A LEAST-SQUARES FINITE ELEMENT METHOD FOR THE MAGNETOSTATIC PROBLEM IN A MULTIPLY CONNECTED LIPSCHITZ DOMAIN

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Abstract. A new least-squares finite element method is developed for the curl-div magnetostatic problem in Lipschitz and multiply connected domains filled with anisotropic nonhomogeneous materials. In order to deal with possibly low regularity of the solution, local $L^2$ projectors are introduced to standard least-squares formulation and applied to both curl and div operators. Coercivity is established by adding suitable mesh-dependent bilinear terms. As a result, any continuous finite elements (lower-order elements are enriched with suitable bubbles) can be employed. A desirable error bound is obtained: $| | u - u_h | |_0 \leq C | | u - \tilde{u} | |_0$, where $u_h$ and $\tilde{u}$ are the finite element approximation and the finite element interpolant of the exact solution $u$, respectively. Numerical tests confirm the theoretical results.

Key words. least-squares continuous finite element method, $L^2$ projector, curl-div magnetostatic problem

AMS subject classifications. 65M60, 74S05, 78A30, 65N30, 65N15

DOI. 10.1137/050640102

1. Introduction. Recently, there has been increasing interest in seeking finite element solutions of Maxwell’s equations; see [5, 7, 16, 21, 22, 23, 26] and references therein. As a typical model, the curl-div magnetostatic problem plays a central role in the study of finite element methods for Maxwell’s equations and other mathematical subjects such as existence-uniqueness and regularity-singularity [1, 7, 15, 28]. Let us first recall this model. For a domain $\Omega$ of $\mathbb{R}^3$ filled with anisotropic nonhomogeneous materials described by a tensor $\varepsilon$, given two functions $g \in (L^2(\Omega))^3$ and $f \in L^2(\Omega)$, with $u$ the unknown field, the curl-div magnetostatic problem reads as follows [6, 7]:

(1.1)\[
\text{curl } u = g, \quad \text{div}(\varepsilon u) = f \quad \text{in } \Omega.
\]

In this paper we shall consider a least-squares $C^0$ finite element method to solve (1.1) numerically. Although (1.1) is simple in appearance, the $C^0$ finite element discretization is not straightforward generally. To illustrate this we consider a simple case below. Let $\Omega$ be a Lipschitz polyhedron of $\mathbb{R}^3$ with boundary $\partial \Omega$, $\varepsilon = 1$, and assuming a boundary condition $u \cdot n = 0$ on $\partial \Omega$. Let $X_T := \{v \in (L^2(\Omega))^3; \text{curl } v \in (L^2(\Omega))^3, \text{div } v \in L^2(\Omega), v \cdot n|_{\partial \Omega} = 0\}$, equipped with the norm $| |v| |_{X_T}^2 = | |\text{curl } v| |^2_0 + | |\text{div } v| |^2_0 + | |v| |^2_0$, where $| | \cdot | |_0$ is the $L^2$ norm. Equation (1.1) can be formulated as a standard least-squares variational problem (see [22, 23]): Find $u \in X_T$ such that, for all $v \in X_T$,

(1.2)\[
L(u, v) := (\text{curl } u, \text{curl } v) + (\text{div } u, \text{div } v) = (g, \text{curl } v) + (f, \text{div } v),
\]
where \((\cdot, \cdot)\) denotes the \(L^2\) inner product. As is well known, \(L\) is coercive on \(X_T\) with respect to the norm \(\| \cdot \|_{X_T}\) (cf. \([2, 15, 20]\)). Then it seems to be natural to employ the \(C^0\) finite element method for problem (1.2). This is indeed true for smooth domains or for convex polyhedra (see \([12, 22, 23, 27]\)). However, when the domain is nonsmooth (\(\Omega\) contains reentrant corners or edges), it turns out that the \(C^0\) finite element method for convex polyhedra (see \([12, 22, 23, 27]\)). However, when the domain is nonsmooth (\(\Omega\) contains reentrant corners or edges), it turns out that the \(C^0\) finite element method of problem (1.2) does not usually yield correct approximations (cf. \([5, 15, 16]\)). Here we provide an intuitive interpretation. A more accurate interpretation may be found in \([5, 15, 16, 24]\). Let \(u_h\) be the \(C^0\) finite element solution of (1.2), where \(h > 0\) denotes the mesh-parameter of the simplex partition of \(\Omega\). The classical \(C^0\) finite element and interpolation theory \([8, 13]\) leads to an error estimate: \(\| u - u_h \|_{X_T} \leq C \| u - \tilde{u} \|_{X_T} \leq C h^r \| u \|_{1+r}, \) for all \(r \geq 0\), where \(\tilde{u}\) is an interpolant of \(u\) in a continuous piecewise \(P_r\) polynomial of order not greater than \(r\). This error estimate indicates that \(u\) should be at least in \(H^1\) in order to have a convergence. But, for nonsmooth domains, \(u\) may not be in \(H^1\) (see \([15]\)).

Nonetheless, this is not necessarily the problem of the use of \(C^0\) finite elements. In fact, any function in \(L^2\) (even in \(L^1\)) can be well approximated by \(C^0\) elements, and we have

\[
(1.3) \quad \| u - \tilde{u} \|_0 \leq C h^s \| u \|_s
\]

when \(u\) is in \(H^s\) with \(0 \leq s < 1\) (see \([3, 8, 13, 14, 29]\)). Since \(C^0\) finite elements are good enough for \(H^s\) (\(0 \leq s < 1\)) functions, the use of the formulation (1.2) is the source of trouble when \(C^0\) finite elements are used for the problem with singular (nonsmooth) solutions whose singularities generally result from nonsmoothness of domains or heterogeneities of materials filling the domain or both. This motivates the design of more suitable formulations to replace (1.2) so that \(C^0\) finite elements may concurrently work for nonsmooth solutions (not in \(H^1\)) as well as smooth solutions (at least in \(H^1\)). Meanwhile, new formulations should still have the same merits as (1.2): (a) the resulting linear system is symmetric and positive definite; (b) a globally continuous solution can be produced. See a survey \([4]\) on least-squares finite element methods.

There are a few modified formulations available. A weighted formulation, proposed in \([16]\), may be employed. See also an earlier paper \([17]\). The weighted least-squares method is theoretically and numerically proven to be convergent correctly. A property of this method is that a positive weight function of nonpolynomials is applied to the div operator appearing in (1.2). In two dimensions the weight function may be taken in the form of \(r^\gamma\), where \(r\) is the distance to the reentrant corner with opening angle greater than \(\pi\), and \(\gamma\) is an index that characterizes the singularity of the exact solution. It becomes, however, rather complicated to determine the weight function in three dimensions, due to the more complex characterization of the singularity information for three-dimensional domains. Another property is that the \(C^0\) finite element space is required to contain the gradient of some \(C^1\) finite element space. This excludes the use of some simpler \(C^0\) finite elements (e.g., the linear element). Several \(C^1\) finite element spaces are available in two dimensions (cf. \([13]\)), but, to our knowledge, few \(C^1\) finite elements are known in three dimensions. So it is unclear how to choose a reasonable three-dimensional \(C^0\) finite element space. There are other least-squares methods available—for example, the FOSLL* method \([11, 25]\) and the negative norm method \([9, 10]\). Of a scaled version of (1.1) (setting \(u^* = \epsilon^{1/2} u\) and \(g = 0\)) the FOSLL* method first seeks the solution of a dual problem associated with the dual operator of the differential operator of the scaled problem of (1.1), and
then the solution $u^*$ is obtained by differentiating the dual solution. This FOSLL$^*$ method [25] is in essence a scalar potential method and the approximation of $u^*$ is always discontinuous because of the differentiation. The negative norm method [9], to accommodate the case of nonsmooth solutions, formulates (1.1) in suitable dual norms (assuming that $\Omega$ is simply connected, $\partial \Omega$ has no disconnected components, $\varepsilon = 1$, and the boundary condition $u \cdot n|\partial \Omega = 0$):

$$\langle \text{curl } u, \text{curl } v \rangle_{(H^{-1}(\Omega))^3} + \langle \text{div } u, \text{div } v \rangle_{(H^{1}(\Omega))^*},$$

where $(H^{-1}(\Omega))^3$ is the dual space of $(H^{1}_0(\Omega))^3$, with $H^{1}_0(\Omega) = \{v \in H^{1}(\Omega); v|\partial \Omega = 0\}$, and $(H^{1}(\Omega))^*$ is the dual space of $H^{1}(\Omega)$, and $(\cdot, \cdot)_{(H^{-1}(\Omega))^3}$ and $(\cdot, \cdot)_{(H^{1}(\Omega))^*}$ denote the inner products of $(H^{-1}(\Omega))^3$ and $(H^{1}(\Omega))^*$, respectively. In the discrete level this method is something like adding the inverse of the discrete Laplace operator or its preconditioner in front of both curl and div operators in (1.2). The error estimate of the finite element approximation for nonsmooth solutions may be obtained from this method, but at the expense of multiple applications of the inverse of the discrete Laplacian or its preconditioner. The programming is rather tricky in practice as well.

In this paper, we develop new least-squares methods with the use of $C^0$ finite elements. The main idea is to apply local $L^2$ projectors to both curl and div operators appearing in (1.2), with a few extra mesh-dependent stabilization terms added. Specifically, let $R_h$ and $\tilde{R}_h$ be $L^2$ projectors defined relative to $L^2$ inner products $(\cdot, \cdot)_h$, and let $S_h(\cdot, \cdot)$ be a mesh-dependent bilinear form; we define a new least-squares bilinear form:

$$L_h(u, v) = (R_h(\text{curl } u), R_h(\text{curl } v))_h + (\tilde{R}_h(\text{div } (\varepsilon u)), \tilde{R}_h(\text{div } (\varepsilon v)))_h + S_h(u, v).$$

These $R_h$ and $\tilde{R}_h$ are defined as local $L^2$ projectors or pseudolocal $L^2$ projectors. Local $L^2$ projectors are defined element-by-element onto the discontinuous piecewise constant finite element spaces and pseudolocal $L^2$ projectors are defined onto the continuous piecewise linear finite element spaces with respect to the trapezoidal quadrature scheme of the standard $L^2$ inner product (Note that the $L^2$ projectors defined in this way are essentially local. See Remark 3.1 of this paper.) We prove that both $L^2$ projected least-squares methods, labeled as the local $L^2$ projection method and the pseudolocal $L^2$ projection method, are coercive:

$$L_h(v, v) \geq C \|v\|_0^2.$$  

As a result, any $C^0$ finite elements can be employed. We also prove that the condition number of the resulting linear system is $O(h^{-2})$. To show the idea, we focus on the error analysis of linear $C^0$ finite elements in three dimensions. We employ the linear element enriched face bubbles for the local $L^2$ projection method and the linear element enriched with element bubbles for the pseudolocal $L^2$ projection method. We can construct an interpolant $\tilde{u}$ of the exact solution $u$ that satisfies not only the usual interpolation error estimation (1.3) but also the exclusive interpolation property:

$$\|R_h(\text{curl } (u - \tilde{u}))\|_h = \|\tilde{R}_h(\text{div } (\varepsilon (u - \tilde{u})))\|_h = 0.$$  

We thus obtain mainly from (1.5) and (1.6) the following desirable error estimates:

$$\|u - u_h\|_0 \leq C \|u - \tilde{u}\|_0,$$
where \( u_h \) is the finite element solution associated with (1.4). Equation (1.7) means that the finite element solution is almost the best approximation to the exact solution. In the case that \( u \in H^s \), an \( L^2 \) optimal error bound \( O(h^s) \) follows directly from (1.7) and (1.3). We point out that the role of the face bubbles and the element bubbles is to make (1.6) hold. For higher order \( C^0 \) finite elements (cubic elements and above), additional bubbles are not always needed, since they usually have face or/and element bubbles of their own.

Before closing this section, we would like to give several remarks. The implementation of the \( L^2 \) projected least-squares method of this paper is almost as easy as that of the standard least-squares method (1.2). But, the former allows less regular solution. In comparison with the weighted least-squares method, it does not need any a priori singularity information of the solution and allows the use of both lower-order (maybe enriched with suitable bubbles) and higher-order \( C^0 \) elements in both two and three dimensions. Also, it is not clear if there is an improved \( L^2 \) error bound one has to resort to the well-known Aubin–Nitsche duality argument \([8, 13]\), but this argument usually requires the domain to be smooth enough in order that the associated auxiliary variational problem admits a solution with an appropriate regularity. Unlike the negative norm least-squares method which involves a preconditioner for second-order elliptic problems, the method here deals only with local \( L^2 \) projectors, so the practical implementation is simpler. Compared with the FOSLL\(^*\) method, the method here avoids the differentiation of approximate solutions of potentials and obtains continuous approximate solutions.

The outline of this paper is as follows. In section 2, we review the curl-div magnetostatic problem and recall the \( L^2 \) orthogonal decomposition of vector fields. In section 3, two \( L^2 \) projected least-squares \( C^0 \) finite element methods are described. In section 4, coercivity is established and condition number is estimated. In section 5, the error estimate of the method is obtained. In the last section, some numerical tests are performed to demonstrate the theoretical results obtained.

2. The magnetostatic problem and \( L^2 \) decomposition. Let \( \Omega \) of \( \mathbb{R}^3 \) be an open, bounded, and possible multiconnected Lipschitz polyhedron, with boundary \( \Gamma = \partial \Omega \) and \( n \) the outward unit normal vector to \( \Gamma \). Let \( \varepsilon = (\varepsilon_{ij}) \in \mathbb{R}^{3 \times 3} \) satisfy \( \varepsilon_{ij} = \varepsilon_{ji}, 1 \leq i, j \leq 3 \), and

\[
C \sum_{i=1}^{3} \xi_i^2 \leq C \varepsilon_{ij} \xi_i \xi_j \leq C^{-1} \sum_{i=1}^{3} \xi_i^2 \quad \text{a.e. in } \bar{\Omega} \quad \forall \xi = (\xi_i) \in \mathbb{R}^3.
\]

For the sake of simplicity, we always assume that \( \varepsilon \) is Lipschitz continuous over \( \bar{\Omega} \). With a few modifications, the method of this paper can deal with the case \( \varepsilon \) being piecewise Lipschitz continuous but not globally continuous (see Remark 5.4).

We now describe the curl-div system of magnetostatic problems.

Given \( g \in (L^2(\Omega))^3 \) and \( f \in L^2(\Omega) \), the curl-div magnetostatic problem is to find \( \mathbf{u} \) such that

\[
\begin{align*}
\text{curl } \mathbf{u} &= \mathbf{g}, \\
\text{div } (\varepsilon \mathbf{u}) &= f & \text{in } \Omega, \\
(\varepsilon \mathbf{u}) \cdot \mathbf{n} &= 0 & \text{on } \Gamma,
\end{align*}
\]

where \( \text{curl } \mathbf{v} = \nabla \times \mathbf{v}, \text{div } \mathbf{v} = \nabla \cdot \mathbf{v}, \) and \( \nabla \) is the gradient operator.
The solution of problem (2.1)–(2.2) may not be unique when Ω is multiconnected.
In that case, to ensure the uniqueness of the solution we need to introduce additional
constraints on the so-called cuts. (Roughly speaking, these cuts “cut” the multiconnected
domain into a simply connected one.) Note that when Ω is simply connected,
problem (2.1)–(2.2) has a unique solution. (Of course, g and f are required to satisfy
necessary compatible conditions, e.g., \( \text{div} g = 0 \), \( \int_{\Omega} f = 0 \), etc.)

To that goal, as done in [2, 28], we assume that there is a set of \( N \) cuts \( \Sigma_j \),
\( 1 \leq j \leq N \), such that \( \Omega = \Omega \setminus \Sigma \) (where \( \Sigma = \bigcup_{j=1}^{N} \Sigma_j \) is pseudoLipschitz and simply
connected, where \( \Sigma_j \subset \Omega \) is a compact and connected two-dimensional Lipschitz
manifold with boundary \( \partial \Sigma_j \subset \Gamma \), and \( \Sigma_i \cap \Sigma_j = \emptyset \) if \( i \neq j \), and \( \Sigma_j \) is globally two-
sided, denoted by \( \Sigma^+_j \) and \( \Sigma^-_j \), and \( \partial \Omega = \Gamma \cup \Sigma^+ \cup \Sigma^- \). As an illustrating example we
consider a multiply connected domain shown in Figure 1. Cutting along \( \Sigma_1 \) and \( \Sigma_2 \)
we get a simply connected domain \( \bar{\Omega} \) with boundary \( \Gamma \cup \Sigma^+_1 \cup \Sigma^-_1 \cup \Sigma^+_2 \cup \Sigma^-_2 \), where
\( \Sigma^+_i \) are the upper and lower (relative to \( n_i \), the unit normals in \( \Sigma_i \)) sheets of \( \Sigma_i \).

Additional constraints can be thus given by [2, 28]

\[
(2.3) \quad \int_{\Sigma_j} (\varepsilon \mathbf{u}) \cdot \mathbf{n} = 0, \quad 1 \leq j \leq N.
\]

Problem (2.1)–(2.3) then admits a unique solution, with compatible conditions satisfied
by \( g \) and \( f \). Readers may refer to [1, 6, 7, 28] for more details.

We next introduce some Hilbert spaces.

Let \( D \subseteq \Omega \). Denote by \( H^1(D), \ H^3(D)/\mathbb{R}, \) and \( H^1_0(D) \) the usual Hilbert spaces.
We also need \( H(\text{div}; \Omega) = \{ \mathbf{v} \in (L^2(\Omega))^3, \text{div} \mathbf{v} \in L^2(\Omega) \} \), \( H(\text{div}^0; \Omega) = \{ \mathbf{v} \in H(\text{div}; \Omega); \text{div} \mathbf{v} = 0 \} \), \( H_\Gamma(\text{div}; \Omega) = \{ \mathbf{v} \in H(\text{div}; \Omega); \mathbf{v} \cdot n_\Gamma = 0 \} \), \( H_\Gamma(\text{div}^0; \Omega) =
H_\Gamma(\text{div}; \Omega) \cap H(\text{div}^0; \Omega) \), \( H(\mathbf{curl}; \Omega) = \{ \mathbf{v} \in (L^2(\Omega))^3, \mathbf{curl} \mathbf{v} \in (L^2(\Omega))^3 \} \), \( H_\Gamma(\mathbf{curl}; \Omega) = \{ \mathbf{v} \in H(\mathbf{curl}; \Omega), \mathbf{v} \times n_\Gamma = 0 \} \), and

\[
(2.4) \quad U = \{ \mathbf{v} \in (L^2(\Omega))^3; \text{div} (\varepsilon \mathbf{v}) \in L^2(\Omega), \mathbf{curl} \mathbf{v} \in (L^2(\Omega))^3, (\varepsilon \mathbf{v}) \cdot n_\Gamma \in L^2(\Gamma) \},
\]

\[
(2.5) \quad H_{\text{flux}, \Sigma}(\text{div}^0; \Omega) = \left\{ \mathbf{v} \in H(\text{div}^0; \Omega); \int_{\Sigma_j} \mathbf{v} \cdot \mathbf{n} = 0, \quad 1 \leq j \leq N \right\},
\]

\[
(2.6) \quad \mathbb{H} = \{ \mathbf{v} \in U; \mathbf{curl} \mathbf{v} = 0, \text{div} (\varepsilon \mathbf{v}) = 0, (\varepsilon \mathbf{v}) \cdot n_\Gamma = 0 \}.
\]

The \( \mathbb{H} \) referred to as “harmonic space,” may not be trivial in the case of multiconnected
domains and accounts for why the solution of problem (2.1)–(2.2) may not be unique. The \( \mathbb{H} \) has a finite dimension and can be characterized as the space of
gradients of a finite number of scalar functions (See Proposition 2.1 below).
we shall describe two \( L^2 \) norm of \( q \).

\[
\nabla q = \Pi \text{ in } \Omega \quad \text{and} \quad \| q \|_{L^2(\Omega)} \leq C \| \Pi \|_{L^2(\Omega)},
\]
where \( q \) satisfies

\[
\nabla (\varepsilon \nabla q) = 0 \quad \text{in } \Omega, \quad (\varepsilon \nabla q) \cdot n \bigg|_{\Gamma} = 0,
\]
\[
[q]_{\Sigma_j} = \text{constant}, \quad [(\varepsilon \nabla q) \cdot n]_{\Sigma_j} = 0, \quad 1 \leq j \leq N,
\]

where \([v]_{\Sigma_j}\) denotes the jump in \( v \) across \( \Sigma_j \).

Denote by \( (\varepsilon \cdot , \cdot) \) the \( \varepsilon \)-weighted \( L^2 \) inner product, i.e.,

\[
(\varepsilon u, v) := \int_\Omega \varepsilon u v.
\]

We now recall the \( L^2 \) orthogonal decomposition.

**Proposition 2.2** (see [2, 28]). For any \( v \in (L^2(\Omega))^3 \), it can be written as the following \( L^2 \) orthogonal decomposition, with respect to \( (\varepsilon \cdot , \cdot) \):

\[
v = \nabla p + \Pi + \varepsilon^{-1} \mathbf{curl} \psi,
\]

where \( p \in H^1(\Omega) / \mathbb{R} \), \( \Pi \in \mathbb{H} \), \( \psi \in H_{\Gamma} \mathbf{curl}; \Omega \cap H_{\text{flux}; \Sigma}(\mathbf{div}; \Omega) \), and

\[
\| \psi \|_0 \leq C \| \mathbf{curl} \psi \|_0, \quad \| \varepsilon^{\frac{1}{2}} v \|_0^2 = \| \varepsilon^{\frac{1}{2}} \nabla p \|_0^2 + \| \varepsilon^{\frac{1}{2}} \Pi \|_0^2 + \| \varepsilon^{-\frac{1}{2}} \mathbf{curl} \psi \|_0^2.
\]

Above and below, the letter \( C \) (with or without subscripts) stands for a generic constant which is independent of the mesh-parameter \( h \) and may take different values at different occurrences. Denote by \( (\cdot , \cdot)_{0, D} \) and \( \| \cdot \|_{0, D} \) the inner product and the norm of \( L^2(D) \) or \( (L^2(D))^3 \), and \( (\cdot , \cdot) := (\cdot , \cdot)_{0, \Omega}, \| \cdot \|_0 := \| \cdot \|_{0, \Omega} \).

We finally recall Green's formula of integrating by parts:

\[(\nabla v, q) = -(v, \nabla q) + \int_\Gamma v \cdot n q \quad \forall v \in H(\mathbf{div}; \Omega), \forall q \in H^1(\Omega);\]

\[(\mathbf{curl} u, v) = (u, \mathbf{curl} v) + \int_\Gamma u \times n \cdot v \quad \forall u \in H(\mathbf{curl}; \Omega), \forall v \in (H^1(\Omega))^3.\]

### 3. The \( L^2 \) projected least-squares finite element methods

In this section, we shall describe two \( L^2 \) projected least-squares finite element methods: (1) the local \( L^2 \) projection method; (2) the pseudolocal \( L^2 \) projection method.

#### 3.1. Bubbles and some finite dimensional spaces

Let \( \mathcal{C}_h \) denote a regular triangulation of \( \Omega \) into tetrahedra, with diameters \( h_K \) for all \( K \in \mathcal{C}_h \) bounded by \( h \) [8, 13]. We assume that the closure of each cut in \( \{ \Sigma_j; 1 \leq j \leq N \} \) is the union of the closure of some faces of tetrahedra in \( \mathcal{C}_h \).

Let \( \mathcal{E}_h \) be the set of all the interior faces in \( \mathcal{C}_h \), \( \mathcal{E}_h^0 \) the set of all the faces on \( \Gamma \), and \( \mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^1 \) the set of all faces in \( \mathcal{C}_h \). We define \( \mathcal{M}_h \) as the collection of macroelements in the following way. Each macroelement in \( \mathcal{M}_h \) corresponds to a face \( F \in \mathcal{E}_h \) one-by-one: (1) if \( F \in \mathcal{E}_h^0 \), then the macroelement in \( \mathcal{M}_h \) is the union of the two tetrahedra sharing \( F \); (2) if \( F \in \mathcal{E}_h^1 \), then the macroelement is the tetrahedron including \( F \) as a face. Note that some macroelements in \( \mathcal{M}_h \), corresponding to different faces, are allowed to be identical and that the number of all macroelements is...
the same as that of all faces. To emphasize the dependence on $F$, sometimes we write $M$ as $M_F$.

We now introduce some bubbles. If $K$ is a tetrahedron with vertices $a_i$, $1 \leq i \leq 4$, we denote by $\lambda_i$ the barycoordinate of $a_i$, and by $F_i$ the face opposite $a_i$, and then we introduce the face bubbles

\begin{align}
(3.1) \quad & b_{F_1} = \lambda_2 \lambda_3 \lambda_4, \quad b_{F_2} = \lambda_3 \lambda_4 \lambda_1, \quad b_{F_3} = \lambda_4 \lambda_1 \lambda_2, \quad b_{F_4} = \lambda_1 \lambda_2 \lambda_3 \\
\text{and the usual element bubble on } K, \quad b_K &= \lambda_1 \lambda_2 \lambda_3 \lambda_4 \in H^1_0(K).
\end{align}

We would have

\begin{align}
(3.2) \quad & b_F \in H^1_0(F_1), \quad b_F|_{F_j} = 0 \quad \forall j \neq i.
\end{align}

For any $M_F \in \mathcal{M}_h$, corresponding to $F \in \mathcal{E}_h$, we introduce the macroelement bubble $b_{M_F}$ as follows.

(1) If $F \in \mathcal{E}^0_h$, i.e., $M_F = K_1 \cup K_2$ with $K_1, K_2 \in \mathcal{C}_h$ sharing $F$, we denote by $b^{K_1}_F$ and $b^{K_2}_F$ the face bubble of $F$ in $K_1$ and $K_2$, respectively. We set

\begin{align}
(3.4) \quad & b_{M_F}(x) = \begin{cases} 
  b^{K_1}_F(x), & x \in K_1, \\
  b^{K_2}_F(x), & x \in K_2, \\
  0 & \text{elsewhere.}
\end{cases}
\end{align}

It can be seen that $b_{M_F} \in H^1_0(M_F)$ and

\begin{align}
(3.5) \quad & b_{M_F}|_F \in H^1_0(F), \quad b_{M_F}|_{F'} = 0 \quad \forall F' (\neq F) \in \mathcal{E}_h.
\end{align}

(2) If $F \in \mathcal{E}^0_\partial$, i.e., $M_F = K$, with $K \in \mathcal{C}_h$ sharing $F$ with $\Gamma$, we set

\begin{align}
(3.6) \quad & b_{M_F}(x) = \begin{cases} 
  b_F(x), & x \in K, \\
  0 & \text{elsewhere.}
\end{cases}
\end{align}

Also, we have (3.5), but $b_{M_F} \notin H^1_0(M_F)$.

Let $\mathcal{P}_r$ be the space of polynomials of order not greater than $r \geq 0$. We define

\begin{align}
(3.7) \quad & \mathcal{P}(M_F) := \text{span}\{\varepsilon (\mathcal{P}_0(M_F))^3, (\mathcal{P}_0(M_F))^3\} = \text{span}\{p_{F;l}; 1 \leq l \leq m_F\}, \\
(3.8) \quad & \mathcal{P}(K) := \text{span}\{\varepsilon (\mathcal{P}_0(K))^3, (\mathcal{P}_0(K))^3\} = \text{span}\{p_l; 1 \leq l \leq m_K\},
\end{align}

where $m_F$ and $m_K$ are positive integers standing for the dimensions of $\mathcal{P}(M_F)$ and $\mathcal{P}(K)$, respectively, and we define the following bubble spaces:

\begin{align}
(3.9) \quad & \Phi_h := \left\{ \mathbf{v} \in (H^1(\Omega))^3; \mathbf{v} = \sum_{F \in \mathcal{E}_h} \sum_{l=1}^{m_F} c_{F;l} p_{F;l} b_{M_F} \forall c_{F;l} \in \mathbb{R} \right\}, \\
(3.10) \quad & \Psi_h := \{ \mathbf{v} \in (H^1_0(\Omega))^3; \mathbf{v} |_{\mathcal{P}(K)} b_K \forall K \in \mathcal{C}_h\}.
\end{align}

We also need some additional finite dimensional spaces which will be used in the next section. First, let

\begin{align}
(3.11) \quad & P_h := \{ q \in H^1(\Omega); q|_K \in \mathcal{P}_1(K) \forall K \in \mathcal{C}_h\}.
\end{align}
Second, let \( q^i_0, 1 \leq i \leq N, \) be the piecewise linear polynomial function, taking value “1” at the nodes on one side of \( \Sigma, \) say, \( \Sigma^+, \) and “0” at all other nodes (including those on \( \Sigma^-, \) the other side of \( \Sigma). \) Let \( A_h := \text{span}\{q^i_0, i = 1, \ldots, N\}; \) we define

\[
V_h := P_h + A_h, \\
W_h := (P_h \cap H^1_0(\Omega))^3,
\]

where \( V_h \) and \( W_h \) will be used only for the pseudolocal \( L^2 \) projection method below.

### 3.2. Finite element method

Let \( U_h \) be the finite element space. The \( L^2 \) projected least-squares finite element method is to find \( u_h \in U_h \) such that

\[
\mathcal{L}_h(u_h, v) = \mathcal{F}_h(v) \quad \forall v \in U_h,
\]

where

\[
\mathcal{L}_h(u, v) := (\tilde{R}_h(\text{div}(\varepsilon u)), \tilde{R}_h(\text{div}(\varepsilon v)))_h \\
+ (R_h(\text{curl} u), R_h(\text{curl} v))_h + S_h(u, v),
\]

\[
\mathcal{F}_h(v) := (f, \tilde{R}_h(\varepsilon v)) + (g, R_h(\text{curl} v)) + Z_h(f, g; v),
\]

\( S_h(u, v) \) is a mesh-dependent semipositive definite bilinear form on \( U_h \times U_h, \) \( Z_h(f, g; v) \) is a mesh-dependent linear form on \( U_h, \) \( (\cdot, \cdot)_h \) is an approximation of \( L^2 \) inner product \( (\cdot, \cdot), \) and \( \tilde{R}_h, R_h \) are \( L^2 \) projectors. These are to be defined below.

We first describe the local \( L^2 \) projection method.

We define \( (\cdot, \cdot)_h := (\cdot, \cdot) \) and

\[
U_h := (P_h)^3 + \Phi_h,
\]

\[
\tilde{R}_h(\text{div}(\varepsilon u))_K := \frac{1}{|K|} \int_K \text{div}(\varepsilon u) \quad \forall K \in \mathcal{C}_h,
\]

\[
R_h(\text{curl} u)_K := \frac{1}{|K|} \int_K \text{curl} u \quad \forall K \in \mathcal{C}_h,
\]

\[
R_h^F((\varepsilon u) \cdot n)_F := \frac{1}{|F|} \int_F (\varepsilon u) \cdot n \quad \forall F \subset \Gamma,
\]

where \( u \) is assumed to belong to \( U \) defined in (2.4), \( |K| \) and \( |F| \) respectively denote the volumes of \( K \) and \( F, \)

\[
S_h(u, v) := \int_\Gamma R_h^F((\varepsilon u) \cdot n) R_h^F((\varepsilon v) \cdot n) + S_{h, \text{div}}(u, v) \\
+ S_{h, \text{curl}}(u, v) + S_{h, \Gamma}(u, v) + S_{\text{flux}, \Sigma}(u, v),
\]

\[
Z_h(f, g; v) := Z_{h, \text{div}}(f; v) + Z_{h, \text{curl}}(g; v) + Z_{h, \Gamma}(f; v),
\]

where the definitions of those mesh-dependent bilinear and linear forms of

\[
S_{h, \text{div}}(u, v), S_{h, \text{curl}}(u, v), S_{h, \Gamma}(u, v), S_{\text{flux}, \Sigma}(u, v)
\]
and
\begin{equation}
Z_{h,\text{div}}(f;v), Z_{h,\text{curl}}(g;v), Z_{h,\Gamma}(f;v)
\end{equation}
will be concretely given in section 3.3.

We next describe the pseudolocal $L^2$ projection method.

We define
\begin{equation}
(p, q)_h := \sum_{K \in \mathcal{C}_h} \frac{|K|}{4} \sum_{i=1}^{4} p(a_i) q(a_i) \quad \text{(trapezoidal quadrature),}
\end{equation}
\begin{equation}
U_h := (P_h)^3 + \Psi_h,
\end{equation}
\begin{equation}
(\tilde{R}_h(\text{div} (\varepsilon u)), q)_h = -\sum_{K \in \mathcal{C}_h} (\varepsilon u, \nabla q)_{0,K} \quad \forall q \in V_h,
\end{equation}
\begin{equation}
(R_h(\text{curl} u), v)_h = (u, \text{curl} v) \quad \forall v \in W_h,
\end{equation}
where $u$ is assumed to belong to $(L^2(\Omega))^3$, $V_h$ and $W_h$ are given by (3.12) and (3.13), and
\begin{equation}
S_h(u, v) := S_{h,\text{div}}(u, v) + S_{h,\text{curl}}(u, v) + S_{h,\Gamma}(u, v),
\end{equation}
\begin{equation}
Z_h(f, g; v) := Z_{h,\text{div}}(f; v) + Z_{h,\text{curl}}(g; v) + Z_{h,\Gamma}(f; v).
\end{equation}

Remark 3.1. Noting that the resulting matrix of the trapezoidal quadrature scheme defined by (3.25) is diagonal, for this reason we call $R_h$ and $\tilde{R}_h$, defined by (3.27) and (3.28), pseudolocal $L^2$ projectors.

3.3. Mesh-dependent bilinear and linear forms. In this subsection we shall define those mesh-dependent bilinear and linear forms as in (3.23) and (3.24).

We need to introduce some local spaces of some suitable functions.

Let $F_K : \hat{K} \rightarrow K$ denote the invertible mapping from the reference element $\hat{K}$ onto $K \in \mathcal{C}_h$, i.e., $K = F_K(\hat{K})$, which associates the function $q$ defined on $K$ with the function $\tilde{q}$ defined on $\hat{K}$ by $q = \tilde{q} \circ F_K^{-1}$. On $\hat{K}$ we introduce three spaces of some suitable functions as follows:
\begin{equation}
\begin{cases}
S_{\text{div}}(\hat{K}) := \text{span}\{\tilde{\omega}_l; 1 \leq l \leq m_{\text{div}}\} \subset L^2(\hat{K}), \\
S_{\Gamma}(\hat{K}) := \text{span}\{\tilde{\gamma}_l; 1 \leq l \leq m_{\Gamma}\} \subset L^2(\hat{K}), \\
S_{\text{curl}}(\hat{K}) := \text{span}\{\tilde{\omega}_l; 1 \leq l \leq m_{\text{curl}}\} \subset (L^2(\hat{K}))^3,
\end{cases}
\end{equation}
where three integers $m_{\text{div}}, m_{\Gamma}, m_{\text{curl}}$ denoting the dimensions of corresponding spaces and these functions $\tilde{\omega}, \tilde{\gamma}, \tilde{\omega}$ are determined according to Hypothesis H1 in section 4.1. Using $F_K$ we obtain on $K \in \mathcal{C}_h$ three local spaces as follows:
\begin{equation}
\begin{cases}
S_{\text{div}}(K) := S_{\text{div}}(\hat{K}) \circ F_K^{-1} = \text{span}\{\omega_{K,l} := \tilde{\omega}_l \circ F_K^{-1}; 1 \leq l \leq m_{\text{div}}\}, \\
S_{\Gamma}(K) := S_{\Gamma}(\hat{K}) \circ F_K^{-1} = \text{span}\{\gamma_{K,l} := \tilde{\gamma}_l \circ F_K^{-1}; 1 \leq l \leq m_{\Gamma}\}, \\
S_{\text{curl}}(K) := S_{\text{curl}}(\hat{K}) \circ F_K^{-1} = \text{span}\{\omega_{K,l} := \tilde{\omega}_l \circ F_K^{-1}; 1 \leq l \leq m_{\text{curl}}\}.
\end{cases}
\end{equation}

Remark 3.2. Consider $\varepsilon = 1$ on $K$. For the local $L^2$ projection method we may choose
\[ S_{\text{div}}(K) = P_2(K), \quad S_{\Gamma}(K) = P_3(K), \quad S_{\text{curl}}(K) = (P_2(K))^3. \]
For the pseudolocal $L^2$ projection method we can choose

$$S_{\text{div}}(K) = \mathcal{P}_0(K) + \text{span} \left\{ \frac{\partial b_K}{\partial x}, \frac{\partial b_K}{\partial y}, \frac{\partial b_K}{\partial z} \right\}, \quad S_\Gamma(K) = \mathcal{P}_1(K),$$

$$S_{\text{curl}}(K) = (\mathcal{P}_0(K))^3$$

$$+ \text{span} \left\{ \left( c_3 \frac{\partial b_K}{\partial y} - c_2 \frac{\partial b_K}{\partial z}, c_1 \frac{\partial b_K}{\partial z} - c_3 \frac{\partial b_K}{\partial x}, c_2 \frac{\partial b_K}{\partial x} - c_1 \frac{\partial b_K}{\partial y} \right)^t \right\}$$

$$\forall (c_1, c_2, c_3) \in \mathbb{R}^3 \right).$$

Note that, in general, $S_{\text{div}}(K)$, $S_\Gamma(K)$, and $S_{\text{curl}}(K)$ are not spaces of polynomials. However, if $\varepsilon$ is piecewise smooth, we may replace $\varepsilon$ by its suitable piecewise polynomial approximation, say, $\varepsilon_h$. With this $\varepsilon_h$, then $S_{\text{div}}(K)$, $S_\Gamma(K)$, and $S_{\text{curl}}(K)$ are of course chosen as piecewise polynomials. See also Remarks 5.4 and 5.5.

We are now in a position to define the mesh-dependent bilinear and linear forms:

$$S_{h,\text{div}}(u, v) := \sum_{K \in \mathcal{C}_h} \sum_{l=1}^{m_{\text{div}}} (\varepsilon u, \nabla (v_{K,l} b_K))_{0,K} (\varepsilon v, \nabla (v_{K,l} b_K))_{0,K},$$

$$Z_{h,\text{div}}(f; v) := -\sum_{K \in \mathcal{C}_h} \sum_{l=1}^{m_{\text{div}}} (f, v_{K,l} b_K)_{0,K} (\nabla v, \nabla (v_{K,l} b_K))_{0,K},$$

$$S_{h,\text{curl}}(u; v) := \sum_{K \in \mathcal{C}_h} \sum_{l=1}^{m_{\text{curl}}} (u, \text{curl}(w_{K,l} b_K))_{0,K} (v, \text{curl}(w_{K,l} b_K))_{0,K},$$

$$Z_{h,\text{curl}}(g; v) := \sum_{K \in \mathcal{C}_h} \sum_{l=1}^{m_{\text{curl}}} (g, w_{K,l} b_K)_{0,K} (v, \text{curl}(w_{K,l} b_K))_{0,K},$$

$$S_{h,\Gamma}(u, v) := \sum_{F \subset \Gamma} \sum_{F \subset \partial K} \sum_{l=1}^{m_{\Gamma}} (\varepsilon u, \nabla (z_{K,l} b_F))_{0,K} (\varepsilon v, \nabla (z_{K,l} b_F))_{0,K},$$

$$Z_{h,\Gamma}(f; v) := -\sum_{F \subset \Gamma} \sum_{F \subset \partial K} \sum_{l=1}^{m_{\Gamma}} (f, z_{K,l} b_F)_{0,K} (\varepsilon v, \nabla (z_{K,l} b_F))_{0,K},$$

$$S_{\text{flux},\Sigma}(u, v) := \sum_{j=1}^{N} \frac{1}{|\Sigma_j|} \int_{\Sigma_j} (\varepsilon u) \cdot n \int_{\Sigma_j} (\varepsilon v) \cdot n.$$

Remark 3.3. Note that $b_K \in H_0^1(K)$ is the element bubble defined as in (3.2) and that $b_F \in H_0^1(F)$ is the face bubble defined as in (3.1); we have $v_{K,l} b_K \in H_0^1(K)$, $w_{K,l} b_K \in (H_0^1(K))^3$ and $z_{K,l} b_F \in H_0^1(F)$, $z_{K,l} b_F |_{F'} = 0$ for $F' \subset \partial K$, so all the denominators above are not zero on any tetrahedron.

4. Coercivity and condition number. In this section we shall investigate the coercivity property and the condition number associated with the finite element problem described in the previous section.
4.1. Mesh-dependent norm. In this subsection we give some properties of the mesh-dependent bilinear forms in Propositions 4.1 and 4.2 below.

**Proposition 4.1.** We have

\[
|S_{h,\text{div}}(u,v)| \leq \|\varepsilon u\|_0 \|\varepsilon v\|_0,
\]

\[
|S_{h,\text{curl}}(u,v)| \leq \|u\|_0 \|v\|_0,
\]

\[
|S_{h,\Gamma}(u,v)| \leq \|\varepsilon u\|_0 \|\varepsilon v\|_0,
\]

\[
0 \leq S_{h,\text{div}}(v,v) \leq C \sum_{K \in C_h} h_K^2 \|\text{div} (\varepsilon v)\|_{0,K}^2,
\]

\[
0 \leq S_{h,\text{curl}}(v,v) \leq C \sum_{K \in C_h} h_K^2 \|\text{curl} v\|_{0,K}^2,
\]

\[
0 \leq S_{h,\Gamma}(v,v) \leq C \left( \sum_{K \in C_h} h_K^2 \|\text{div} (\varepsilon v)\|_{0,K}^2 + \sum_{F \subset \Gamma} h_F \int_F |(\varepsilon v) \cdot \text{n}|^2 \right).
\]

Here \(h_F\) stands for the diameter of \(F\).

**Proof.** The first three inequalities and the left-hand sides of the last three inequalities easily follow from the definitions given as in (3.33), (3.35), and (3.37). Here we show only the right-hand side of the last inequality as an example. Using Green’s formula of integrating by parts, we have

\[
S_{h,\Gamma}(v,v) = \sum_{F \subset \partial K} \sum_{l=1}^{m_F} \frac{((\varepsilon v, \nabla (z_{K,l} b_F)_{0,K})^2}{\sum_{l=1}^{m_F} \|\nabla (z_{K,l} b_F)\|_{0,K}^2}
\]

\[
= \sum_{F \subset \partial K} \sum_{l=1}^{m_F} \frac{((\varepsilon v, \nabla (z_{K,l} b_F)_{0,F})^2}{\sum_{l=1}^{m_F} \|\nabla (z_{K,l} b_F)\|_{0,K}^2}
\]

\[
\leq C \sum_{F \subset \partial K} \sum_{l=1}^{m_F} \frac{((\varepsilon v) \cdot \text{n}, z_{K,l} b_F)_{0,F}^2}{\sum_{l=1}^{m_F} \|\nabla (z_{K,l} b_F)\|_{0,K}^2}
\]

\[
+ C \sum_{F \subset \partial K} \sum_{l=1}^{m_F} \frac{((\varepsilon v, z_{K,l} b_F)_{0,K})^2}{\sum_{l=1}^{m_F} \|\nabla (z_{K,l} b_F)\|_{0,K}^2},
\]

where, by a standard scaling argument [13, 8],

\[
\sum_{l=1}^{m_F} ((\varepsilon v, z_{K,l} b_F)_{0,K})^2 \leq \|\text{div} (\varepsilon v)\|_{0,K}^2 \sum_{l=1}^{m_F} \|z_{K,l} b_F\|_{0,K}^2,
\]

\[
\leq C h_K^2 \|\text{div} (\varepsilon v)\|_{0,K}^2.
\]
\[
\sum_{l=1}^{m_F} ((\varepsilon \cdot n, z_{K,l} b_F)_{0,F})^2 \leq \|((\varepsilon \cdot n) \cdot n\|_{0,F}^2 \sum_{l=1}^{m_F} \|z_{K,l} b_F\|_{0,F}^2 \\
\leq C h_F^2 \|((\varepsilon \cdot n) \cdot n\|_{0,F}^2, \\
\sum_{l=1}^{m_F} \|\nabla (z_{K,l} b_F)\|_{0,K}^2 \geq C h_K,
\]

where the last three constants \(C\) depend on the \(L^2\) or \(H^1\) norms of given functions \(\hat{z}_l b_F\) on the reference element \(\hat{K}\), \(1 \leq l \leq m_F\), with \(b_F = b_F \circ (\mathcal{F}_K|_F)\), but they are independent of \(h\) and \(K\). It follows that the right-hand side of the last inequality in Proposition 4.1 holds.

We define

\[
\|v\|_{C_h}^2 := \sum_{K \in \mathcal{C}_h} h_K^2 \|\text{div} (\varepsilon v)\|_{0,K}^2 + \sum_{K \in \mathcal{C}_h} h_K^2 \|\text{curl} v\|_{0,K}^2 \\
+ \sum_{F \subset \Gamma} h_F \int_F |(\varepsilon v) \cdot n|^2.
\]

**Hypothesis H1.** We assume that, for any \(u \in U_h\), the following local inclusions hold:

\[
\text{div} (\varepsilon u) |_K \in S_{\text{div}}(K) = \text{span}\{v_{K,l}; 1 \leq l \leq m_{\text{div}}\} \quad \forall K \in \mathcal{C}_h,
\]

\[
(\varepsilon u) \cdot n |_F \in S_{\varepsilon}(K)|_F = \text{span}\{z_{F,l} = z_{K,l}|_F; 1 \leq l \leq m_F\} \quad \forall F \subset \Gamma,
\]

\[
\text{curl} (u) |_K \in S_{\text{curl}}(K) = \text{span}\{w_{K,l}; 1 \leq l \leq m_{\text{curl}}\} \quad \forall K \in \mathcal{C}_h.
\]

We additionally assume that \(v_{K,l}, 1 \leq l \leq m_{\text{div}},\) constitutes a group of linearly independent basis and assume the same for \(z_{F,l}, 1 \leq l \leq m_F,\) and \(w_{K,l}, 1 \leq l \leq m_{\text{curl}}.\)

**Remark 4.1.** Considering the example in Remark 3.2, we see that Hypothesis H1 holds.

**Proposition 4.2.** Assume Hypothesis H1 holds. We have on \(U_h\)

\[
S_{h,\text{div}}(v,v) \geq C \sum_{K \in \mathcal{C}_h} h_K^2 \|\text{div} (\varepsilon v)\|_{0,K}^2,
\]

\[
S_{h,\text{curl}}(v,v) \geq C \sum_{K \in \mathcal{C}_h} h_K^2 \|\text{curl} v\|_{0,K}^2,
\]

\[
S_{h,\Gamma}(v,v) \geq C_1 \sum_{F \subset \Gamma} h_F \int_F |(\varepsilon v) \cdot n|^2 - C_2 \sum_{K \in \mathcal{C}_h} h_K^2 \|\text{div} (\varepsilon v)\|_{0,K}^2.
\]

Consequently, we have

\[
S_{h,\text{div}}(v,v) + S_{h,\text{curl}}(v,v) + S_{h,\Gamma}(v,v) \geq C \|v\|_{C_h}^2 \quad \forall v \in U_h.
\]
Proof. We first show (4.5). From (4.2) we write $\text{div}(\varepsilon \mathbf{v})_K = \sum_{l=1}^{m_{\text{div}}} c_l v_{K,l}$. We have
\[
\sum_{l=1}^{m_{\text{div}}} ((\varepsilon \mathbf{v}, \nabla (v_{K,l} b_K))_{0,K})^2 = \sum_{l=1}^{m_{\text{div}}} ((\text{div}(\varepsilon \mathbf{v}), v_{K,l} b_K)_{0,K})^2
\]
\[
= \sum_{l=1}^{m_{\text{div}}} (c_l^t d_l)^2 = c_l^t A_K^2 c_l,
\]
where $c = (c_1, \ldots, c_{m_{\text{div}}})^t \in \mathbb{R}^{m_{\text{div}}}$, $d_l = (d_{1l}, \ldots, d_{m_{\text{div}}})^t \in \mathbb{R}^{m_{\text{div}}}$, with $d_{il} = (v_{K,i}, v_{K,l} b_K)_{0,K}$, $i = 1, \ldots, m_{\text{div}}$, and $A_K$ is the mass matrix and $A_K = [d_1, \ldots, d_{m_{\text{div}}}] \in \mathbb{R}^{m_{\text{div}} \times m_{\text{div}}}$. Clearly, $A_K$ is symmetric and positive definite. Let $T \in \mathbb{R}^{m_{\text{div}} \times m_{\text{div}}}$ be the orthogonal matrix such that $A_K = T^t \text{diag}(\lambda_1, \ldots, \lambda_{m_{\text{div}}}) T$, where $0 < \lambda_1 < \cdots < \lambda_{m_{\text{div}}}$ are the eigenvalues of $A_K$. It can be easily seen that $\lambda_1 \geq C |K|$. Let $\bar{c} := Tc = (\bar{c}_1, \ldots, \bar{c}_{m_{\text{div}}})^t \in \mathbb{R}^{m_{\text{div}}}$; we have $\sum_{l=1}^{m_{\text{div}}} ((\varepsilon \mathbf{v}, \nabla (v_{K,l} b_K))_{0,K})^2 = \sum_{l=1}^{m_{\text{div}}} (\bar{c}_l \lambda_l)^2$. By a similar argument we have $\langle \text{div}(\varepsilon \mathbf{v}), \text{div}(\varepsilon \mathbf{v}) b_K \rangle_{0,K} = \sum_{l=1}^{m_{\text{div}}} (\bar{c}_l \lambda_l)^2$. We then obtain
\[
\sum_{l=1}^{m_{\text{div}}} ((\varepsilon \mathbf{v}, \nabla (v_{K,l} b_K))_{0,K})^2 = \sum_{l=1}^{m_{\text{div}}} (\bar{c}_l \lambda_l)^2 \geq \lambda_1 \sum_{l=1}^{m_{\text{div}}} (\bar{c}_l)^2 \lambda_l
\]
\[
= \lambda_1 \langle \text{div}(\varepsilon \mathbf{v}), \text{div}(\varepsilon \mathbf{v}) b_K \rangle_{0,K} \geq C |K| \langle \text{div}(\varepsilon \mathbf{v}), \text{div}(\varepsilon \mathbf{v}) \rangle_{0,K}.
\]
Here we used the equivalence $C^{-1} \int_K |g| \leq \int_K \|v_{K,l} b_K\|_{0,K} \leq C \int_K |g|$ for any function $g$ in deriving the last inequality. Noting that $\sum_{l=1}^{m_{\text{div}}} \|\nabla (v_{K,l} b_K)\|_{0,K}^2 \leq C h_K^2$ (where $C$ depends on the $H^1$ norms of $\hat{b}$, $\bar{b}$, $\hat{b} \circ F_K$ and $1 \leq l \leq m_{\text{div}}$, but it is independent of $h$ and $K$), over $C_h$ we take the sum of $\sum_{l=1}^{m_{\text{div}}} \|\nabla (v_{K,l} b_K)\|_{0,K}^2 = C h_K^2 |\text{div}(\varepsilon \mathbf{v})|_{0,K}^2$ to get (4.5), with $S_{h,\text{div}}(\cdot, \cdot)$ defined by (3.33).

The inequality (4.6) can be similarly established from the local inclusion condition (4.4). We next show (4.7). From the definition of $S_{h,\Gamma}(\cdot, \cdot)$ as in (3.37), using Green’s formula of integrating by parts, we have
\[
S_{h,\Gamma}(\mathbf{v}, \mathbf{w}) = \sum_{\mathcal{F} \subseteq K} \sum_{\mathcal{F} \subseteq \partial K} \left( \sum_{l=1}^{m_{\text{pr}}} \frac{((\varepsilon \mathbf{v}, \nabla (z_{K,l} b_F))_{0,F})^2}{\sum_{l=1}^{m_{\text{pr}}} \|\nabla (z_{K,l} b_F)\|_{0,K}^2} \right)
\]
\[
= \sum_{\mathcal{F} \subseteq K} \sum_{\mathcal{F} \subseteq \partial K} \left( \sum_{l=1}^{m_{\text{pr}}} \frac{- (\langle \text{div}(\varepsilon \mathbf{v}), z_{K,l} b_F \rangle_{0,K} + (\langle \varepsilon \mathbf{v}, \mathbf{n} \cdot z_{K,l} b_F \rangle_{0,F})^2}{\sum_{l=1}^{m_{\text{pr}}} \|\nabla (z_{K,l} b_F)\|_{0,K}^2} \right)
\]
\[
\geq C_3 \sum_{\mathcal{F} \subseteq K} \sum_{\mathcal{F} \subseteq \partial K} \left( \sum_{l=1}^{m_{\text{pr}}} \frac{((\varepsilon \mathbf{v}) \cdot \mathbf{n}, z_{K,l} b_F)_{0,F})^2}{\sum_{l=1}^{m_{\text{pr}}} \|\nabla (z_{K,l} b_F)\|_{0,K}^2} \right)
\]
\[
- C_4 \sum_{\mathcal{F} \subseteq K} \sum_{\mathcal{F} \subseteq \partial K} \left( \sum_{l=1}^{m_{\text{pr}}} \frac{((\varepsilon \mathbf{v}), z_{K,l} b_F)_{0,K})^2}{\sum_{l=1}^{m_{\text{pr}}} \|\nabla (z_{K,l} b_F)\|_{0,K}^2} \right),
\]
where, using a similar argument for proving (4.5), we have from the local inclusion condition (4.3)
\[
\sum_{\mathcal{F} \subseteq K} \sum_{\mathcal{F} \subseteq \partial K} \left( \sum_{l=1}^{m_{\text{pr}}} \frac{((\varepsilon \mathbf{v}) \cdot \mathbf{n}, z_{F,l} b_F)_{0,F})^2}{\sum_{l=1}^{m_{\text{pr}}} \|\nabla (z_{K,l} b_F)\|_{0,K}^2} \right) \geq C \sum_{\mathcal{F} \subseteq K} h_F \|((\varepsilon \mathbf{v}) \cdot \mathbf{n})\|_{0,F}^2.
\]
and from Proposition 4.1 we have
\[ \sum_{F \in \mathcal{F}} \sum_{i=1}^{m_F} \left( (\text{div} (\varepsilon v), z_{K,i,b_F}^0)_{0,K} \right)^2 \leq C \sum_{K \in \mathcal{C}_h} h_K^2 \| \text{div} (\varepsilon v) \|_{0,K}^2. \]

The estimate (4.7) thus follows. \( \square \)

Remark 4.2. Propositions 4.1 and 4.2 imply that, instead of using \( S_{h,\text{div}}(u, v) \), \( S_{h,\text{curl}}(u, v) \), and \( S_{h,\Gamma}(u, v) \), one may equivalently use \( (u, v)_{\mathcal{C}_h} \) which corresponds to the mesh-dependent norm (4.1). However, when deriving the error estimates of \( C^0 \) finite elements, one can bound only \( \| \text{curl} \tilde{u} \|_0 \) by \( \| u \|_1 \); here \( \tilde{u} \) is the \( C^0 \) finite element interpolant of \( u \), but \( u \) is not generally in \( H^1 \) in the case of nonsmooth domains. The use of \( S_{h,\text{div}}(u, v) \), \( S_{h,\text{curl}}(u, v) \), and \( S_{h,\Gamma}(u, v) \) allows the solution to be in \( H^s \) with \( 0 \leq s < 1 \). This can easily be seen from the first three inequalities in Proposition 4.1.

Lemma 4.1. Under the same hypotheses as in Proposition 4.2, for the local \( L^2 \) projection method we have
\[ S_h(v, v) \geq C \left\{ \| R_h^0((\varepsilon v) \cdot n) \|_{0,\Gamma}^2 + S_{\text{flux,}\Sigma}(v, v) + \| v \|_{\mathcal{C}_h}^2 \right\} \quad \forall v \in U_h. \]
We can have a similar estimate for the pseudolocal \( L^2 \) projection method.

4.2. Coercivity. This subsection is devoted to the coercivity property of \( \mathcal{L}_h \).

Hypothesis H2. We assume that for any \( \psi \in H^1_{\Gamma}(\text{curl}; \Omega) \cap H_{\text{flux,}\Sigma}(\text{div}; \Omega) \), it can be written as a regular-singular decomposition,
\[ \psi = \psi^0 + \psi^1, \quad \psi^0 \in H^1_{\Gamma}(\text{curl}; \Omega) \cap (H^1(\Omega))^3, \quad \text{curl} \psi^1 = 0, \]
where \( \psi^0 \) is the regular part and \( \psi^1 \) the singular part, with
\[ \| \psi^0 \|_1 \leq C \left\{ \| \psi \|_0 + \| \text{curl} \psi \|_0 \right\}. \]

Remark 4.3. From [5] any \( \psi \in H^1_{\Gamma}(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \) can be written as the following “regular-singular” decomposition:
\[ \psi = \psi^* + \nabla p, \quad p \in H^1_0(\Omega), \]
with \( \psi^* \in H^1_{\Gamma}(\text{curl}; \Omega) \cap (H^1(\Omega))^3 \) and
\[ \| \psi^* \|_1 \leq C \left\{ \| \psi \|_0 + \| \text{curl} \psi \|_0 + \| \text{div} \psi \|_0 \right\}. \]
We may take \( \psi^0 := \psi^* \) and \( \psi^1 := \nabla p \), and then verify Hypothesis H2.

Theorem 4.1. Let Hypotheses H1 and H2 hold. We have
\[ \mathcal{L}_h(u, u) \geq C \| u \|_0^2 \quad \forall u \in U_h. \]
Therefore, the finite element problem has a unique solution.

Proof. We consider only the local \( L^2 \) projection method. The following argument can easily be applied to the pseudolocal \( L^2 \) projection method, with minor modifications.

We first show
\[ (4.9) \quad \mathcal{L}_h(u, u) \geq C \| u \|_0^2 \quad \forall u \in U_h. \]

We consider only the local \( L^2 \) projection method. The following argument can easily be applied to the pseudolocal \( L^2 \) projection method, with minor modifications.

We first show
\[ (4.10) \quad \left\{ \begin{array}{l}
\| \tilde{R}_h(\varepsilon u) \|_0^2 + \| R_h(\text{curl} u) \|_0^2 \\
\geq C_5 \| u \|_0^2 - C_6 (\| R_h^0((\varepsilon u) \cdot n) \|_{0,\Gamma}^2 + S_{\text{flux,}\Sigma}(u, u) + \| u \|_{\mathcal{C}_h}^2 ).
\end{array} \right. \]
and then use Lemma 4.1 to obtain (4.9).

To show (4.10), we need to use the $L^2$ orthogonal decomposition of $u$. From Proposition 2.2 we write $u$ as

(4.11) \[ u = \nabla p + \Pi + \varepsilon^{-1} \text{curl} \psi, \]

with $p \in H^1(\Omega)/\mathbb{R}$, $\Pi \in \mathbb{H}$, $\psi \in H_\Gamma(\text{curl}; \Omega) \cap H_{\text{flux}, \Sigma}(\text{div}^0; \Omega)$, and

(4.12) \[ \|\psi\|_0 \leq C \|\text{curl} \psi\|_0, \quad \|\varepsilon^{1/2} u\|_0^2 = \|\varepsilon^{1/2} \nabla p\|_0^2 + \|\varepsilon^{1/2} \Pi\|_0^2 + \|\varepsilon^{-1/2} \text{curl} \psi\|_0^2. \]

From Hypothesis H2 we further write $\psi$ as

(4.13) \[ \psi = \psi^0 + \psi^1, \]

where

(4.14) \[ \psi^0 \in H_\Gamma(\text{curl}; \Omega) \cap (H^1(\Omega))^3, \quad \text{curl} \psi^1 = 0, \]

and

(4.15) \[ \|\psi^0\|_1 \leq C \{\|\psi\|_0 + \|\text{curl} \psi\|_0\} \leq C \|\text{curl} \psi\|_0. \]

In what follows we divide the proof of (4.10) into three steps according to the three components $(p, \Pi, \psi)$ from the decomposition (4.11).

**Step 1.** We consider $\Pi$.

From Proposition 2.1, let $q \in H^1(\Omega)$ such that $\nabla q = \Pi$ in $\Omega$ and $\|q\|_{H^1(\Omega)} \leq C \|\Pi\|_0$. Let $\tilde{q}$ and $\bar{q}$ be piecewise constants such that [18, 19, 20]

(4.16) \[ \bar{q}_{K} = \frac{1}{|K|} \int_K q, \quad \|\tilde{q}\|_0, K \leq \|q\|_0, K \quad \forall K \in \mathcal{C}_h, \]

and

(4.17) \[ \left( \sum_{K \in \mathcal{C}_h} h_K^{-2} \|\tilde{q} - q\|_{0, K}^2 \right)^{1/2} \leq C \|q\|_{H^1(\Omega)}, \]

and

(4.18) \[ \bar{q}_{F} = \frac{1}{|F|} \int_F q, \quad \|\tilde{q}\|_{0, F} \leq \|q\|_{0, F} \quad \forall F \subset \Gamma, \]

and

(4.19) \[ \left( \sum_{F \subset \Gamma} h_F^{-1} \|\tilde{q} - q\|_{0, F}^2 \right)^{1/2} \leq C \|q\|_{H^1(\Omega)}. \]
Let $\epsilon_1 > 0$ be a constant to be determined. We have

\begin{equation}
(4.20) \quad \|\bar{R}_h(\text{div}(\epsilon u))\|^2=\|\bar{R}_h(\text{div}(\epsilon u)) + \epsilon_1 \bar{q}\|^2 - \epsilon_1^2 \|\bar{q}\|^2 - 2\epsilon_1 (\bar{R}_h(\text{div}(\epsilon u)), \bar{q}),
\end{equation}

\begin{equation}
(4.21) \quad \|\bar{q}\|_0 \leq C \|\bar{q}\|_{L^2(\hat{\Omega})} \leq C \|q\|_{H^1(\hat{\Omega})} \leq C \|\Pi\|_0 \leq C \|\epsilon^\frac{1}{2} \Pi\|_0,
\end{equation}

\begin{equation}
(4.22) \quad -2\epsilon_1 (\bar{R}_h(\text{div}(\epsilon u)), \bar{q}) = -2\epsilon_1 \sum_{K \in \mathcal{C}_h} (\text{div}(\epsilon u), \bar{q})_0,K
\end{equation}

\begin{equation}
(4.23) \quad 2\epsilon_1 \sum_{K \in \mathcal{C}_h} (\epsilon u, \nabla q)_0,K = 2\epsilon_1 (\epsilon u, \Pi) = 2\epsilon_1 (\epsilon \Pi, \Pi) = 2\epsilon_1 \|\epsilon^\frac{1}{2} \Pi\|_0^2,
\end{equation}

\begin{equation}
(4.24) \quad -2\epsilon_1 \sum_{F \subset \Gamma} \int_F (\epsilon u) \cdot nq = 2\epsilon_1 \sum_{F \subset \Gamma} \int_F (\epsilon u) \cdot n(\bar{q} - q) - 2\epsilon_1 \sum_{F \subset \Gamma} \int_F (\epsilon u) \cdot n\bar{q},
\end{equation}

and

\begin{equation}
(4.25) \quad -2\epsilon_1 \sum_{F \subset \Gamma} \int_F (\epsilon u) \cdot n\bar{q} = -2\epsilon_1 \int_{\Gamma} R^F_h((\epsilon u) \cdot n) \bar{q}
\end{equation}

\begin{align*}
&\geq -\epsilon_1 C \|R^F_h((\epsilon u) \cdot n)\|_{0,r} \|\bar{q}\|_{0,r} \\
&\geq -\epsilon_1 C \|R^F_h((\epsilon u) \cdot n)\|_{0,r} \|q\|_{0,r} \\
&\geq -\epsilon_1 C \|R^F_h((\epsilon u) \cdot n)\|_{0,r} \|q\|_{H^1(\hat{\Omega})} \\
&\geq -\epsilon_2 \|R^F_h((\epsilon u) \cdot n)\|_{0,r}^2 - \frac{C \epsilon^2}{\epsilon_2} \|\epsilon^\frac{1}{2} \Pi\|_0^2.
\end{align*}

Here we have used $\|q\|_{H^1(\hat{\Omega})} \leq C \|\Pi\|_0 \leq C \|\epsilon^\frac{1}{2} \Pi\|_0$ and the trace theorem,

\begin{equation}
\|q\|_{0,r} \leq C \|q\|_{L^2(\partial \hat{\Omega})} \leq C \|q\|_{H^1(\hat{\Omega})},
\end{equation}

and Young’s inequality,

$$|a| |b| \leq \epsilon |a|^2 + \frac{1}{4\epsilon} |b|^2 \quad \forall a, b \in \mathbb{R}, \forall \epsilon > 0,$$

with $\epsilon = \epsilon_2$ a constant to be determined.
We also have
\[
2 \epsilon_1 \sum_{K \in \mathcal{C}_h} (\text{div} (\varepsilon \mathbf{u}), q - \tilde{q})_{0,K} + 2 \epsilon_1 \sum_{F \subset \Gamma} \int_F (\varepsilon \mathbf{u}) \cdot \mathbf{n} (\tilde{q} - q)
\geq - \epsilon_1 C \left( \sum_{K \in \mathcal{C}_h} h_K^2 \|\text{div} (\varepsilon \mathbf{u})\|_{0,K}^2 + \sum_{F \subset \Gamma} h_F \|\varepsilon \mathbf{u}\|_{0,F}^2 \right)^{\frac{1}{2}}
\times \left( \sum_{K \in \mathcal{C}_h} h_K^{-2} \|q - \tilde{q}\|_{0,K}^2 + \sum_{F \subset \Gamma} h_F^{-1} \|\tilde{q} - q\|_{0,F}^2 \right)^{\frac{1}{2}}
\geq - \epsilon_1 C \|\varepsilon \mathbf{u}\|_{0,h} \|q\|_{H^1(\tilde{\Omega})} \geq - \epsilon_1 C \|\varepsilon \mathbf{u}\|_{0,h} \|\varepsilon \frac{1}{2} \Pi\|_0
\]
and
\[
-2 \epsilon_1 \sum_{j=1}^N \int_{\Sigma_j} (\varepsilon \mathbf{u}) \cdot \mathbf{n} [q] \geq -2 \epsilon_1 \sum_{j=1}^N |\Sigma_j|^{-1/2} \left( \int_{\Sigma_j} (\varepsilon \mathbf{u}) \cdot \mathbf{n} \left| \frac{1}{|\Sigma_j|} \right| |q| \right) \left( \int_{\Sigma_j} (\varepsilon \mathbf{u}) \cdot \mathbf{n} \right)^{1/2}
\geq -2 \epsilon_1 \left( \sum_{j=1}^N \frac{1}{|\Sigma_j|} \left( \int_{\Sigma_j} (\varepsilon \mathbf{u}) \cdot \mathbf{n} \right)^{1/2} \right)
\times \left( \sum_{j=1}^N \frac{1}{|\Sigma_j|} \right)^{1/2}
= -2 \epsilon_1 (S_{\text{flux},\Sigma}(\mathbf{u}, \mathbf{u}))^{1/2} \left( \sum_{j=1}^N \left( \int_{\Sigma_j} |q|^2 \right)^{1/2} \right)
\geq -2 \epsilon_1 C (S_{\text{flux},\Sigma}(\mathbf{u}, \mathbf{u}))^{1/2} \|q\|_{L^2(\tilde{\Omega})}
\geq -2 \epsilon_1 C (S_{\text{flux},\Sigma}(\mathbf{u}, \mathbf{u}))^{1/2} \|q\|_{H^1(\tilde{\Omega})}
\geq -\epsilon_2 C (S_{\text{flux},\Sigma}(\mathbf{u}, \mathbf{u}))^{1/2} \|\varepsilon \frac{1}{2} \Pi\|_0^2.
\]
Equations (4.20)–(4.28) yield
\[
\|\bar{R}_h(\text{div} (\varepsilon \mathbf{u}))\|_0^2 \geq \epsilon_1 \left( 2 - C \epsilon_1 - \frac{C \epsilon_1}{\epsilon_2} \right) \|\varepsilon \frac{1}{2} \Pi\|_0^2
- \epsilon_2 \left( \|\bar{R}_h(\varepsilon \mathbf{u}) \cdot \mathbf{n}\|_{0,\Gamma}^2 + S_{\text{flux},\Sigma}(\mathbf{u}, \mathbf{u}) + \|\mathbf{u}\|_{0,h}^2 \right).
\]

Step 2. We consider \( p \).
By a similar argument as that used in proving (4.29) with II replaced by $\nabla p$, we have

\begin{equation}
(4.30)
\begin{cases}
\| \tilde{R}_h(\text{div } (\varepsilon \mathbf{u})) \|_0^2 
\geq \epsilon_1 \left( 2 - C_\epsilon \right) \| \varepsilon^{\frac{1}{2}} \nabla p \|_0^2 \\
- \epsilon_2 \left( \| R_h^L ( (\varepsilon \mathbf{u}) \cdot \mathbf{n}) \|_{0,\Gamma}^2 + \| \mathbf{u} \|_{\mathcal{C}_h}^2 \right).
\end{cases}
\end{equation}

**Step 3.** We consider $\psi$.

Let $\tilde{\psi}^0$ be a piecewise constant vector such that

\begin{equation}
(4.31)
\tilde{\psi}^0|_K = \frac{1}{|K|} \int_K \psi^0, \quad \| \tilde{\psi}^0 \|_{0,K} \leq \| \psi^0 \|_{0,K} \quad \forall K \in \mathcal{C}_h,
\end{equation}

and using Young’s inequality,

\begin{equation}
(4.32)
\left( \sum_{K \in \mathcal{C}_h} h_K^{-2} \| \psi^0 - \tilde{\psi}^0 \|_{0,K}^2 \right)^{1/2} \leq C \| \psi^0 \|_1.
\end{equation}

We have

\begin{equation}
(4.33)
\| R_h(\text{curl } \mathbf{u}) \|_0^2 = \| R_h(\text{curl } \mathbf{u}) - \epsilon_1 \tilde{\psi}^0 \|_0^2 - \epsilon_1^2 \| \tilde{\psi}^0 \|_0^2 + 2 \epsilon_1 \left( R_h(\text{curl } \mathbf{u}), \tilde{\psi}^0 \right),
\end{equation}

\begin{equation}
(4.34)
\| \tilde{\psi}^0 \|_0 \leq \| \psi^0 \|_1 \leq C \| \text{curl } \psi \|_0 \leq C \| \varepsilon^{-\frac{1}{2}} \text{curl } \psi \|_0,
\end{equation}

\begin{equation}
(4.35)
2 \epsilon_1 \left( R_h(\text{curl } \mathbf{u}), \tilde{\psi}^0 \right) = 2 \epsilon_1 \sum_{K \in \mathcal{C}_h} (\text{curl } \mathbf{u}, \tilde{\psi}^0)_{0,K}
= 2 \epsilon_1 \left( \sum_{K \in \mathcal{C}_h} (\text{curl } \mathbf{u}, \tilde{\psi}^0 - \psi^0)_{0,K}
+ 2 \epsilon_1 \left( \varepsilon^{-1} \text{curl } \psi, \text{curl } \psi \right)
= 2 \epsilon_1 \left( \varepsilon^{-1} \text{curl } \psi, \text{curl } \psi \right) \right.
= 2 \epsilon_1 \| \varepsilon^{-\frac{1}{2}} \text{curl } \psi \|_0.
\end{equation}

We have

\begin{equation}
(4.36)
2 \epsilon_1 \sum_{K \in \mathcal{C}_h} (\tilde{\psi}^0 - \psi^0, \text{curl } \mathbf{u}) \geq - \epsilon_1 C \left( \sum_{K \in \mathcal{C}_h} h_K^{-2} \| \tilde{\psi}^0 - \psi^0 \|_{0,K}^2 \right)^{1/2}
\times \left( \sum_{K \in \mathcal{C}_h} h_K^2 \| \text{curl } \mathbf{u} \|_{0,K}^2 \right)^{1/2}
\geq - \epsilon_1 C \| \psi^0 \|_1 \| \mathbf{u} \|_{\mathcal{C}_h}
\geq - \epsilon_1 C \| \varepsilon^{-\frac{1}{2}} \text{curl } \psi \|_0 \| \mathbf{u} \|_{\mathcal{C}_h}
\geq - \epsilon_2 \| \mathbf{u} \|_{\mathcal{C}_h}^2 - \frac{C \epsilon_1^2}{\epsilon_2} \| \varepsilon^{-\frac{1}{2}} \text{curl } \psi \|_0^2.
\end{equation}

Summarizing (4.33)–(4.37), we have

\begin{equation}
(4.38)
\| R_h(\text{curl } \mathbf{u}) \|_0^2 \geq \epsilon_1 \left( 2 - C \epsilon_1 - \frac{C \epsilon_1^2}{\epsilon_2} \right) \| \varepsilon^{-\frac{1}{2}} \text{curl } \psi \|_0^2 - \epsilon_2 \| \mathbf{u} \|_{\mathcal{C}_h}^2.
\end{equation}
Combining (4.29), (4.30), and (4.38), we obtain

\[(4.39) \quad 2 \| \tilde{R}_h(\nabla (\varepsilon u)) \|_0^2 + \| R_h(\nabla u) \|_0^2 \]
\[\geq \epsilon_1 \left( 2 - C \epsilon_1 - \frac{C \epsilon_1}{\epsilon_2} \right) \times \left( \| \varepsilon^{\frac{1}{2}} \nabla p \|_0^2 + \| \varepsilon^{\frac{1}{2}} \Pi \|_0^2 + \| \varepsilon^{-\frac{1}{2}} \nabla \psi \|_0^2 \right) \]
\[- \epsilon_2 C \| \tilde{R}_h^1((\varepsilon u) \cdot n) \|_{0,\Gamma}^2 + S_{\text{flux},\Sigma}(u, u) + \| u \|_{C_h}^2. \]

We therefore obtain (4.10), taking suitable values for \( \epsilon_i, i = 1, 2. \)

4.3. Condition number. We finally estimate the condition number of the resulting linear system. We here again consider only the local \( L^2 \) projection method.

**Theorem 4.2.** In addition to the same hypotheses as in Theorem 4.1, we assume uniform meshes. The condition number, associated with the resulting linear system, is \( \mathcal{O}(h^{-2}). \)

**Proof.** Since, for all \( v \in U_h, \)

\[(4.40) \quad (S_{h,\text{div}}(v, v))^{1/2} + (S_{h,\text{curl}}(v, v))^{1/2} + (S_{h,\Gamma}(v, v))^{1/2} \leq C \| v \|_0, \]
\[(4.41) \quad \| R_h(\nabla v) \|_0 \leq \| \nabla v \|_0 \leq C h^{-1} \| v \|_0, \]
\[(4.42) \quad \| \tilde{R}_h(\nabla v) \|_0 \leq C \| v \|_\infty \left( \sum_{K \in C_h} h_K^{-1} \| v \|_{0,\partial K}^2 \right)^{\frac{1}{2}} \leq C h^{-1} \| v \|_0, \]
\[(4.43) \quad \| R_h^1((\varepsilon v) \cdot n) \|_{0,\Gamma} \leq \left( \sum_{F \subseteq \Gamma} \| (\varepsilon v) \cdot n \|_{0,F}^2 \right)^{1/2} \leq C h^{-1} \| v \|_0, \]
\[(4.44) \quad (S_{\text{flux},\Sigma}(v, v))^{1/2} \leq C h^{-1} \| v \|_0, \]

we have

\[(4.45) \quad \mathcal{L}_h(v, v) \leq C h^{-2} \| v \|_0^2 \quad \forall v \in U_h, \]

which, together with the coercivity property (4.9) and the symmetry property of \( \mathcal{L}_h, \) leads to the conclusion. \( \square \)

5. Error bounds. In this section we shall establish the error bounds. We analyze only the local \( L^2 \) projection method. The analysis is similar for the pseudolocal \( L^2 \) projection method.

**Lemma 5.1.** Let \( u \in U \) and \( u_h \in U_h \) be the exact solution and the finite element solution to the local \( L^2 \) projection method. We have

\[(5.1) \quad \mathcal{L}_h(u - u_h, v_h) = 0 \quad \forall v_h \in U_h. \]

**Proof.** Equation (5.1) holds, since

\[(5.2) \quad S_h(u, v_h) = Z_h(f, g; v_h), \]
\[(5.3) \quad (\tilde{R}_h(\nabla (\varepsilon u)), \tilde{R}_h(\nabla (\varepsilon v_h))) = (\nabla (\varepsilon u), \tilde{R}_h(\nabla (\varepsilon v_h))) = (f, \tilde{R}_h(\nabla (\varepsilon v_h))), \]
\[(5.4) \quad (R_h(\nabla u), R_h(\nabla v_h)) = (\nabla u, R_h(\nabla v_h)) = (g, R_h(\nabla v_h)). \]
Lemma 5.2. Assume that $\mathbf{u} \in (H^s(\Omega))^3$ with $s > \frac{1}{2}$. Then, there exists a $\tilde{\mathbf{u}} \in U_h$ such that

\begin{equation}
\|\tilde{R}_h(\text{div} (\varepsilon (\mathbf{u} - \tilde{\mathbf{u}})))\|_0^2 = \|R_h(\text{curl} (\mathbf{u} - \tilde{\mathbf{u}}))\|_0^2 = 0.
\end{equation}

\begin{equation}
\|R_h^\ell ((\varepsilon (\mathbf{u} - \tilde{\mathbf{u}})) \cdot \mathbf{n})\|_{0,T}^2 = S_{\text{flux}}(\mathbf{u} - \tilde{\mathbf{u}}, \mathbf{u} - \tilde{\mathbf{u}}) = 0,
\end{equation}

\begin{equation}
\|\mathbf{u} - \tilde{\mathbf{u}}\|_0 \leq C h^s \|\mathbf{u}\|_s.
\end{equation}

Proof. We first let $\mathbf{u}^0 \in (P_h)^3$ be such that \cite{3, 14, 29}

\begin{equation}
\|\mathbf{u} - \mathbf{u}^0\|_0 + \left( \sum_{F \in \mathcal{E}_h} h_F \|\mathbf{u} - \mathbf{u}^0\|_{0,F}^2 \right)^{\frac{1}{2}} \leq C h^s \|\mathbf{u}\|_s, \quad s > \frac{1}{2}.
\end{equation}

We then define $\tilde{\mathbf{u}} \in U_h$ such that

\begin{equation}
\tilde{\mathbf{u}}(a) = \mathbf{u}^0(a) \quad \text{for all vertices } a,
\end{equation}

\begin{equation}
\int_F (\tilde{\mathbf{u}} - \mathbf{u}) \cdot \mathbf{p} = 0 \quad \forall \mathbf{p} \in \mathbf{P}(M_F), \forall F \in \mathcal{E}_h,
\end{equation}

where $\mathbf{P}(M_F)$ is defined by (3.7). We write

\begin{equation}
\tilde{\mathbf{u}} = \mathbf{u}^0 + \sum_{F \in \mathcal{E}_h} \sum_{l=1}^{m_F} c_{F,l} \mathbf{p}_{F,l} b_{M_F}.
\end{equation}

Noting that for any given $F \in \mathcal{E}_h$,

\begin{equation}
\sum_{F' \in \mathcal{E}_h, F' \neq F} \int_F \sum_{l=1}^{m_{F'}} c_{F',l} \mathbf{p}_{F',l} b_{M_{F'}} = 0,
\end{equation}

from (5.10) we obtain

\begin{equation}
\sum_{l=1}^{m_F} c_{F,l} \int_F \mathbf{p}_{F,l} \cdot \mathbf{p}_{F,l} b_{M_F} = \int_F (\mathbf{u} - \mathbf{u}^0) \cdot \mathbf{p}_{F,i}, \quad 1 \leq i \leq m_F \forall F \in \mathcal{E}_h,
\end{equation}

which uniquely determines the coefficients $c_{F,l}$, $1 \leq l \leq m_F$, $F \in \mathcal{E}_h$. Also, note that $\sum_{F \in \mathcal{E}_h} \sum_{i=1}^{m_F} c_{F,i} \mathbf{p}_{F,i} b_{M_F}$ is zero at all vertices, and (5.9) and (5.10) uniquely determine $\tilde{\mathbf{u}}$.

Let $M$ be any given macroelement in $\mathcal{M}_h$ corresponding to a face $F \in \mathcal{E}_h$. We write $M$ and $F$ as $M_{F_1}$ and $F_1$. We first consider the case $M_{F_1} = K_1 \cup K_2$, with $K_1 \cap K_2 = F_1 \in \mathcal{E}_h^0$. For convenience we number all the faces in $\partial M_{F_1}$ by $F_1$, one-to-one corresponding to $M_{F_1}$, $2 \leq i \leq 7$. Denote $\mathbf{c}_i := (c_{F_1,1}, \ldots, c_{F_1,m_{F_1}}) \in \mathbb{R}^{m_{F_1}}$, and $\mathbf{L}_i := [\mathbf{p}_{F_1,1}, \ldots, \mathbf{p}_{F_1,m_{F_1}}] \in \mathbb{R}^{m_{F_1} \times m_{F_1}}$; by a standard scaling argument we then have
from (5.12) and (5.13)

\[
\left( \sum_{F \in \mathcal{E}_h} \sum_{l=1}^{m_F} c_{F,l} p_{F,l} b_{F,l} \right)^2 \left( \sum_{i=1}^{7} \sum_{l=1}^{m_{F_i}} c_{F_i,l} p_{F_i,l} b_{M_{F_i}} \right)^2 = \int_{M_{F_i}} \left( \sum_{i=1}^{7} \sum_{l=1}^{m_{F_i}} c_{F_i,l} p_{F_i,l} b_{M_{F_i}} \right)^2
\]

\[
\leq C \sum_{i=1}^{7} |c_i|^2 \int_{M_{F_i}} \left( \sum_{l=1}^{m_{F_i}} |p_{F_i,l} b_{M_{F_i}}|^2 \right) \leq C |M_{F_i}| \sum_{i=1}^{7} |c_i|^2 \leq C \sum_{i=1}^{7} h_{F_i}^4 \| u - u^0 \|_{0,F_i}^2.
\]

We thus obtain

\[
\| u - \hat{u} \|_{0,M} \leq C \| u - u^0 \|_{0,M} + C \sum_{i=1}^{7} h_{F_i}^4 \| u - u^0 \|_{0,F_i}.
\]

Similarly, if $M$ is the tetrahedron sharing an $F$ with $\Gamma$, we have

\[
\| u - \hat{u} \|_{0,M} \leq C \| u - u^0 \|_{0,M} + C \sum_{F \subset \partial M} h_{F}^4 \| u - u^0 \|_{0,F}.
\]

Hence, from (5.15), (5.16), and (5.8), we obtain

\[
\| u - \hat{u} \|_{0,M} \leq C \| u - u^0 \|_{0,M} + C \sum_{F \subset \mathcal{E}_h} h_{F}^4 \| u - u^0 \|_{0,F}^2 \leq C h^s \| u \|_s.
\]

Finally, noting that $n$ is a constant vector and

\[
\mathcal{P}_0(M) \subset n \subset \mathcal{P}(M), \quad (\mathcal{P}_0(M))^3 \times n \subset \mathcal{P}(M),
\]

by virtue of (5.10) we can easily verify (5.5) and (5.6). \[\Box\]

**Theorem 5.1.** Assume that Hypotheses H1 and H2 hold and that $u \in (H^s(\Omega))^3$ with $s > \frac{1}{2}$. Let $u \in U$ and $u_h \in U_h$ be the exact solution and the finite element solution to the local $L^2$ projection method. We have

\[
\| u - u_h \|_{0} \leq C h^s \| u \|_s.
\]

**Proof.** Let $\hat{u} \in U_h$ be constructed as in Lemma 5.2. We have from Lemma 5.1

\[
\| u - \hat{u} \|_{0,M} \leq C h^s \| u \|_s.
\]

(5.19)

That is,

\[
\| u - \hat{u} \| \leq \| u - \hat{u} \|_M.
\]

(5.20)
On the other hand, from Lemma 5.2 and Proposition 4.1,

\begin{equation}
\|\|u - \bar{u}\|\| = L_h(u - \bar{u}, u - \bar{u})
\end{equation}

\begin{align*}
&= \|R_h(\text{div}(\varepsilon (u - \bar{u})))\|_0^2 + \|R_h(\text{curl}(u - \bar{u}))\|_0^2 \\
&\quad + \|R_h^k((\varepsilon (u - \bar{u}) \cdot n))\|_{0, \Gamma}^2 + S_h(\text{div}(u - \bar{u}, u - \bar{u}) \\
&\quad + S_{h, \text{curl}}(u - \bar{u}, u - \bar{u}) \\
&\quad + S_h(\text{div}(u - \bar{u}, u - \bar{u}) + S_{h, \text{curl}}(u - \bar{u}, u - \bar{u}) \\
&\quad + S_{h, r}(u - \bar{u}, u - \bar{u}) \\
&\leq C \|u - \bar{u}\|_0^2.
\end{align*}

Using the triangle inequality, Theorem 4.1, (5.21), (5.22), and Lemma 5.2, we obtain

\begin{equation}
\|u - u_h\|_0 \leq \|u - \bar{u}\|_0 + \|u_h - \bar{u}\|_0 \\
\leq \|u - \bar{u}\|_0 + C \|u_h - \bar{u}\|_0 \\
\leq \|u - \bar{u}\|_0 + C \|u - \bar{u}\|_0 \\
\leq C \|u - \bar{u}\|_0 \leq C h^s \|u\|_s. \quad \Box
\end{equation}

Remark 5.1. From [2] we know that, in the case of $\varepsilon = 1$, the solution of problem (2.1)–(2.3) is in $(H^s(\Omega))^3$ with $s > \frac{1}{2}$.

Remark 5.2. Regarding the pseudolocal $L^2$ projected method, following a similar argument in Lemma 5.2, we can find an interpolant $\hat{u} \in U_h$ of $u \in (H^s(\Omega))^3$ with $s \geq 0$ such that $\|u - \hat{u}\|_0 \leq C h^s \|u\|_s$, and $\hat{u}$ satisfies interpolation properties similar to (5.5) and (5.6). Let $u$ and $u_h$ be the exact solution and the finite element solution to the pseudolocal $L^2$ projection method. Following the same argument in Theorem 5.1, we can obtain

\begin{equation}
\|u - u_h\|_0 \leq C h^s \|u\|_s \quad \forall s \geq 0.
\end{equation}

Remark 5.3. The pseudolocal $L^2$ projection method admits $u \in (H^s(\Omega))^3$ for all $s \geq 0$. The local $L^2$ projection method requires a little more regularity of $u$, i.e., $u \in (H^s(\Omega))^3$ for all $s > \frac{1}{2}$. This is because the latter error estimates on element faces are involved with the trace theorem that requires $s > \frac{1}{2}$; see [12, 14]. However, it is allowed that $u$ has weaker regularity; i.e., $u \in (W^{s, r}(\Omega))^3$ with $s > \frac{1}{r}$ and $2 \leq r \leq \infty$. Then (5.19) would become

\begin{equation}
\|u - u_h\|_0 \leq C h^s \|u\|_{s, r}, \quad s > \frac{1}{r}, \quad \text{and} \quad 2 \leq r \leq \infty.
\end{equation}

Remark 5.4. In the case when $\varepsilon$ is not globally continuous, we assume that $\varepsilon$ is a piecewise Lipschitz continuous function. This determines a partition $\mathcal{P}$ of $\Omega$ into a finite set of subdomains $\Omega_1, \ldots, \Omega_L$ (which are assumed to be polyhedra). In each $\Omega_i$ the restriction of $\varepsilon$ is Lipschitz continuous. We denote by $\Gamma_{ij}$ the faces of $\Omega_i \cap \Omega_j$ and assume that all $\Gamma_{ij}$ are contained in $\Omega$. As usual, the triangulation $\mathcal{C}_h$ should be
conformed with these material interfaces so that each material interface is the union of the element faces in \( C_h \). Let \( \mathcal{E}_{\text{inter},
abla} \subset \mathcal{E}_h \) denote the set of all element faces on \( \Gamma \) and on interfaces \( \Gamma_{ij} \), and let \( \mathcal{M}_{\text{inter},
abla} \subset \mathcal{M}_h \) be the set of macroelements deduced from \( \mathcal{E}_{\text{inter},
abla} \) in a similar way as that for \( \mathcal{M}_h \) in subsection 3.1. The few modifications to our \( L^2 \) projected methods for the case of discontinuous materials are as follows.

We need only modify (3.20) by

\[
R_{h,\text{inter},\nabla}^\Gamma (\| (\varepsilon \mathbf{u}) \cdot \mathbf{n} \|_{L^2} ) := \frac{1}{|F|} \int_F \| (\varepsilon \mathbf{u}) \cdot \mathbf{n} \| \quad \forall F \in \mathcal{E}_{\text{inter},
abla},
\]

modify the mesh-dependent terms \( S_{h,\text{inter},
abla} (\cdot, \cdot) \), \( Z_{h,\text{inter},
abla} (\cdot, \cdot) \) in (3.37) and (3.38) by \( S_{h,\text{inter},
abla} (\cdot, \cdot) \), \( Z_{h,\text{inter},
abla} (\cdot, \cdot) \):

\[
S_{h,\text{inter},
abla} (\mathbf{u}, \mathbf{v}) := \sum_{F \in \mathcal{E}_{\text{inter},
abla}} \sum_{m=1}^{m_{\text{inter}, \nabla}} \left( \varepsilon \mathbf{u} \cdot \nabla (z_{M,l} b_M) \right)_{0,M} \left( \varepsilon \mathbf{v} \cdot \nabla (z_{M,l} b_M) \right)_{0,M} \sum_{l=1}^{m_{\text{inter}, \nabla}} \| \nabla (z_{M,l} b_M) \|_{0,M}^2,
\]

\[
Z_{h,\text{inter},
abla} (f; \mathbf{v}) := - \sum_{F \in \mathcal{E}_{\text{inter},
abla}} \sum_{m=1}^{m_{\text{inter}, \nabla}} \left( f \cdot z_{M,l} b_M \right)_{0,M} \left( \varepsilon \mathbf{v} \cdot \nabla (z_{M,l} b_M) \right)_{0,M} \sum_{l=1}^{m_{\text{inter}, \nabla}} \| \nabla (z_{M,l} b_M) \|_{0,M}^2,
\]

modify the term \( \int_F R_{h,\text{inter},\nabla}^\Gamma ((\varepsilon \mathbf{u}) \cdot \mathbf{n}) R_{h,\text{inter},\nabla}^\Gamma ((\varepsilon \mathbf{v}) \cdot \mathbf{n}) \) in (3.21) by

\[
\sum_{F \in \mathcal{E}_{\text{inter},
abla}} \left( R_{h,\text{inter},\nabla}^\Gamma ((\varepsilon \mathbf{u}) \cdot \mathbf{n}), R_{h,\text{inter},\nabla}^\Gamma ((\varepsilon \mathbf{v}) \cdot \mathbf{n}) \right)_{0,F},
\]

and finally modify the notation \( S_{\Gamma} \) in (3.31), (3.32), and (4.3) by \( S_{\text{inter},\nabla} \). With these modifications, one may follow the same routine in the previous sections to obtain similar stability results and error estimates.

**Remark 5.5.** In Remark 3.2, we mentioned that, to consider that \( S_{\text{div}}(K) \), \( S_{\Gamma}(K) \), and \( S_{\text{curl}}(K) \) are simpler polynomial spaces, one may replace \( \varepsilon \) by a suitable piecewise polynomial approximation, say, \( \varepsilon_h \). This replacement does not affect the theory of stability analysis and error estimates. Simply, one need only work with \( \varepsilon_h \), but note that such replacement introduces inconsistent error terms in Lemma 5.1 as follows:

\[
(\text{div } ((\varepsilon - \varepsilon_h) \mathbf{u}), \tilde{R}_h(\text{div} (\varepsilon_h \mathbf{v}_h))) \quad \text{for the local } L^2 \text{ projection method},
\]

\[
\sum_{K \in \mathcal{C}_h} ((\varepsilon - \varepsilon_h) \mathbf{u}, \nabla \tilde{R}_h(\text{div}(\varepsilon_h \mathbf{v}_h)))_{0,K} \quad \text{for the pseudolocal } L^2 \text{ projection method},
\]

\[
\sum_{K \in \mathcal{C}_h} \sum_{l=1}^{m_{\text{div}}} ((\varepsilon - \varepsilon_h) \mathbf{u}, \nabla (v_{K,l} b_K))_{0,K} (\varepsilon_h \mathbf{v}_h, \nabla (v_{K,l} b_K))_{0,K} \sum_{l=1}^{m_{\text{div}}} \| \nabla (v_{K,l} b_K) \|_{0,K}^2,
\]

\[
\sum_{F \subset K} \sum_{l=1}^{m_{\Gamma}} ((\varepsilon - \varepsilon_h) \mathbf{u}, \nabla (z_{K,l} b_F))_{0,K} (\varepsilon_h \mathbf{v}_h, \nabla (z_{K,l} b_F))_{0,K} \sum_{l=1}^{m_{\Gamma}} \| \nabla (z_{K,l} b_F) \|_{0,K}^2,
\]

\[
\sum_{j=1}^{N} \frac{1}{|\Sigma_j|} \int_{\Sigma_j} ((\varepsilon - \varepsilon_h) \mathbf{u}) \cdot \mathbf{n} \int_{\Sigma_j} (\varepsilon_h \mathbf{v}_h) \cdot \mathbf{n} \quad \text{for the local } L^2 \text{ projection method}.
\]
Assume that \( \varepsilon \) is smooth enough, say, \( \varepsilon_{ij} \in H^{\frac{s}{2}}, 1 \leq i, j \leq 3 \). Let \( \varepsilon_h \) be taken as a \( C^0 \) finite element interpolant to \( \varepsilon \); for the pseudolocal \( L^2 \) projection method, \( \varepsilon_{ij,h} | K \in P^+_2(K) \) for all \( K \in \mathcal{C}_h \), where \( P^+_2(K) \) denotes the quadratic element \( P_2(K) \) plus one element bubble, while for the local \( L^2 \) projection method, \( \varepsilon_{ij,h} | K \in P^\square_2(K) \) for all \( K \in \mathcal{C}_h \), where \( P^\square_2(K) \) denotes the quadratic element \( P_2(K) \) plus four face bubbles. Recall that \( K \in \mathcal{C}_h \) is a tetrahedron. These bubbles ensure that \( \varepsilon_{ij,h}, 1 \leq i, j \leq 3 \), satisfy the interpolation property

\[
\begin{align*}
\int_K (\varepsilon_{ij} - \varepsilon_{ij,h}) = 0 & \quad \forall K \in \mathcal{C}_h \quad \text{for the pseudolocal } L^2 \text{ projection method} \\
\int_F (\varepsilon_{ij} - \varepsilon_{ij,h}) = 0 & \quad \forall F \in \partial K, \forall K \in \mathcal{C}_h, \quad \text{for the local } L^2 \text{ projection method.}
\end{align*}
\]

We have from [13, 20] that

\[
\| \varepsilon - \varepsilon_h \|_{0,K} + h_K \| \varepsilon - \varepsilon_h \|_{1,K} \leq C h_K^{\frac{5}{2}} \| \varepsilon \|_{2,K} \quad \forall K \in \mathcal{C}_h,
\]

where \( | \cdot | \) denotes the seminorm of \( H^1 \). Note that \( \varepsilon_h \) satisfies the same uniform ellipticity property as \( \varepsilon \) for a suitably small \( h \).

It suffices to explain how to estimate (5.30) for the local \( L^2 \) projection method and (5.31) for the pseudolocal \( L^2 \) projection method. Error terms (5.32)–(5.34) can be estimated similarly. We first consider (5.30). Since we have assumed that \( u \in (H^s(\Omega))^3 \) with \( s > \frac{1}{2} \), letting

\[
\bar{u}|_K = \frac{1}{|K|} \int_K u \quad \forall K \in \mathcal{C}_h,
\]

we can obtain

\[
\begin{align*}
(\text{div}(\varepsilon - \varepsilon_h) u, \tilde{R}_h(\text{div}(\varepsilon_h v_h))) &= \sum_{K \in \mathcal{C}_h} \tilde{R}_h(\text{div}(\varepsilon_h v_h)) \sum_{F \subset \partial K} \int_F ((\varepsilon - \varepsilon_h) u) \cdot n \\
&= \sum_{K \in \mathcal{C}_h} \tilde{R}_h(\text{div}(\varepsilon_h v_h)) \sum_{F \subset \partial K} \int_F ((\varepsilon - \varepsilon_h) (u - \bar{u})) \cdot n \\
&\leq C h^s |u|_s ||\tilde{R}_h(\text{div}(\varepsilon_h v_h))||_0.
\end{align*}
\]

We next consider (5.31). Since \( u \in (H^s(\Omega))^3 \) with \( s \geq 0 \), we have

\[
\begin{align*}
\sum_{K \in \mathcal{C}_h} ((\varepsilon - \varepsilon_h) u, \nabla \tilde{R}_h(\text{div}(\varepsilon_h v_h)))_{0,K} &= \sum_{K \in \mathcal{C}_h} \nabla \tilde{R}_h(\text{div}(\varepsilon_h v_h)) \cdot \int_K (\varepsilon - \varepsilon_h) u \\
&= \sum_{K \in \mathcal{C}_h} \nabla \tilde{R}_h(\text{div}(\varepsilon_h v_h)) \cdot \int_K (\varepsilon - \varepsilon_h) (u - \bar{u}) \\
&\leq C h^s |u|_s ||\tilde{R}_h(\text{div}(\varepsilon_h v_h))||_h,
\end{align*}
\]
where we have used the norm equivalence between \( \| \cdot \|_0 \) and \( \| \cdot \|_h \) (since \( \| \cdot \|_h \) is a norm induced from the trapezoidal quadrature scheme in (3.25)); see [13, 8].

If assuming \( \varepsilon_{ij} \in W^{1,\infty}, \) \( 1 \leq i, j \leq 3, \) then for the pseudolocal \( L^2 \) projection method, we may choose \( \varepsilon_h \) as a piecewise linear (plus one element bubble) continuous approximation with \( \varepsilon_{ij,h}|_K \in \mathcal{P}_1^+(K) \), where \( \mathcal{P}_1^+(K) \) denotes the linear polynomial space plus one element bubble (so \( \varepsilon_{ij,h}|_K \) can satisfy (5.35)). From [3, 19, 13] we have \( \| \varepsilon - \varepsilon_h \|_{0,\infty,K} \leq C h_K \| \varepsilon \|_{1,\infty,K} \) for all \( K \in \mathcal{C}_h \), and we can obtain an estimate similar to (5.40); while for the local \( L^2 \) projection method, we may choose \( \varepsilon_h \) as a piecewise linear (plus four face bubbles) continuous approximation of \( \varepsilon \) with \( \varepsilon_{ij,h}|_K \in \mathcal{P}_0(K) \), where \( \mathcal{P}_0(K) \) denotes the linear polynomial space plus four face bubbles (so \( \varepsilon_{ij,h}|_K \) can satisfy (5.36)). From [3, 19, 13] we have \( \| \varepsilon - \varepsilon_h \|_{0,K} + h_K \| \varepsilon - \varepsilon_h \|_{1,K} \leq C h_K^{5/2} \| \varepsilon \|_{1,\infty,K} \) for all \( K \in \mathcal{C}_h \), and we can obtain an estimate similar to (5.39).

Note that we may also choose \( \varepsilon_h \) as a suitable discontinuous piecewise polynomial approximation to \( \varepsilon \), provided that (5.35) (or (5.36)) and the corresponding interpolation error estimates are satisfied. For example, for the pseudolocal \( L^2 \) projection method, we may choose a discontinuous piecewise constant \( \varepsilon_h \), with \( \varepsilon_{ij,h}|_K \in \mathcal{P}_0(K) \) for all \( K \in \mathcal{C}_h \). In that case, one should work with \( \varepsilon_h \) following a similar modifying routine for discontinuous materials as highlighted in Remark 5.4.

6. Numerical experiments. In this section we perform some numerical tests for the local \( L^2 \) projection method. We consider a two-dimensional problem, with an \( L \) domain \( \Omega = [-1,1] \times [-1,1] \setminus [0,1] \times [-1,0] \) (see Figure 2).

The continuous problem reads: Find \( u \) such that

\[
\text{curl} \, u = g, \quad \text{div} \, u = f \quad \text{in} \, \Omega, \quad u \cdot n = \chi \quad \text{on} \, \Gamma = \partial \Omega.
\]

We first consider a case of nonsmooth solution and take

\[
u = \nabla \left( r^{3/2} \cos \left( \frac{2 \theta}{3} \right) \right),
\]

where \( x = r \cos(\theta), y = r \sin(\theta) \), and \( r \) is the distance to the reentrant corner \((0,0)\) (at the origin) of opening angle \( 3\pi/2 \). We determine \( g := \text{curl} \, u, f := \text{div} \, u, \) and \( \chi := u \cdot n|_\Gamma \). We also consider a case of smooth solution and take

\[
u = (\sin(\pi x) \cos(\pi y)/2\pi, \cos(\pi x) \sin(\pi y)/2\pi)^T.
\]

The regularity for the first \( u \) is \( (H^{3/2} - \epsilon)^2 \) for any \( \epsilon \in (0,1) \), and from the theoretical result obtained the error reduction ratio should be approximately \( 2^{2/3} \approx 1.586 \). The
second $\mathbf{u}$ is infinitely smooth; the error reduction ratio should be around 8 since $U_h$ corresponds to a quadratic element. (In two dimensions, the linear element enriched with edge bubbles is none other than the quadratic element.) The calculated results are listed in Tables 1 and 2 as follows. From Table 1 we see that the error reduction ratio is consistent with the predicted value 1.586, and from Table 2 we see that the error reduction ratio is approximately the predicted value 8 as $h$ decreases. These computational results confirm our theoretical estimates.

**Acknowledgment.** The authors are very thankful to the anonymous referees for valuable comments which led to the improved presentation of this paper.

**REFERENCES**


