma_n^2 \int_{0}^{\infty} (u - \mu_{\hat{\Phi}_n})^2 \left| \hat{\Psi}_n (u/\sigma_n) \right|^2 du}{\int_{0}^{\infty} \left| \hat{\Psi}_n (u/\sigma_n) \right|^2 du}
\]
\[
= \frac{\int_{0}^{\infty} (u - \sigma_n \mu_{\hat{\Phi}_n})^2 \left| \hat{\Psi}_n (u/\sigma_n) \right|^2 du}{\int_{0}^{\infty} \left| \hat{\Psi}_n (u/\sigma_n) \right|^2 du}
\]
\[
\rightarrow \frac{\int_{-\infty}^{\infty} u^2 e^{-u^2} du}{\int_{-\infty}^{\infty} e^{-u^2} du} = \frac{1}{2}, \quad (4.27)
\]
as \( n \rightarrow \infty \), by (4.26), (4.25) and (4.22). The result then follows from (4.23) and (4.27). □

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